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DIVISIBILITY OF SUMS OF POWERS OF EVEN INTEGERS

T. Aaron Gulliver
Department of Electrical and Computer Engineering
University of Victoria
P.O. Box 3055, STN CSC, Victoria, BC, V8W 3P6, CANADA
e-mail: agullive@ece.uvic.ca

Abstract: A number of sequences based on sums of powers of integers is presented. This approach provides a simple derivation of some well known sequences, as well as the construction of many new sequences.

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The sums of powers of integers have been the subject of research for hundreds of years. This is because of well-known combinatorial expressions such as

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}, \tag{1}$$

for the sum of the first n natural numbers and

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = (\sum_{i=1}^{n} i)^2. \tag{2}$$

In this paper we examine the sums themselves, and in particular the integer sequences that result. Consider the sum of the first n m-th powers

$$\sum_{i=1}^{n} i^m. \tag{3}$$

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For $n = 4$, the values are

$$\begin{gathered} 10, 30, 100, 354, 1300, 4890, 18700, 72354, 282340, \\ 1108650, 4373500, 17312754, \ldots \end{gathered} \tag{4}$$

This is sequence A103438 in the online encyclopedia of integer sequences maintained by Sloane [2]. Sequences with other values of n and m were considered in [1]. In this paper, we generalize (3) to even values of i, giving

$$\sum_{i=1}^{n} (2i)^m. \tag{5}$$

For $n = 1$, we have 2^m. Looking at these numbers, $2, 4, 8, 16, 32, 64, 128, 256, \ldots$ one will note that the units digit repeats in a sequence 2,4,8,6. For $n = 2$ to 10, the values are

$$\begin{array}{ccccccc} 6, & 20, & 72, & 272, & 1056, & 4160, & \ldots \\ 12, & 56, & 288, & 1568, & 8832, & 50816, & \ldots \\ 20, & 120, & 800, & 5664, & 41600, & 312960, & \ldots \\ 30, & 220, & 1800, & 15664, & 141600, & 1312960, & \ldots \\ 42, & 364, & 3528, & 36400, & 390432, & 4298944, & \ldots \\ 56, & 560, & 6272, & 74816, & 928256, & 11828480, & \ldots \\ 72, & 816, & 10368, & 140352, & 1976832, & 28605696, & \ldots \\ 90, & 1140, & 16200, & 245328, & 3866400, & 62617920, & \ldots \\ 110, & 1540, & 24200, & 405328, & 7066400, & 126617920, & \ldots \end{array} \tag{6}$$

The first two sequences in this list are A063376 and A074533, respectively, see [2]. The other sequences are new. Comparing (4) and the third row in (6), one notices a regularity in the units digit, namely

$$10 \mid \sum_{i=1}^{4} i^m \iff m \neq 0 \bmod 4. \tag{7}$$

In order to show this more clearly, the values of (5) for $n = 1$ to 10 (number of terms) and $m = 1$ to 12 (power) reduced modulo 10 are given in Table (of course the last two columns are identical because we are adding a power of 10).

Table 1 shows that for $n = 4, 5, 9$ and 10, it appears that

$$10 \mid \sum_{i=1}^{n} i^m \iff m \neq 0 \bmod 4.$$

To prove these results, one could consider expressions such as

$$\sum_{i=1}^{n} 2i = 2 + 4 + 6 + \ldots + 2n = n(n+1), \tag{8}$$

power	number of terms n									
m	1	2	3	4	5	6	7	8	9	10
1	2	6	2	0	0	2	6	2	0	0
2	4	0	6	0	0	4	0	6	0	0
3	8	2	8	0	0	8	2	8	0	0
4	6	2	8	4	4	0	6	2	8	8
5	2	6	2	0	0	2	6	2	0	0
6	4	0	6	0	0	4	0	6	0	0
7	8	2	8	0	0	8	2	8	0	0
8	6	2	8	4	4	0	6	2	8	8
9	2	6	2	0	0	2	6	2	0	0
10	4	0	6	0	0	4	0	6	0	0
11	8	2	8	0	0	8	2	8	0	0
12	6	2	8	4	4	0	6	2	8	8

Table 1

and the following for $m = 2, 3$, and 4

$$\sum_{i=1}^{n}(2i)^2 = \frac{2}{3}n(n+1)(2n+1), \tag{9}$$

$$\sum_{i=1}^{n}(2i)^3 = 2n^2(n+1)^2, \tag{10}$$

$$\sum_{i=1}^{n}(2i)^4 = \frac{8}{15}n(n+1)(2n+1)(3n^2+3n-1), \tag{11}$$

respectively. Fortunately, there is a much simpler way.

Consider the residues modulo 10 of the powers of the integers $2, 4, 6, 8$ and 10, which are given in Table 2.

The periods of these residues are:

6,10	period 1
4	period 2
2,8	period 4

which are all factors of $\phi(10) = 4$. These values show that the periods of the units digits of (5) must be 4. Determining the units digits for the sums in the table can simply be done by summing the first n columns of Table 2 (taking

power	integer				
	2	4	6	8	10
1	2	4	6	8	0
2	4	6	6	4	0
3	8	4	6	2	0
4	6	6	6	6	0
5	2	4	6	8	0
6	4	6	6	4	0
⋮	⋮	⋮	⋮	⋮	⋮

Table 2

power	number of terms n									
m	1	2	3	4	5	6	7	8	9	$\cdots$
1	2	6	2	0	0	2	6	2	0	$\cdots$
2	4	0	6	0	0	4	0	6	0	$\cdots$
3	8	2	8	0	0	8	2	8	0	$\cdots$
4	6	2	8	4	4	0	6	2	8	$\cdots$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 3

power	number of terms n									
m	1	2	3	4	5	6	7	8	9	10
1	2	0	0	2	0	0	2	0	0	2
2	1	2	2	0	1	1	2	0	0	1
3	2	0	0	2	0	0	2	0	0	2
4	1	2	2	0	1	1	2	0	0	1
5	2	0	0	2	0	0	2	0	0	2
6	1	2	2	0	1	1	2	0	0	1
7	2	0	0	2	0	0	2	0	0	2
8	1	2	2	0	1	1	2	0	0	1
9	2	0	0	2	0	0	2	0	0	2
10	1	2	2	0	1	1	2	0	0	1
11	2	0	0	2	0	0	2	0	0	2
12	1	2	2	0	1	1	2	0	0	1

Table 4

columns modulo 5), which are given in Table 3. Since Table 3 will repeat for powers greater than 4, this proves the expressions above.

For other residues, it is a simple matter to form the tables and determine how the sequences of units digits repeat. For example, taking the values of (5) modulo 3 gives Table 4. Note that in this case columns 2 and 3, 5 and 6, and 8 and 9, are identical because we are adding a power of 3. In addition, the units digits in base 3 repeats every second power! A simple proof of this requires only the residues modulo 3 of the powers of the even integers, which are given in Table 5.

This shows that the periods of the residues is either 1 or 2, which is expected

	integer								
power	2	4	6	8	10	12	14	16	18
1	2	1	0	2	1	0	2	1	0
2	1	1	0	1	1	0	1	1	0
3	2	1	0	2	1	0	2	1	0
4	1	1	0	1	1	0	1	1	0
5	2	1	0	2	1	0	2	1	0
6	1	1	0	1	1	0	1	1	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 5

power	number of terms n									
m	1	2	3	4	5	6	7	8	9	10
1	2	1	2	0	0	2	1	2	0	0
2	4	0	1	0	0	4	0	1	0	0
3	3	2	3	0	0	3	2	3	0	0
4	1	2	3	4	4	0	1	2	3	3
5	2	1	2	0	0	2	1	2	0	0
6	4	0	1	0	0	4	0	1	0	0
7	3	2	3	0	0	3	2	3	0	0
8	1	2	3	4	4	0	1	2	3	3
9	2	1	2	0	0	2	1	2	0	0
10	4	0	1	0	0	4	0	1	0	0
11	3	2	3	0	0	3	2	3	0	0
12	1	2	3	4	4	0	1	2	3	3

Table 6

since $\phi(3) = 2$. In terms of divisibility, Table 4 shows that for $n = 8$ and 9

$$3 \mid \sum_{i=1}^{n} (2i)^m.$$

Another interesting case is the residues modulo 5, which are given in Table 6. Note that columns 4, 5, 9 and 10 contain mostly zeros. From this table it is

power	2	4	6	8	10	12	14	16	18	20
1	2	4	1	3	0	2	4	1	3	0
2	4	1	1	4	0	4	1	1	4	0
3	3	4	1	2	0	3	4	1	2	0
4	1	1	1	1	0	1	1	1	1	0
5	2	4	1	3	0	2	4	1	3	0
6	4	1	1	4	0	4	1	1	4	0
7	3	4	1	2	0	3	4	1	2	0
8	1	1	1	1	0	1	1	1	1	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 7

obvious that for these columns

$$5 \mid \sum_{i=1}^{n} (2i)^m \iff m \neq 0 \bmod 4.$$

This is confirmed by the results in Table 7, which gives the residues modulo 5 of the powers of the even integers, $(2i)^m$. This table shows that for $i = 0$ or 3 modulo 5, the period is 1, for $i = 2$ modulo 5, the period is 2, and otherwise the period is 4. These values are all factors of $\phi(5) = 4$.

It is left to the reader to determine the results modulo other values, in particular, how does Table 4 differ from the values taken modulo 7? The period of the integer powers will be a factor of $\phi(7) = 6$.

It is much more difficult to determine the divisibility when the values of (5) are considered with m fixed. However, (9) shows that

$$5 \mid \sum_{i=1}^{n} (2i)^2,$$

when $n = 5r, 5r - 1$ or $5r - 3$, $r \geq 1$, and (10) shows that

$$5 \mid \sum_{i=1}^{n} (2i)^3,$$

when $n = 5r$ or $5r - 1$. The proof comes from the fact that 5 is not a factor of the denominator of the constant in these expressions. The divisibility by other primes is easily established.

More can be said for $m = 2$. Since the constant is $2/3$, and n or $n + 1$ is even

$$4 \mid \sum_{i=1}^{n} (2i)^2.$$

In addition, with a constant of 2 for $m = 3$, we have

$$8 \mid \sum_{i=1}^{n} (2i)^3,$$

when $n = 2r$ or $2r - 1$, in other words, for all values of n. This is also true for $m = 4$, as the constant in (11) is 8/15, and the terms must be integers. In fact, in this case

$$16 \mid \sum_{i=1}^{n} (2i)^4,$$

for all n. Of course, one can easily show these results by just factoring out the constant.

Many other divisibility identities can be established for fixed n or m simply by looking at the sums modulo a number, or by examining the closed form expressions for

$$\sum_{i=1}^{n} (2i)^m,$$

for fixed m. In addition, one could consider

$$\sum_{i=1}^{n} (pi)^m,$$

for p other than 2.

Finally, we consider the sequences from the columns in (6). This corresponds to fixed m in (5). For $m = 1$ and 2 we have (8) and (9), respectively, which are sequences A002378 and A002492 in [2]. For $m = 3$ and 4 we have (10) and (11), which are new, as are the sequences for other values of m. The sequence for $m = 5$ is given by

$$\sum_{i=1}^{n} (2i)^5 = \frac{8}{3} n^2 (2n^2 + 2n - 1)(n+1)^2$$

and the sequence for $m = 6$ by

$$\sum_{i=1}^{n} (2i)^6 = \frac{32}{21} n(n+1)(2n+1)(3n^4 + 6n^3 - 3n + 1).$$

The sequences for other values of m can easily be found.

References

[1] T.A. Gulliver, Divisibility of sums of powers of integers, *International Mathematical Journal*, **3** (2003), 699-704.

[2] N.J.A. Sloane, *On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/~njas/sequences/index.html.