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OSCILLATION CRITERIA FOR n-TH ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH "MAXIMA"

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Abstract: In this paper differential equations with "maxima" of the type

$$L_n x(t) + f(t, \max_{[\sigma(t),\tau(t)]} x(s)) = 0 \tag{E}$$

is considered, where $n \geq 2$, $\sigma(t), \tau(t)$ are continuous functions and $\sigma(t) \leq \tau(t) \leq t$.

The oscillation of equation (E) is reduced to the oscillation of certain set of second order comparison differential equations.

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1. Introduction

We consider the n-th order differential equation with "maxima"

$$L_n x(t) + f(t, \max_{[\sigma(t),\tau(t)]} x(s)) = 0, \tag{1}$$

where $\sigma(t), \tau(t)$ are continuous functions and $\sigma(t) \leq \tau(t) \leq t$. Here $n \geq 2$ is an integer, $t \in J = [\alpha, +\infty) \subseteq R_+ = [0, +\infty)$ and

$$L_0 x(t) = x(t), \quad L_k x(t) = r_k(t)(L_{k-1}x(t))', \quad k = 1, ..., n.$$

The domain $D(L_n)$ of L_n is defined to be the set of all functions $x : [t_0, +\infty) \to R$ such that $L_k x(t), k+1, ..., n$ exist and are continuous on the interval $[t_0, +\infty) \subseteq J$. By a *proper* solution of equation (1) is meant a function $x \in D(L_n)$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)| : t \geq T\} > 0$ for $T \geq t_0$. A proper solution of equation (1) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *nonoscillatory*. Equation (1) is said to be *oscillatory* if all its proper solutions are oscillatory. Equation (1) is said to be *almost oscillatory* if it every proper solution $x(t)$ of equation (1) is either oscillatory or $|L_i x(t)| \to 0$ monotonically as $t \to +\infty, i = 0, 1, ..., n-1$.

We suppose that

$$\int^{\infty} \frac{ds}{r_i(s)} = +\infty \quad \text{and} \quad f(t, x)\operatorname{sgn} x \geq a(t)\,|x|\,,$$

for $i = 1, ..., n, t \in J$ and $x \neq 0$, where $a \in C(J, R_+)$.

In the present paper we are interested in conditions under which equation (1) is oscillatory (for n even) and almost oscillatory (for n odd). Various oscillation criteria are obtained for higher order differential equations. For typical results on the subject we refer to the papers [2], [4], [5], [6], [7], [8], [9], [10], [12], [13], [14],[15], [17], [20], [22]. In the papers [4], [5] and [15] the oscillation of equation (1) is compared with the oscillation of one or several comparison equations (see [15]). The main results (Theorems 1, 2, 3 and 4) generalize and improve the results of [15].

2. Preliminary Notes

Introduce the following conditions:

H1. $r_i \in C(J,(0,+\infty)), i=1,...,n-1, r_n \equiv 1$ and

$$\int^{\infty} \frac{ds}{r_i(s)} = +\infty, \quad i=1,...,n-1. \tag{2}$$

H2. $f \in C(J \times R, R)$ and there exists a function $a \in C(J, R_+)$ such that

$$f(t,x)\operatorname{sgn} x \geq a(t)\,|x| \quad \text{for} \quad t \in J, x \neq 0. \tag{3}$$

H3. $\sigma, \tau \in C(J,R)$ $\lim_{t\to+\infty} \sigma(t) = +\infty$ and $\sigma(t) \leq \tau(t) \leq t$ for $t \in J$.

H4. $\sigma \in C^1(J,R), \sigma'(t) \geq 0$ for $t \in J$ and $\sigma(t)$ has an inverse function.

In order to formulate our results we use the following notations:

$$I_0 = 1,$$

$$I_j(t,s;r_j,...,r_1) = \int_s^t \frac{1}{r_j(u)} I_{j-1}(u,s;r_{j-1},...,r_1)du, \quad j=1,...,n-1.$$

It is easy to verify that for $j=1,...,n-1$

$$I_j(t,s;r_j,...,r_1) = (-1)I_j(s,t;r_1,...,r_j), \tag{4}$$

$$I_j(t,s;r_j,...,r_1) = \int_s^t \frac{1}{r_1(u)} I_{j-1}(t,u;r_j,...,r_2)du \tag{5}$$

and the following is valid (see [20]), Lemma 1 (v)):

$$\int_T^t \frac{1}{r_j(s)} I_{j-1}(u,s;r_1,...,r_{j-1})ds \geq \frac{I_1(t,T;r_j)}{I_1(u,T;r_j)} I_j(u,T;r_1,...,r_j) \tag{6}$$

for $\alpha \leq T \leq t < u$.

We will need the following lemmas.

Lemma 1. *If $x \in D(L_n)$, then for $t,s \in J$ and $0 \leq i < \nu \leq n$:*

$$(i) \qquad L_i x(t) = \sum_{j=i}^{\nu-1} I_{j-i}(t,s;r_{i+1},...,r_j)L_j x(s) + \int_s^t I_{\nu-i-1}(t,u;r_{i+1},...,r_{\nu-1})\frac{L_\nu x(u)}{r_\nu(u)}du;$$

$$(ii) \qquad L_i x(t) = \sum_{j=i}^{\nu-1} (-1)^{j-i} I_{j-i}(s,t;r_j,...,r_{i+1})L_j x(s)$$

$$+(-1)^{\nu-i}\int_t^s I_{\nu-i-1}(u,t;r_{\nu-1},...,r_{i+1})\frac{L_\nu x(u)}{r_\nu(u)}du.$$

Lemma 2. *Suppose that condition (2) holds and the functions $L_n x$ and $x \in D(L_n)$ are of constant sing and not identically zero for $t \geq t_* \geq \alpha$. Then:*

(i) There exist a $t_k \geq t_*$ and an integer $k, 0 \leq k < n$ with $n+k$ odd for $x(t)L_n x(t)$ nonpositive such that for every $t \geq t_k$

$$x(t)L_i x(t) > 0, \quad i = 0,1,...,k, \tag{7}$$

$$(-1)^{k-i}x(t)L_i x(t) > 0, \quad i = k+1,...,n-1; \tag{8}$$

(ii) The following inequality is valid

$$\frac{I_\nu(t,t_0;r_{k-\nu+1},...,r_k)}{|L_{k-\nu}x(t)|} \geq \frac{I_{\nu+1}(t,t_0;r_{k-\nu},...,r_k)}{|L_{k-\nu-1}x(t)|} \tag{9}$$

for $t \geq t_0 \geq t_k$ and $\nu = 0,1,...,k-1$.

Lemma 1 is a generalization of Taylor's formula with reminder encountered in calculus. The proof is immediate.

Lemma 2 generalizes the well-known lemma of Kiguradze [12] and can be proved similarly. The proof of part (ii) is given in [20]. From inequality (9) with $\nu = 1,...,k-1$ it follows

$$|x(t)| \geq \frac{I_k(t,t_0;r_1,...,r_k)}{I_1(t,t_0;r_k)}\left|L_{k-1}x(t)\right|, \quad t > t_0 \geq t_k. \tag{10}$$

Consider the equations

$$(r(t)x'(t))' + p(t)x(\sigma(t)) = 0, \tag{11}$$

$$(r(t)x'(t))' + q(t)x(\tau(t)) = 0, \tag{12}$$

and the inequality

$$\{(r(t)x'(t))' + q(t)x(\tau(t))\}\operatorname{sgn} x(t) \leq 0, \tag{13}$$

where

$$r,p,q,\sigma,\tau \in C(J,R_+), \quad \int^\infty \frac{ds}{r(s)} = +\infty, \quad \lim_{t\to+\infty}\sigma(t) = +\infty \tag{14}$$

and $p(t) \leq q(t), \quad \sigma(t) \leq \tau(t)$ for $t \in J$.

Lemma 3. *Let conditions (14) hold. Then:*

(i) Equation (11) has a nonoscillatory solution if inequality (13) has a nonoscillatory solution;

(ii) Equation (12) is oscillatory if equation (11) is oscillatory.

We omit here the proof of Lemma 3, because it is the same as the proofs of analogous assertions (see [1], [3]).

Remark 1. Various oscillation criteria for particular cases of equation (11) are given in [6], [11], [13], [16], [21], [22], [24], [26].

We note that equation (11) is oscillatory, if

$$\int^{\infty} p(t)dt < +\infty \tag{15}$$

and one of the following inequalities holds:

$$\limsup_{t\to+\infty} \int_{\alpha}^{\sigma(t)} \frac{ds}{r(s)} \int_{t}^{\infty} p(s)ds > 1, \tag{16}$$

$$\liminf_{t\to+\infty} \int_{\alpha}^{\sigma(t)} \frac{ds}{r(s)} \int_{t}^{\infty} p(s)ds > \frac{1}{4}. \tag{17}$$

These criteria generalize the criteria given in [11], [13], [21], [24] and are proved by Dzurina [6] and Ohridska [19] in the case when $\sigma(t)$ is differentiable and $\sigma'(t) > 0$.

3. Main Results

Consider the following second order comparison equation

$$(r_k(t)x'(t))' + A_k(t)x(\sigma(t)) = 0, \tag{18;k}$$

where

$$A_k(t) = \frac{1}{r_{k+1}(t)} \times \int_{t}^{\infty} I_{k-1}(\sigma(u), \sigma(t); r_1, ..., r_{k-1}) I_{n-k-2}(u, t; r_{n-1}, ..., r_{k+2}) a(u)du, \tag{19}$$

for $k = 1, 2, ..., n-3$ and

$$A_{n-1}(t) = \frac{\sigma'(t)}{r_{n-2}(\sigma(t))} \int_{t}^{\infty} I_{n-3}(\sigma(u), \sigma(t); r_1, ..., r_{n-3}) a(u)du. \tag{20}$$

Theorem 1. *Assume that:*

1. Conditions H1-H4 hold and $n \geq 4$ is even.

2. Equations (18;k), $k \in \{1,3,...,n-1\}$ are oscillatory.

Then equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality we suppose that $x(t) > 0, t \geq t_0 \geq \alpha$. Then there exists a $t_1 \geq t_0$ such that $\max_{[\sigma(t),\tau(t)]} x(s) > 0$ for $t \geq t_1$.

By Lemma 2(i) there exist a $t_k \geq t_1$ and an odd integer $k \in \{1,3,...,n-1\}$ such that (7) and (8) hold for $t \geq t_k$.

Suppose $1 \leq k \leq n-3$. Then from Lemma 1(ii) (with $i = k+1$ and $\nu = n$) it follows that

$$L_{k+1}x(t) = \sum_{j=k+1}^{n-1} (-1)^{j-k-1} I_{j-k-1}(s,t;r_j,...,r_{k+2})L_j x(s) + (-1)^{n-k-1} \int_t^s I_{n-k-2}(u,t;r_{n-1},...,r_{k+2})L_n x(u)du.$$

Using (1), (8) and (3) and letting $s \to +\infty$ we have

$$-L_{k+1}x(t) \geq \int_t^\infty I_{n-k-2}(u,t;r_{n-1},...,r_{k+2})a(u) \max_{[\sigma(u).\tau(u)]} x(s)du, \quad t \geq t_k. \tag{21}$$

Since $k \geq 1$, then $x(t)$ is increasing and $\max_{[\sigma(u),\tau(u)]} x(s) \geq x(\sigma(u))$ for $t \geq t_2$, where $t_2 \geq t_k$ is such that $\sigma(t) \geq t_k, t \geq t_2$. In this case

$$-L_{k+1}x(t) \geq \int_t^\infty I_{n-k-2}(u,t;r_{n-1},...,r_{k+2})a(u)x(\sigma(u))du, \quad t \geq t_2. \tag{22}$$

If $k \geq 3$, then using Lemma 1(i) (with $i = 0, \nu = k-1, s = t_2$ and $t \geq t_2$) we get

$$x(t) = \sum_{j=0}^{k-2} I_j(t,t_2;r_1,...,r_j)L_j x(t_2) + \int_{t_2}^t I_{k-2}(t,u;r_1,...,r_{k-2})\frac{L_{k-1}x(u)}{r_{k-1}(u)}du, \quad t \geq t_2.$$

Then using (7) we obtain

$$x(\sigma(t)) \geq \int_{t_2}^{\sigma(t)} I_{k-2}(\sigma(t),u;r_1,...,r_{k-2})\frac{L_{k-1}x(u)}{r_{k-1}(u)}du, \quad t \geq t_2. \tag{23}$$

Let $T \geq t_2$ be such that $\sigma(t) \geq t_2$ for $t \geq T$. Then combining (21), (23) and taking in mind that $L_{k-1}x(u)$ is increasing we have for $t \geq T$

$$
\begin{aligned}
&- L_{k+1}x(t) \\
&\geq \int_t^\infty I_{n-k-2}(u,t;r_{n-1},...,r_{k+2})a(u) \int_{t_2}^{\sigma(u)} I_{k-2}(\sigma(u),s;r_1,...,r_{k-2}) \frac{L_{k-1}x(s)}{r_{k-1}(s)} dsdu \\
&\geq \int_t^\infty I_{n-k-2}(u,t;r_{n-1},...,r_{k+2})a(u) \int_{\sigma(t)}^{\sigma(u)} I_{k-2}(\sigma(u),s;r_1,...,r_{k-2}) \frac{L_{k-1}x(s)}{r_{k-1}(s)} dsdu \\
&\geq L_{k-1}x(\sigma(t)) \int_t^\infty I_{n-k-2}(u,t;r_{n-1},...,r_{k+2})a(u) \\
&\qquad \times \int_{\sigma(t)}^{\sigma(u)} \frac{1}{r_{k-1}(s)} I_{k-2}(\sigma(u),s;r_1,...,r_{k-2}) dsdu.
\end{aligned}
$$

The above inequality and equalities (5) and (19) imply

$$
-\frac{L_{k+1}x(t)}{r_{k+1}(t)} \geq A_{k+1}(t) L_{k-1}x(\sigma(t)) \tag{24}
$$

for $3 \leq k \leq n-1$ and $t \geq T$. Inequality (24) with $k = 1$ follows immediately from (22).

The function $y(t) = L_{k-1}x(t)$ is positive for $t \geq T$ and from (24) we conclude that

$$
(r_k(t)y'(t))' + A_k(t)y(\sigma(t)) \leq 0, \quad t \geq T.
$$

Now from Lemma 3 (i) it follows that for each $k \in \{1, 3, ..., n-3\}$ equation (18;k) has a nonoscillatory solution which leads to a contradiction.

Suppose $k = n-1$. Integrating (1) we have

$$
L_{n-1}x(t) \geq \int_t^\infty f(u, \max_{[\sigma(u),\tau(u)]} x(s))du \geq \int_t^\infty a(u)x(\sigma(u))du, \quad t \geq T. \tag{25}
$$

From Lemma 1 (i) (with $i = 0, \nu = n-2, s = t_2$ and $t \geq t_2$) we have

$$
\begin{aligned}
&x(t) \\
&\geq \sum_{j=1}^{n-3} I_j(t,t_2;r_1,...,r_j)L_jx(t_2) + x(t_2) + \int_{t_2}^t I_{n-3}(t,u;r_1,...,r_{n-3}) \frac{L_{n-2}x(u)}{r_{n-2}(u)} du
\end{aligned}
$$

$$\geq \int_{t_2}^{t} I_{n-3}(t,u;r_1,...,r_{n-3})\frac{L_{n-2}x(u)}{r_{n-2}(u)}du, \quad t \geq t_2. \quad (26)$$

Combining (25) and (26) and changing the order of integration we get

$$L_{n-1}x(t) \geq \int_{t}^{\infty} a(u) \int_{t_2}^{\sigma(u)} I_{n-3}(\sigma(u),s;r_1,...,r_{n-3})\frac{L_{n-2}x(s)}{r_{n-2}(s)}dsdu \geq \int_{\sigma(t)}^{\infty}$$

and

$$(s)L_{n-2}x(s)ds, \quad t \geq T,$$

where $(s) = \frac{1}{r_{n-2}(s)} \int_{\psi(s)}^{\infty} I_{n-3}(\sigma(u),s;r_1,...,r_{n-3})a(u)du$ and $\psi(s)$ is the inverse function of $\sigma(s)$.

Hence the positive function $w(t) = L_{n-2}x(t)$ satisfies the inequality

$$w'(t) \geq \frac{1}{r_{n-1}(t)} \int_{\sigma(t)}^{\infty} (s)w(s)ds, \quad t \geq T. \quad (27)$$

Integrating (27) from T to t we have

$$w(t) \geq w(T) + \int_{T}^{t} \frac{1}{r_{n-1}(u)} \int_{\sigma(u)}^{\infty} (s)w(s)dsdu, \quad t \geq T. \quad (28)$$

Denote the right-hand side of (28) by $y(t)$. By differentiation

$$(r_{n-1}(t)y'(t))' + \sigma'(t)(\sigma(t))w(\sigma(t)) = 0.$$

Taking into account (28), (20) and the identity $\psi(\sigma(t)) = t$ we obtain

$$(r_{n-1}(t)y'(t))' + A_{n-1}(t)y(\sigma(t)) \leq 0.$$

By Lemma 3 (i) equation (18; $n-1$) has a nonoscillatory solution which leads to a contradiction. □

Theorem 2. *Assume that:*

1. Conditions H1-H4 hold and $n \geq 3$ is odd.

2. Equations (18;k), $k \in \{2,4,...,n-1\}$ are oscillatory and

$$\int^{\infty} I_{n-1}(s,T;r_{n-1},...,r_1)a(s)ds = +\infty \quad (29)$$

for each $T \geq \alpha$.

Then equation (1) is almost oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality we suppose that $x(t) > 0, t \geq t_0 \geq \alpha$. Then there exists a $t_1 \geq t_0$ such that $\max_{[\sigma(t),\tau(t)]} x(s) > 0, x(\sigma(t)) > 0$ for $t \geq t_1$.

By Lemma 2 (i) there exist a $t_k \geq t_1$ and an even integer $k \in \{0, 2, 4, ..., n-1\}$ such that (7) and (8) hold for $t \geq t_k$.

For $k \geq 2$ the proof of Theorem 2 is the same as the proof of Theorem 1.

Suppose $k = 0$. Then applying Lemma 1 (ii) (with $i = 0, \nu = n$) we get

$$x(t) \geq \int_t^\infty I_{n-1}(u, t; r_{n-1}, ..., r_1)a(u) \max_{[\sigma(u),\tau(u)]} x(s)du, \quad t \geq t_k. \tag{30}$$

Since $x(t)$ is decreasing and positive for $t \geq t_k$ there exists $\lim_{t\to+\infty} x(t) = c \geq 0$.

If $c > 0$, then there exists $T \geq t_k$ such that $2c \geq \max_{[\sigma(u),\tau(u)]} x(s) \geq c, u \geq T$. Then (30) implies the inequality

$$2c \geq \int_T^\infty I_{n-1}(u, T; r_{n-1}, ..., r_1)a(u)du.c,$$

which contradicts (29). Hence $c = 0$ and taking in mind (8) with $k = 0$ we conclude that $\lim_{t\to+\infty} |L_k x(t)| = 0$ monotonically for $k = 0, 1, ..., n-1$. □

Consider the n-th order differential equation

$$x^{(n)}(t) + f(t, \max_{[\sigma(t),\tau(t)]} x(s)) = 0 \tag{31}$$

and the second order comparison equations

$$x''(t) + a_k(t)x(\sigma(t)) = 0, \tag{32,k}$$

where

$$a_k(t) = \int_t^\infty \frac{(u-t)^{n-k-2}}{(n-k-2)!} \frac{(\sigma(u) - \sigma(t))^{k-1}}{(k-1)!} a(u)du \tag{33}$$

for $k = 1, 2, ..., n-3$ and

$$a_{n-1}(t) = \sigma'(t) \int_t^\infty \frac{(\sigma(u) - \sigma(t))^{n-3}}{(n-3)!} a(u)du. \tag{34}$$

As a consequence of Theorems 1 and 2 we obtain the following corollaries.

Corollary 1. *Let conditions H1, H3 and H4 hold and $n \geq 4$ be even.*

Then equation (31) is oscillatory, if equations (32,k) $k \in \{2, 4, ..., n-1\}$ are oscillatory.

Corollary 2. *Let H2, H3 and H4 hold and $n \geq 3$ be odd. Then equation (31) is almost oscillatory, if equations (32,k) $k \in \{2, 4, ..., n-1\}$ are oscillatory and*

$$\int^{\infty} s^{n-1} a(s) ds = +\infty. \tag{35}$$

In the following two theorems we will use the following set of second order comparison equations

$$(r_k(t)x'(t))' + B_k(t)x(\sigma(t)) = 0, \tag{36;k}$$

where

$$B_k(t) = \frac{I_{n-k}(t, T; r_{n-1}, ..., r_k)}{I_1(t, T; r_k)} . \frac{I_k(\sigma(t), T; r_1, ..., r_k)}{I_1(\sigma(t), T; r_k)} a(t) \tag{37}$$

for $k \in \{1, 2, 3, ..., n-3\}$ and

$$B_{n-1}(t) = \frac{I_{n-1}(\sigma(t), T; r_1, ..., r_{n-1})}{I_1(\sigma(t), T; r_{n-1})} a(t). \tag{38}$$

Theorem 3. *Assume that:*

1. Conditions H1-H3 hold and $n \geq 2$ is even.

2. Equations (36;k), $k \in \{1, 3, ..., n-1\}$ are oscillatory.

Then equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality we suppose that $x(t) > 0, t \geq t_0 \geq \alpha$. Then there exists a $t_1 \geq t_0$ such that $\max_{[\sigma(t), \tau(t)]} x(s) > 0, x(\sigma(t)) > 0$ for $t \geq t_1$.

By Lemma 2 (i) there exist a $t_k \geq t_1$ and an odd integer $k \in \{1, 3, ..., n-1\}$ such that (7) and (8) hold for $t \geq t_k$. Applying Lemma 1(ii) (with $i = k$ and $\nu = n$) we have

$$L_k x(t) = \sum_{j=k}^{n-1} (-1)^{j-k} I_{j-k}(s, t; r_j, ..., r_{k+1}) L_j x(s)$$

$$+ (-1)^{n-k} \int_t^s I_{n-k-1}(u, t; r_{n-1}, ..., r_{k+1}) L_n x(u) du, \quad t \leq s.$$

Using (1), (8) and (3) and letting $s \to +\infty$ we have

$$L_k x(t) \geq \int_t^\infty I_{n-k-1}(u,t;r_{n-1},...,r_{k+1})a(u) \max_{[\sigma(u),\tau(u)]} x(s)du, \quad t \geq t_k.$$

Since $k \geq 1$, then $x(t)$ is increasing for $t \geq t_k$ and $\max_{[\sigma(u),\tau(u)]} x(s) \geq x(\sigma(u))$ for $t \geq T$, where $T \geq t_k$ is such that $\sigma(t) \geq t_k$, $t \geq T$. In this case

$$L_k x(t) \geq \int_t^\infty I_{n-k-1}(u,t;r_{n-1},...,r_{k+1})a(u)x(\sigma(u))du, \quad t \geq T. \tag{39}$$

Integrating (39) from T to t we get

$$L_{k-1}x(t) = L_{k-1}x(T)$$

$$+ \int_T^t \frac{1}{r_k(s)} \int_s^\infty I_{n-k-1}(u,s;r_{n-1},...,r_{k+1})a(u)x(\sigma(u))duds$$

$$= L_{k-1}x(T) + \int_T^t \left[\int_T^u \frac{1}{r_k(s)} I_{n-k-1}(u,s;r_{n-1},...,r_{k+1})ds \right] a(u)x(\sigma(u))du$$

$$+ \int_t^\infty \left[\int_T^t \frac{1}{r_k(s)} I_{n-k-1}(u,s;r_{n-1},...,r_{k+1})ds \right] a(u)x(\sigma(u))du. \tag{40}$$

From (5) and (6) it follows

$$\int_T^u \frac{1}{r_k(s)} I_{n-k-1}(u,s;r_{n-1},...,r_{k+1})ds = I_{n-k}(u,T;r_{n-1},...,r_k)$$

for $t \geq u \geq T$ and

$$\int_T^t \frac{1}{r_k(s)} I_{n-k-1}(u,s;r_{n-1},...,r_{k+1})ds \geq \frac{I_1(t,T;r_k)}{I_1(u,T;r_k)} I_{n-k}(u,T;r_{n-1},...,r_k)$$

for $T \leq t < u$.

This together with (40) and (7) implies

$$L_{k-1}x(t) \geq \int_T^t I_{n-k}(u,T;r_{n-1},...,r_k)a(u)x(\sigma(u))du$$

$$+ I_1(t,T;r_k)\int\limits_t^\infty \frac{I_{n-k}(u,T;r_{n-1},...,r_k)}{I_1(u,T;r_k)}a(u)x(\sigma(u))du, \quad t\geq T. \quad (41)$$

Denote the right-hand side of (41) by $z(t)$. It is easy to verify that $z(t)$ is positive and satisfies the equation

$$(r_k(t)z'(t))' + \frac{I_{n-k}(t,T;r_{n-1},...,r_k)}{I_1(t,T;r_k)}a(t)x(\sigma(t)) = 0. \quad (42)$$

But from (10)

$$x(\sigma(t)) \geq \frac{I_k(\sigma(t),T;r_1,...,r_k)}{I_1(\sigma(t),T;r_k)}L_{k-1}x(\sigma(t)) \quad (43)$$

for $t \geq t_2$, where $t_2 \geq T$ is such that $\sigma(t) > T$ for $t \geq t_2$. Taking in mind (42), (43), (41) and (37), (38) we obtain

$$(r_k(t)z'(t))' + B_k(t)z(\sigma(t)) \leq 0, \quad t \geq t_2.$$

By Lemma 3 (i) equation (36;k) has an eventually positive solution. This contradicts 2 of Theorem 3. □

Theorem 4. *Assume that:*

1. Conditions H1-H3 hold and $n \geq 3$ is odd.

2. Equations (36;k), $k \in \{2,4,...,n-1\}$ are oscillatory and (29) holds.

Then equation (1) is almost oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), which without loss of generality, is eventually positive. Then proceeding as in the proof of Theorem 2 we conclude that there exist a $t_k \geq \alpha$ and an even integer $k \in \{0,2,...,n-1\}$ such that (7) and (8) hold for $t \geq t_k$ and $\max_{[\sigma(t),\tau(t)]} x(s) > 0, x(\sigma(t)) > 0, t \geq t_k$. Further on the proof is the same as the proof of Theorem 3 if $k \in \{2,4,...,n-1\}$ and as the proof of Theorem 2, if $k = 0$.

Consider equation (31) and the second order comparison equations

$$x''(t) + b_k(t)x(\sigma(t)) = 0, \quad (44;k)$$

where

$$b_k(t) = \frac{(t-T)^{n-k-1}}{(n-k)!}.\frac{(\sigma(t)-T)^{k-1}}{k!}.a(t) \quad (45)$$

for $k = 1,2,...,n-1$.

As a consequence of Theorems 3 and 4 we obtain the following corollaries.

Corollary 3. *Let conditions H2 and H3 hold and $n \geq 2$ be even. Then equation (31) is oscillatory if equations (44;k), $k \in \{1,3,...,n-1\}$ are oscillatory.*

Corollary 4. *Let conditions H2 and H3 hold and $n \geq 3$ be odd. Then equation (31) is almost oscillatory if equations (44;k), $k \in \{2, 4, ..., n-1\}$ are oscillatory and (35) holds.*

Consider the Euler equation

$$(t^{m-\beta}x^{(m)})^{(m)} + ct^{-\beta-m}x = 0, \quad t \geq 1, \tag{46}$$

where β and $c > 0$ are real constants and $\beta \geq m-1$. Here $n = 2m$ is even, $r_0 = r_1 = ... = r_{m-1} = r_{m+1} = ..., , = r_n = 1, r_m(t) = t^{m-\beta}, a(t) = ct^{-\beta-m}$.

Applying Theorem 1 and oscillation criterion (17) one can conclude (see [15]) that equation (46) is oscillatory provided c is so large that:

(i) when $m = 2, c > k_1 = \frac{1}{4}\beta(\beta+1)$;

(ii) when $m > 2$ is even,

$$c > k_2 = \frac{1}{4}\max\{(m-1)!\beta(\beta+1)...(\beta+m-1),$$

$$\frac{(m-1)!(m-2)!}{(2m-3)!}(\beta-m+2)(\beta-m+3)...(\beta+m-1)\ \};$$

(iii) when $m > 2$ is odd,

$$c > k_3 = \frac{1}{4}\max\{(m-1)!\beta(\beta+1)...(\beta+m-1,$$

$$\frac{(m-1)!(m-2)!}{(2m-3)!}(\beta-m+1)^2(\beta-m+2)(\beta-m+3)...(\beta+m-1)\ \}.$$

Applying Theorem 3 to equation (46) we obtain the following result.

Collorary 5. *Assume that:*

(i) if $m = 2$, then the equation

$$z''(t) + \frac{a(t)}{t-T}\int_T^t s^{\beta-2}(t-s)(s-T)ds.z(t) = 0 \tag{47}$$

is oscillatory;

(ii) if $m > 2$ is even, then the equation

$$z''(t) + \frac{a(t)}{t-T}\int_T^t s^{\beta-m}\frac{(t-s)^{m-1}}{(m-1)!}.\frac{(s-T)^{m-1}}{(m-1)!}ds.z(t) = 0 \tag{48}$$

is oscillatory;

(iii) if $m > 2$ is odd, then equation (48) and the equation

$$(t^{m-\beta}z')' + A_m(t)z(t) = 0 \tag{49}$$

are oscillatory, where

$$A_m(t) = a(t)\left[\frac{\int_T^t s^{\beta-m}\frac{(t-s)^{m-1}}{(m-1)!}ds}{\int_T^t s^{\beta-m}ds}\right]^2 . \qquad (50)$$

Then equation (46) is oscillatory.

From Corollary 5 and oscillation criterion (17) we conclude that equation (46) is oscillatory provided c is so large that:

(j) when $m = 2, c > d_1 = \frac{1}{4}\beta(\beta + 1)$;

(jj) when $m > 2$ is even,

$$c > d_2 = \frac{1}{4}(m-1)!\beta(\beta+1)...(\beta+m-1);$$

(jjj) when $m > 2$ is odd,

$$c > d_3 = \frac{1}{4}\max\left\{(m-1)!\beta(\beta+1)...(\beta+m-1), [\beta(\beta-1)...(\beta-m+1)]^2\right\}.$$

Remark 2. If $n = 2$ $(m = 1)$ then by Theorem 3 equation (46) is oscillatory if the comparison equation (49) is oscillatory. But equation (49) with $m = 1$ coincides with equation (46) which has the form

$$(t^{1-\beta}x')' + ct^{-\beta-1}x = 0. \qquad (51)$$

Applying the oscillation criterion (17) we obtain that equation (51) is oscillatory if $c > \frac{\beta^2}{4}$. This condition is also necessary for equation (51) to be oscillatory.

Comparing the oscillation criteria (i)-(iii) and (j)-(jjj) we conclude that equation (46) is oscillatory provided c is so large that:

1. when $m = 2, c > k_1 = d_1 = \frac{1}{4}\beta(\beta + 1)$;
2. when $m > 2$ is even, $c > d_2$;
3. when $m > 2$ is odd, $c > k_3$.

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