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ON THE SYMMETRIC TENSOR RANK OF NON-TANGENTIAL POINTS OF PROJECTIVE CURVES

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Abstract: Let $X \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve with unramified normalization map. Fix $P \in \mathbb{P}^n$ not contained in any tangent line of X. Here we prove the existence of $S \subset X$ such that $\sharp(S) \leq n-1$ and $P \in \langle S \rangle$.

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Fix an integral and non-degenerate variety $X \subseteq \mathbb{P}^n$ defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle\ \ \rangle$ denotes the linear span. Hence $r_X(P) = 1$ if and only if $P \in X$. Since X is non-degenerate, the X-rank is defined and $r_X(P) \leq n+1$ for all $P \in \mathbb{P}^n$ (see [2], [4], [3], [6]). If $\mathrm{char}(\mathbb{K}) = 0$, then a use of Bertini's Theorem for base point free linear systems gives $r_X(P) \leq n+1-\dim(X)$ for all $P \in \mathbb{P}^n$ (see [6], 5.1). Form now on we assume $\dim(X) = 1$ and call $u : C \to X$ the normalization map. Assume that u is unramified. This assumption implies that for every $Q \in C$ the effective degree 2 Cartier divisor $2Q$ of X is mapped by u isomorphically onto a length 2 subscheme $u(2Q)$ of X (and hence of $\mathbb{P}^n$) with $u(Q)$ as its reduction. There is a unique line of $\mathbb{P}^n$ spanned by the subscheme $u(2Q)$ and we say that this line is the tangent line of X at the branch determined by Q. Let T_QX

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denote this line. Set $TX := \cup_{Q\in C} T_QX$. Here we prove the following result.

Theorem 1. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve whose normalization map $u : C \to X$ is unramified. Then $r_X(P) \leq n-1$ for all $P \in \mathbb{P}^n \setminus TX$.*

Remark 1. Let $G(1,n)$ denote the Grassmannian of all lines of $\mathbb{P}^n$. Take X as in the statement of Theorem 1. Let $\mathbb{L}(X) \subset G(1,n)$ the set of the lines T_QX, $Q \in C$. For any integer $z > 0$ let $\mathrm{Hilb}^z(X)$ denote the Hilbert scheme of all length z zero-dimensional subschemes of X. Let $\Sigma(X) \subseteq \mathrm{Hilb}^2(X)$ be the set of the schemes $u(2Q)$. The inclusion $\Sigma(X) \subseteq \mathrm{Hilb}^2(X)$ is an equality if and only if X is smooth. $\Sigma(X)$ is a partial normalization of X, i.e. the normalization map $u : C \to X$ factors through the surjection $C \to \Sigma(X)$, whose existence is just the definition of $\Sigma(X)$. Quite often $C = \Sigma(X)$ (e.g. if X has only seminormal or ordinary planar singularities with arbitrary multiplicities). For curves with inramified normalization the condition $C = \Sigma(X)$ would be a generalization of the condition "no tacnode". We have $\mathbb{L}(X) \cong \Sigma(X)$. The set $\Sigma(X)$ is the closure inside $\mathrm{Hilb}^2(X)$ of the set η of all reduced subschemes of $X_{reg}(X)$ with length 2. Hence $S^2(X)$ is the union of TX and the union of all lines $D \subset \mathbb{P}^n$ such that $\sharp((X \cap D)_{red}) \geq 2$.

Lemma 1. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve whose normalization $u : C \to X$ is unramified. Set $d := \deg(X)$. Fix a general $O \in X$ and let $E \subset \mathbb{P}^{n-1}$ the image of X by the linear projectiion from O. Then E is a degree $d-1$ curve birational to X and with unramified normalization.*

Proof. A theorem of Kaji says that $(T_QX \cap X)_{red} = \{u(Q)\}$ for a general $Q \in C$ (see [5], Theorem 3.1 and Remark 3.8). Hence the set $B := \{Q \in C : (T_QX \cap X)_{red} \neq \{u(Q)\}\}$ is finite. Let $\ell_O : \mathbb{P}^n \setminus \{O\} \to \mathbb{P}^{n-1}$ be the linear projection from O. Since O is general in X, it is a smooth point of X. Thus ℓ_O induces a morphism $v : X \to \mathbb{P}^{n-1}$ such that $\deg(v) \cdot \deg(v(X)) = d-1$. Since $\mathrm{char}(\mathbb{K}) = 0$, a general secant line of X is not a multisecant line. Hence the generality of O gives $\deg(v) = 1$. Hence $E := v(X)$ has degree $d-1$ and $v \circ u : C \to E$ is the normalization map. Since $\mathrm{char}(\mathbb{K}) = 0$, a general point of X is not a Weierstrass point of X. Thus the osculating plane of X at O has order of contact 3 with X at O. Thus v is unramified at O. Thus to check that $v \circ u$ is unramified it is sufficient to prove that $O \notin T_QX$ for all $Q \in C\setminus\{O\}$. This is true, because O is general in X and B is finite. □

Proof of Theorem 1. Set $d := \deg(X)$. Fix $P \in \mathbb{P}^n \setminus TX$.

(a) Here we assume $n = 3$. Since $S^2(X) = \mathbb{P}^3$ (see [1], Remark 1.6) and $P \notin TX$, there is $D \in G(1,3)$ such that $P \in D$ and $\sharp((X \cap D)_{red}) \geq 2$ (Remark

1). Hence $r_X(P) = 2$.

(b) Now assume $n > 3$. Fix a general $A \subset X$ such that $\sharp(A) = n-3$ and set $M := \langle A \rangle$. Since A is general, $\dim(M) = n-4$. Let $Y \subset \mathbb{P}^3$ be the curve obtained from X making the linear projection from M. Applying $n-3$ times Lemma 1 we get $\deg(Y) = d-n+3$, Y birational to X and that the normalization map of Y is unramified. We also need that the proof of Lemma 1 gives the existence of a morphism $v : X \to Y$ such that $v \circ u : C \to Y$ is the normalization map and $v^{-1}(v(O)) = \{O\}$ for all $O \in A$. If $P \in M$, then $r_X(P) \le n-3$. Hence we may assume $P \notin M$. Thus $P' := \ell_M(P)$ is a point of $\mathbb{P}^3$. Fix any finite $S \subset Y$ such that $P' \in \langle S \rangle$ and take any $s' \subset Y$ such that $v(S') = S$ and $\sharp(S') = \sharp(S)$. Notice that $P \in \langle S' \cup A \rangle$. Thus $r_X(P) \le n-3+r_Y(P')$. The case $n=3$ applied to Y gives $r_Y(P') \le 2$. Hence $r_X(P) \le n-1$. □

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