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SOME FOURIER SERIES EXPANSIONS AND THEIR APPLICATIONS

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Abstract: In this paper various Fourier series expansions and their applications for generating some special integrals, sums of real series and combinatorial identities are presented.

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1. Introduction

Sums of many series may be expressed in terms of „elementary" integrals by using Parseval's Formula (the case of even functions):

$$\sum_{n=1}^{\infty} \hat{f}(n)\hat{g}(n) = \frac{2}{\pi}\int_0^{\pi} f(t)g(t)dt - \frac{1}{2}\hat{f}(0)\hat{g}(0) \tag{1.1}$$

for $f, g \in L^2(0,\pi)$, where $\hat{f}(n) := \frac{2}{\pi}\int_0^{\pi} f(t)\cos(nt)dt, n = 0, 1, \ldots$.

The respective formula for odd functions has the form

$$\sum_{n=1}^{\infty} \hat{f}(n)\hat{g}(n) = \frac{2}{\pi}\int_0^{\pi} f(t)g(t)dt \tag{1.2}$$

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for $f, g \in L^2(0,\pi)$, where $\hat{f}(n) := \frac{2}{\pi}\int\limits_0^\pi f(t)\sin(nt)dt, n = 1, 2, \ldots$.

The aim of this paper is to collect many of the classical and new Fourier series expansions and to generate, with the aid of them, some original trigonometric identities and special integrals, especially by applying formulae (1.1) and (1.2). The nature of this work is both investigative and review. The numerous bibliography, used for preparing this paper, had a basic influence for its final form.

In Section 2 some trigonometric polynomials are discussed, which were earlier considered by Wituła and Słota in papers [32] and [31]. In Section 3 authors receive the analytical description of the sums of Fourier series, having the following form ($N \in \mathbb{N}_0$):

$$\sum_{k=1}^{\infty} \frac{\cos\big((k+N)y\big)}{k(k+1)\ldots(k+N)} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{\sin\big((k+N)y\big)}{k(k+1)\ldots(k+N)}. \tag{1.3}$$

In Section 3.1 various applications of the series (1.3) are proposed, for example in generating the integrals written below

$$\int\limits_0^\pi \ln^2\left(2\sin\frac{x}{2}\right)dx = \frac{\pi^3}{12},$$

$$\int\limits_0^\pi (6x^2 - 6\pi x + \pi^2)\ln\left(\cot\frac{x}{2}\right)dx = 0,$$

$$\int\limits_0^1 \ln\left(2\sin\frac{x}{2}\right)dx = -\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2},$$

$$\int\limits_0^1 \ln\left(\cot\frac{x}{2}\right)dx = 2\sum_{n=1}^{\infty}\frac{\sin(2n-1)}{(2n-1)^2}.$$

In Section 4 some formulae for the sine integral function are presented. Section 5 deals with a certain Salaev's Fourier series expansion and some of its applications. Moreover, Tveritin's generalizations of (1.3) for $N = 0$ are given. Finally, Section 6 treats on the Fourier sine series expansion of the exponent function.

In the current paper the following theorem on trigonometric series [34] is of great importance.

Theorem 1.1. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, such that*

$$\sum_{n=1}^{\infty} \big(|a_n| + |b_n|\big) < \infty.$$

Then the sum of the following trigonometric series

$$F(x) := \sum_{n=1}^{\infty} \big(a_n \, \cos(n\,x) + b_n \, \sin(n\,x)\big) \tag{1.4}$$

is a continuous function, due to uniform convergence. Thus the series on the right side of (1.4) is the Fourier series expansion of its sum $F(x)$.

2. Certain Finite Sums

Let f be a trigonometric polynomial. Then the interesting identities of combinatorial nature can be generated from (1.1) and (1.2). First, let us apply Parseval's Formula (1.1) to the following classical trigonometric polynomials (see [32] and [31]):

$$2^{n-2}\, C_n^{+}(x,\varphi) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \cos\big((n-2k)\varphi\big) \cos\big((n-2k)x\big) - \frac{1}{2}\left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor\right) \binom{n}{\lfloor \frac{n}{2} \rfloor}, \tag{2.1}$$

$$2^{n-2}\, C_n^{-}(x,\varphi) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{k} \sin\big((n-2k)\varphi\big) \sin\big((n-2k)x\big), \tag{2.2}$$

where

$$C_n^{+}(x,\varphi) := \cos^n(x+\varphi) + \cos^n(x-\varphi), \tag{2.3}$$

$$C_n^{-}(x,\varphi) := \cos^n(x-\varphi) - \cos^n(x+\varphi). \tag{2.4}$$

We note that (see also Remark 2.3):

$$\frac{2}{\pi}\int_0^{\pi} \left(2^{n-2} C_n^{+}(x,\varphi)\right)^2 dx = \frac{2^{2n-2}}{\pi}\int_0^{\pi} \left(\frac{1}{2} C_{2n}^{+}(x,\varphi) + (\cos^2 x - \sin^2\varphi)^n\right) dx = \overset{(2.1)}{=} \frac{1}{4}\binom{2n}{n} + \frac{2^{2n-2}}{\pi}\int_0^{\pi} (\cos^2 x - \sin^2\varphi)^n dx. \tag{2.5}$$

On the other hand, by (1.1) we obtain

$$\frac{2}{\pi}\int\limits_0^\pi \left(2^{n-2}C_n^+(x,\varphi)\right)^2 dx =$$

$$= \begin{cases} \sum\limits_{k=0}^{(n-1)/2} \binom{n}{k}^2 \cos^2\left((n-2k)\varphi\right), & \text{when } n \text{ is odd,} \\ \sum\limits_{k=0}^{(n-2)/2} \binom{n}{k}^2 \cos^2\left((n-2k)\varphi\right) + \frac{1}{2}\binom{n}{n/2}^2, & \text{when } n \text{ is even.} \end{cases} \tag{2.6}$$

Similarly, by using equality (1.1) for (2.2) we receive

$$\frac{1}{4}\binom{2n}{n} - \frac{2^{2n-2}}{\pi}\int\limits_0^\pi (\cos^2 x - \sin^2\varphi)^n dx = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \binom{n}{k}^2 \sin^2\left((n-2k)\varphi\right). \tag{2.7}$$

By adding (2.5) and (2.7), according to (2.6), we derive

$$(n \mapsto 2n) \quad \Longrightarrow \quad \binom{4n}{2n} - \binom{2n}{n}^2 = 2\sum_{k=0}^{n-1}\binom{2n}{k}^2, \tag{2.8}$$

$$(n \mapsto 2n-1) \quad \Longrightarrow \quad \binom{4n-2}{2n-1} = 2\sum_{k=0}^{n-1}\binom{2n-1}{k}^2. \tag{2.9}$$

In turn, by subtracting (2.7) from (2.5) and taking into account (2.6), we get

$$(n \mapsto 2n-1) \quad \Longrightarrow \quad \int\limits_0^\pi (\cos^2 x - \sin^2\varphi)^{2n-1} dx =$$

$$= \pi 2^{3-4n}\sum_{k=0}^{n-1}\binom{2n-1}{k}^2 \cos\left(2(2(n-k)-1)\varphi\right), \tag{2.10}$$

$$(n \mapsto 2n) \quad \Longrightarrow \quad \int\limits_0^\pi (\cos^2 x - \sin^2\varphi)^{2n} dx =$$

$$= \pi 2^{1-4n}\sum_{k=0}^{n-1}\binom{2n}{k}^2 \cos\left(4(n-k)\varphi\right) + \pi 2^{-4n}\binom{2n}{n}^2. \tag{2.11}$$

Hence, in particular, we receive for $\varphi = \frac{\pi}{4}$:

$$\int\limits_0^\pi \left(\cos^2 x - \frac{1}{2}\right)^{2n-1} dx = 2^{1-2n}\int\limits_0^\pi \cos^{2n-1}(2x)dx = 0,$$

and

$$\int\limits_0^{\pi}\left(\cos^2 x-\frac{1}{2}\right)^{2n}dx=\pi 2^{1-4n}\sum_{k=0}^{n-1}(-1)^{n-k}\binom{2n}{k}^2+\pi 2^{-4n}\binom{2n}{n}^2;$$

for $\varphi=\frac{\pi}{2}$:

$$\int\limits_0^{\pi}\sin^{2n}x dx=2\int\limits_0^{\frac{\pi}{2}}\sin^{2n}x dx=\pi 2^{-2n}\binom{2n}{n};$$

for $\varphi=\frac{\pi}{8}$:

$$\int\limits_0^{\pi}\left(\cos^2 x-\sin^2\frac{\pi}{8}\right)^{2n}dx-\pi 2^{-4n}\binom{2n}{n}^2=$$

$$=\pi 2^{1-4n}\sum_{k=0}^{n-1}\binom{2n}{k}^2\cos\left((n-k)\frac{\pi}{2}\right)=$$

$$=\pi 2^{1-4n}\left(-\binom{2n}{n-2}^2+\binom{2n}{n-4}^2-\binom{2n}{n-6}^2+\dots\right)=$$

$$=\pi 2^{1-4n}\sum_{k=1}^{\lfloor (n-1)/2\rfloor}(-1)^k\binom{2n}{n-2k}^2.$$

Remark 2.1. In paper [32] the following attracting decomposition of functions $C_n^+(x,\varphi)$ and $C_n^-(x,\varphi)$ are also presented:

$$C_n^+(x,\varphi)=\sum_{k=0}^{\lfloor n/2\rfloor}A_k^{(n)}(\cos\varphi)\,(\cos x)^{n-2k}, \qquad (2.12)$$

where

$$A_0^{(n)}(y):=2\,T_n(y), \qquad (2.13)$$

$$A_k^{(n)}(y):=\frac{2n}{(2k)!!}\,(1-y^2)^k\,U_{n-k-1}^{(k-1)}(y), \qquad (2.14)$$

and

$$C_n^-(x,\varphi)=C_1^-(x,\varphi)\sum_{k=0}^{\lfloor (n-1)/2\rfloor}B_k^{(n)}(\cos\varphi)(\cos x)^{n-2k-1}$$

$$=2\sin(\varphi)\sin(x)\sum_{k=0}^{\lfloor (n-1)/2\rfloor}B_k^{(n)}(\cos\varphi)(\cos x)^{n-2k-1}, \qquad (2.15)$$

where

$$B_k^{(n)}(x) := \frac{1}{(2k)!!}\,(1-x^2)^k\,U_{n-k-1}^{(k)}(x). \tag{2.16}$$

In above formulae $T_n(y)$ and $U_n(y)$ denote the n-th Chebyshev polynomial of the first and second kind, respectively. We note, that from (2.12) it could be deduced the relations

$$\frac{1}{n}\big(\cos^n\varphi - \cos(n\varphi)\big) = \sum_{k=1}^{\lfloor n/2\rfloor} \frac{\sin^{2k}\varphi}{(2k)!!} U_{n-k-1}^{(k-1)}(\cos\varphi),$$

$$2^{n-1}C_n^+\left(\frac{\pi}{3};\varphi\right) = \cos(n\varphi) + n\sum_{k=1}^{\lfloor n/2\rfloor} \frac{(2\sin\varphi)^{2k}}{(2k)!!} U_{n-k-1}^{(k-1)}(\cos\varphi),$$

$$2^{(n-2)/2}C_n^+\left(\frac{\pi}{4};\varphi\right) = \cos(n\varphi) + n\sum_{k=1}^{\lfloor n/2\rfloor} \frac{(2\sin^2\varphi)^{k}}{(2k)!!} U_{n-k-1}^{(k-1)}(\cos\varphi).$$

Remark 2.2. By differentiating (2.1) (taken for $\varphi = 0$ and $n := 2n$) we obtain

$$\frac{d}{dx}\cos^{2n}(x) = \frac{1}{2}\frac{d}{dx}C_{2n}^+(x,0) = -\frac{1}{2^{2n-2}}\sum_{k=0}^{n}\binom{2n}{k}(n-k)\sin\big(2(n-k)x\big).$$

Analogically, from (2.2) we have

$$\frac{d}{dx}\sin^{2n}(x) = \frac{1}{2}\frac{d}{dx}S_{2n}^+(x,0) = -\frac{1}{2^{2n-2}}\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{k}(n-k)\sin\big(2(n-k)x\big).$$

Placing above relations into the equality

$$\frac{d}{dx}\cos^{2n}(x)\cdot\frac{d}{dx}\sin^{2n}(x) = -4n^2\cos^{2n}(x)\sin^{2n}(x) = -\frac{4n^2}{2^{2n}}\sin^{2n}(2x)$$

and substituting $y = 2x$ we receive

$$\left(\sum_{k=0}^{n}\binom{2n}{k}(n-k)\sin\big((n-k)y\big)\right)\cdot$$
$$\cdot\left(\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{k}(n-k)\sin\big((n-k)y\big)\right) = -\big(2^{n-1}n\sin^n(y)\big)^2. \tag{2.17}$$

Let us notice, that if we denote by a_k the kth term in the first sum on the left side of (2.17), then we can use the identity

$$\big(a_0 + a_1 + a_2 + \ldots + a_n\big)\big((-1)^n a_0 + (-1)^{n-1}a_1 + (-1)^{n-2}a_2 - \ldots + a_n\big) =$$

$$= (-1)^n \Big((a_0 + a_2 + a_4 + \ldots + a_s)^2 - (a_1 + a_2 + a_3 + \ldots + a_t)^2 \Big),$$

where s (or t, respectively) is the greatest even (or odd, respectively) number, not greater than n. Hence, we obtain (for every $n \in \mathbb{N}$):

$$(-1)^n \big(2^{n-1} n \sin^n(y)\big)^2 = \\ = \left(\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2n}{2k+1} (n-2k-1) \sin\big((n-2k-1)y\big) \right)^2 - \\ - \left(\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n}{2k} (n-2k) \sin\big((n-2k)y\big) \right)^2 . \quad (2.18)$$

In consequence, we obtain the sinus-Pitagoras type identities:

$(n \mapsto 2n) \quad \Longrightarrow$

$$\big(n(2\sin y)^{2n}\big)^2 + \left(2 \sum_{k=0}^{n} \binom{4n}{2k} (n-k) \sin\big(2(n-k)y\big) \right)^2 = \\ = \left(\sum_{k=1}^{n} \binom{4n}{2k-1} \big(2(n-k)+1\big) \sin\big((2(n-k)+1)y\big) \right)^2 , \quad (2.19)$$

$(n \mapsto 2n-1) \quad \Longrightarrow$

$$\big(2^{2n-2}(2n-1)\sin^{2n-1} y\big)^2 + \left(2 \sum_{k=1}^{n} \binom{4n-2}{2k-1} (n-k) \sin\big(2(n-k)y\big) \right)^2 = \\ = \left(\sum_{k=0}^{n-1} \binom{4n-2}{2k} \big(2(n-k)-1\big) \sin\big((2(n-k)-1)y\big) \right)^2 . \quad (2.20)$$

Remark 2.3. The following identities hold true:

$$C_n^+(x,\varphi) C_n^-(x,\varphi) = C_{2n}^-(x,\varphi),$$

$$\big(C_n^+(x,\varphi)\big)^2 + \big(C_n^-(x,\varphi)\big)^2 = 2C_{2n}^+(x,\varphi),$$

$$\big(C_n^+(x,\varphi)\big)^4 = C_{4n}^+(x,\varphi) + 4(\cos^2 x - \sin^2\varphi)^n C_{2n}^+(x,\varphi)^n + 6(\cos^2 x - \sin^2\varphi)^{2n} = \\ = \big(C_{2n}^+(x,\varphi)\big)^2 + 2C_{2n}^+(x,\varphi)\big(C_n^+(x,\varphi)\big)^2 - 2C_{4n}^+(x,\varphi),$$

$$\big(C_n^+(x,\varphi)\big)^4 + \big(C_n^-(x,\varphi)\big)^4 = 2C_{4n}^+(x,\varphi) + 12(\cos^2 x - \sin^2\varphi)^{2n} = \\ = 6\big(C_{2n}^+(x,\varphi)\big)^2 - 4C_{4n}^+(x,\varphi),$$

$$\big(C_n^+(x,\varphi)\big)^6 = C_{6n}^+(x,\varphi) + 6(\cos^2 x - \sin^2\varphi)^n C_{4n}^+(x,\varphi) + \\ + 15(\cos^2 x - \sin^2\varphi)^{2n} C_{2n}^+(x,\varphi) + 20(\cos^2 x - \sin^2\varphi)^{3n} = \\ = C_{6n}^+(x,\varphi) + \frac{5}{2}\big(C_{2n}^+(x,\varphi)\big)^2\Big(C_{2n}^+(x,\varphi) + 2\big(C_n^+(x,\varphi)\big)^2\Big) - \\ - \frac{1}{2}C_{4n}^+(x,\varphi)\Big(4\big(C_n^+(x,\varphi)\big)^2 + 11C_{2n}^+(x,\varphi)\Big),$$

$$\big(C_n^+(x,\varphi)\big)^6 + \big(C_n^-(x,\varphi)\big)^6 = 2C_{6n}^+(x,\varphi) + 30(\cos^2 x - \sin^2\varphi)^{2n} C_{2n}^+(x,\varphi) = \\ = 2C_{6n}^+(x,\varphi) + 15\big(C_{2n}^+(x,\varphi)\big)^3 - 15C_{2n}^+(x,\varphi)C_{4n}^+(x,\varphi),$$

$$\big(C_n^+(x,\varphi)\big)^8 + \big(C_n^-(x,\varphi)\big)^8 = \\ = -68C_{8n}^+(x,\varphi) + 42\big(C_{4n}^+(x,\varphi)\big)^2 + 28C_{4n}^+(x,\varphi)\big(C_{2n}^+(x,\varphi)\big)^2.$$

3. Sums of the Selected Trigonometric Series

Let us take

$$F(z) := z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots \tag{3.1}$$

The above series is of course absolutely convergent for $|z| < 1$ and, as it is well known, $F(z) = \ln(1+z)$, $|z| < 1$, where $\ln(\xi) := \ln|\xi| + i\arg\xi$ (in other words, $\ln(\xi)$ denotes the main branch of the complex natural logarithm). Let us consider the trigonometric series, written below

$$\cos x - \frac{\cos 2x}{2} + \frac{\cos 3x}{3} - \dots \tag{3.2}$$

and

$$\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots. \tag{3.3}$$

The following two theorems come from [34] (see also Remark 3.6).

Theorem 3.1. *If $\{a_k\}$ tends to 0 and it is of bounded variation (in particular, if $\{a_k\}$ tends monotonically to 0), then both series*

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k\cos(kx), \quad \sum_{k=1}^{\infty} a_k\sin(kx)$$

and so also the series $\sum\limits_{k=1}^{\infty} a_k e^{ikx}$, converge uniformly in each interval $\varepsilon \le x \le 2\pi - \varepsilon$ $(\varepsilon > 0)$.

Thus, replacing x by $x+\pi$ we obtain:

Theorem 3.2. *If $\{a_k\}$ is of bounded variation and tends to 0, the series*

$$\frac{1}{2}a_0+\sum_{k=1}^{\infty}(-1)^k a_k\cos(kx),\quad \sum_{k=1}^{\infty}(-1)^k a_k\sin(kx)$$

converge uniformly for $|x|\le \pi-\varepsilon$ ($\varepsilon>0$).

By Theorem 3.2, the series (3.2) and (3.3) converge for $x\notin\pi(2\mathbb{Z}+1)$.

Let $C(x)$ and $S(x)$ denote the sums of the series (3.2) and (3.3), respectively, for $x\notin\pi(2\mathbb{Z}+1)$. This implies the equality

$$C(x)+iS(x)=F\big(e^{ix}\big) \tag{3.4}$$

for $x\notin\pi(2\mathbb{Z}+1)$. Since the function $\ln(1+z)$ is holomorphic in the Gaussian plane without the point $z=-1$, then, according to the Abel theorem (complex version), we receive the equality ($x\notin\pi(2\mathbb{Z}+1)$):

$$\ln\big(1+e^{ix}\big)=\lim_{z\to e^{ix}}\ln(1+z)=\lim_{z\to e^{ix}}F(z)=F\big(e^{ix}\big), \tag{3.5}$$

where the convergence $z\to e^{ix}$ holds in some angle sector, included in the circle $|z|<1$, with the vertex e^{ix}. Moreover we have ($-\pi<x<\pi$):

$$\ln\big(1+e^{ix}\big)=\ln\left(2\cos\frac{x}{2}e^{i\frac{x}{2}}\right)=\ln\left(2\cos\frac{x}{2}\right)+i\frac{x}{2}, \tag{3.6}$$

which implies the identities

$$\ln\left(2\cos\frac{x}{2}\right)=C(x)\quad\text{and}\quad\frac{x}{2}=S(x). \tag{3.7}$$

The convolution multiplication (meaning the Cauchy product of power series) leads to the following equality

$$-p(z)F(-z)=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\min\{n-1,N\}}\frac{a_k}{n-k}\right)z^n, \tag{3.8}$$

where $p(z):=\sum\limits_{n=0}^{N}a_nz^n$ and $|z|\leqslant 1$, $z\neq 1$, $a_n\in\mathbb{C}$, $n=1,\dots,N$. Hence, by replacing $z=-e^{ix}$, $-\pi<x<\pi$, and comparing the real and imaginary parts of both sides we obtain

$$\mathcal{R}e\big(p\big(-e^{ix}\big)\big)\ln\left(2\cos\frac{x}{2}\right)-\mathcal{I}m\big(p\big(-e^{ix}\big)\big)\frac{x}{2}=$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\sum_{k=0}^{\min\{n-1,N\}}\frac{a_k}{n-k}\right)\cos(nx) \tag{3.9}$$

and

$$\mathcal{R}e\big(p\big(-e^{ix}\big)\big)\frac{x}{2}+\mathcal{I}m\big(p\big(-e^{ix}\big)\big)\ln\left(2\cos\frac{x}{2}\right)=$$
$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\sum_{k=0}^{\min\{n-1,N\}}\frac{a_k}{n-k}\right)\sin(nx),\quad(3.10)$$

from where, by substituting $x+\pi=y$, $0<y<2\pi$, we receive the alternative identities

$$\mathcal{I}m\big(p\big(e^{iy}\big)\big)\frac{y-\pi}{2}-\mathcal{R}e\big(p\big(e^{iy}\big)\big)\ln\left(2\sin\frac{y}{2}\right)=$$
$$=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\min\{n-1,N\}}\frac{a_k}{n-k}\right)\cos(ny),\quad(3.11)$$

and

$$\mathcal{R}e\big(p\big(e^{iy}\big)\big)\frac{\pi-y}{2}-\mathcal{I}m\big(p\big(e^{iy}\big)\big)\ln\left(2\sin\frac{y}{2}\right)=$$
$$=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\min\{n-1,N\}}\frac{a_k}{n-k}\right)\sin(ny).\quad(3.12)$$

In particular, from formulae (3.11) and (3.12) the following identities can be obtained for $p(z)=z-1$:

$$\sum_{k=1}^{\infty}\frac{\cos\big((k+1)y\big)}{k(k+1)}=(1-\cos y)\ln\left(2\sin\frac{y}{2}\right)-\frac{\pi-y}{2}\sin y+\cos y,\quad(3.13)$$

$$\sum_{k=1}^{\infty}\frac{\sin\big((k+1)y\big)}{k(k+1)}=\frac{\pi-y}{2}(\cos y-1)-(\sin y)\ln\left(2\sin\frac{y}{2}\right)+\sin y,\quad(3.14)$$

and generally for $p(z)=\frac{1}{N!}(z-1)^N, N\in\mathbb{N}$:

$$\sum_{k=1}^{\infty}\frac{\cos\big((k+N)y\big)}{k(k+1)\dots(k+N)}=-\ln\left(2\sin\frac{y}{2}\right)\sum_{k=0}^{N}\frac{(-1)^{N-k}\cos(ky)}{k!(N-k)!}+$$
$$+\frac{y-\pi}{2}\sum_{k=0}^{N}\frac{(-1)^{N-k}\sin(ky)}{k!(N-k)!}-\sum_{n=1}^{N}\left(\sum_{k=0}^{n-1}\frac{(-1)^{N-k}}{k!(N-k)!(n-k)}\right)\cos(ny),\quad(3.15)$$

$$\sum_{k=1}^{\infty}\frac{\sin\big((k+N)y\big)}{k(k+1)\dots(k+N)}=-\ln\left(2\sin\frac{y}{2}\right)\sum_{k=0}^{N}\frac{(-1)^{N-k}\sin(ky)}{k!(N-k)!}+$$

$$+\frac{y-\pi}{2}\sum_{k=0}^{N}\frac{(-1)^{N-k}\cos(ky)}{k!(N-k)!}-\sum_{n=1}^{N}\left(\sum_{k=0}^{n-1}\frac{(-1)^{N-k}}{k!(N-k)!(n-k)}\right)\sin(ny). \quad (3.16)$$

We note, that from identity (3.16) by using (1.2) we obtain (only the case $N=2$ is presented here):

$$\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)^2(k+2)^2}=$$

$$=\frac{2}{\pi}\int\limits_0^{\pi}\left((t-\pi)\sin^2\frac{t}{2}\cos t+\sin^2\frac{t}{2}\sin t\ln\left(2\sin\frac{t}{2}\right)-\frac{1}{2}\sin t+\frac{1}{2}\sin 2t\right)^2 dt=$$

$$=\frac{168\pi^2-1423}{1152}+\frac{1}{\pi}\int\limits_0^{\pi}(1-\cos t)\sin t\ln\left(2\sin\frac{t}{2}\right)\times$$

$$\times\left(\frac{1}{2}(1-\cos t)\sin t\ln\left(2\sin\frac{t}{2}\right)-\sin t+\sin 2t+(t-\pi)(1-\cos t)\cos t\right)dt.$$

On the other hand we have

$$\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)^2(k+2)^2}=\frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{1}{k^2}+\frac{4}{(k+1)^2}+\frac{1}{(k+2)^2}-\frac{3}{k}+\frac{3}{k+2}\right)=$$

$$=\frac{1}{4}\left(\frac{\pi^2}{6}+4\left(\frac{\pi^2}{6}-1\right)+\frac{\pi^2}{6}-\frac{5}{4}-\frac{9}{2}\right)=\frac{1}{4}\left(\pi^2-\frac{39}{4}\right).$$

Summarizing we derive

$$\frac{120\pi^2-1385}{1152}\pi=\int\limits_0^{\pi}(1-\cos t)\sin t\ln\left(2\sin\frac{t}{2}\right)\times$$

$$\times\left(\frac{1}{2}(1-\cos t)\sin t\ln\left(2\sin\frac{t}{2}\right)-\sin t+\sin 2t+(t-\pi)(1-\cos t)\cos t\right)dt.$$

Remark 3.3. In paper [23] application of the psi function $\psi(x):=\frac{\Gamma'(x)}{\Gamma(x)}$, $x\in\mathbb{R}\setminus\{0,-1,-2,\ldots\}$ (see [1]) for determining sums of the series

$$S=\sum_{k=1}^{\infty}\frac{w(k)}{\prod\limits_{j=1}^{n}(a_j\,k+b_j)},$$

is described, where $w\in\mathbb{R}[x]$, $\deg w\leqslant n-2$, $a_j,b_j\in\mathbb{Z}$, $a_j\neq 0$, $j=1,\ldots,n$. Additionally we assume, that no two vectors $[a_j,b_j]$ and $[a_l,b_l]$ are parallel whenever $j\neq l$.

Remark 3.4. Executing in identities

$$\ln\left(2\cos\frac{x}{2}\right) = C(x) \quad \text{i} \quad \frac{x}{2} = S(x), \tag{3.17}$$

the substitution $x + \pi = y$, $0 < y < 2\pi$, we obtain the dual identities

$$\cos y + \frac{\cos 2y}{2} + \frac{\cos 3y}{3} + \ldots = -\ln\left(2\sin\frac{y}{2}\right) \tag{3.18}$$

and

$$\sin y + \frac{\sin 2y}{2} + \frac{\sin 3y}{3} + \ldots = \frac{\pi - y}{2}. \tag{3.19}$$

Whereas, the addition of identity (3.17) with identities (3.18) and (3.19) leads to the new relations

$$\cos x + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \ldots = \frac{1}{2}\ln\left(\cot\frac{x}{2}\right) \tag{3.20}$$

for $0 < x < \pi$, which, after some tiny modification:

$$\cos x + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \ldots = \frac{1}{2}\ln\left|\cot\frac{x}{2}\right| \tag{3.21}$$

holds true for every $x \in \mathbb{R}$, $x \notin \pi\mathbb{Z}$, and

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots = \frac{\pi}{4} \tag{3.22}$$

for $0 < x < \pi$. From this one can easily follow, that

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots = \begin{cases} \frac{\pi}{4}, & 0 < x < \pi, \\ -\frac{\pi}{4}, & \pi < x < 2\pi. \end{cases} \tag{3.23}$$

Irving Kaplansky in [15] proposed to derive the formulae (3.21) and (3.23) in the purely formal way (in the spirit of the XVIII century mathematics – like he wrote himself). In fact, by using the power series expansion of the function $\ln(1 \pm x)$ we receive

$$f(x) := x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right). \tag{3.24}$$

This equality gives (we consider only the main branch of logarithm), that

$$f(e^{i\varphi}) = \frac{1}{2}\ln\frac{1+e^{i\varphi}}{1-e^{i\varphi}} = \frac{1}{2}\ln\left(i\cot\frac{\varphi}{2}\right) = \begin{cases} \frac{1}{2}\ln(\cot\frac{\varphi}{2}) + \frac{i\pi}{4}, 0 < \varphi < \pi, \\ \frac{1}{2}\ln(-\cot\frac{\varphi}{2}) - \frac{i\pi}{4}, \pi < \varphi < 2\pi, \end{cases}$$

which, in the quite trivial way, implies (3.21) and (3.23).

Remark 3.5. Let us notice, that a small correction of the formula (3.18):

$$\cos y + \frac{1}{2}\cos 2y + \frac{1}{3}\cos 3y + \ldots = -\ln\left|2\sin\frac{y}{2}\right| \tag{3.25}$$

makes it true for every $y \in \mathbb{R}$, $y \notin 2\pi\mathbb{Z}$. It is well known, that partial sums of this series are $\geqslant -1$ (W. H. Young, 1913, however the proof, given there, is incorrect – for more details see [4]).

Remark 3.6. Chaundy and Jolliffe [34] proved the following result:

Theorem 3.7. *Suppose, that $b_n \geq b_{n+1}$ and $b_n \to 0$. Then a necessary and sufficient condition for uniform convergence of the series*

$$\sum_{n=1}^{\infty} b_n \sin(nx)$$

is $nb_n \to 0$.

J. R. Nurcombe in paper [22] generalized this result for the, so called, quasi-monotonic sequences. Further generalizations, essentially enriching the subject of research, were obtained by L. Leindler [19] and B. Szal [26], [27] and [28].

Extension of the Theorem 3.7 for the complex series was investigated by Xie and Zhou in paper [33]. Analytical properties of the series sums

$$G_\alpha(x) := \sum_{n=1}^{\infty} n^{-\alpha} \cos(nx) \quad \text{oraz} \quad H_\alpha(x) := \sum_{n=1}^{\infty} n^{-\alpha} \sin(nx),$$

where $\alpha > 0, x \in \mathbb{R}$, were discussed by Brown and others in [7].

Remark 3.8. C. Jordan in the classical monograph [16] and M. Eie in monograph [14], by using the trigonometric Fourier series expansions of the Bernoulli polynomials $B_n(x)$, derived the following relation

$$\sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^{2n}} = \frac{(-1)^{n-1}(2\pi)^{2n} B_{2n}(\{x\})}{2(2n)!}, \tag{3.26}$$

for $n \in \mathbb{N}$, $x \in \mathbb{R}$, where $\{x\} := x - \lfloor x \rfloor$ is the fractional part of number x. Bernoulli polynomials $B_n(x)$, $n = 0, 1, 2, \ldots$, can be defined, with the aid of the formula given below [14]:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \tag{3.27}$$

where B_k denote Bernoulli numbers, defined, by the following recurrence relation:

$$B_0 = 1$$

and

$$\binom{n}{n-1} B_{n-1} + \binom{n}{n-2} B_{n-2} + \ldots + \binom{n}{0} B_0 = 0, \quad \text{for} \quad n \geqslant 2. \tag{3.28}$$

Similarly like in formula (3.26), one can generate the dual relation of the form (see [2]):

$$\sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2n+1}} = \frac{(-1)^{n-1}(2\pi)^{2n+1} B_{2n+1}\big(\{x\}\big)}{2(2n+1)!}. \tag{3.29}$$

D. Cvijović and J. Klinowski in [11] deduced from (3.29) the following representations for the values of the zeta function

$$\zeta(2n+1) = \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} = (-1)^{n+1} \frac{(2\pi)^{2n+1}}{2\delta(2n+1)!} \int_0^{\delta} B_{2n+1}(t)\cot(\pi t)dt$$
$$= \frac{(-1)^{n+1}(2\pi)^{2n+1}}{2\delta(1-2^{-2n})(2n+1)!} \int_0^{\delta} B_{2n+1}(t)\tan(\pi t)dt, \tag{3.30}$$

where $\delta = 1$ or $\delta = \frac{1}{2}$.

Formulae (3.26) and (3.29) were used in work [2] for explaining an intriguing phenomenon. Difference between the values

$$J_N := \int_0^{\infty} \operatorname{sinc}^N(x)dx \tag{3.31}$$

and

$$S_N := \sum_{n=1}^{\infty} \operatorname{sinc}^N(x), \tag{3.32}$$

where

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin x}{x}, & \text{gdy} \quad x \neq 0, \\ 1, & \text{gdy} \quad x = 0. \end{cases} \tag{3.33}$$

is constant for $N = 1, 2, \ldots, 6$ and is equal to $1/2$. Whereas for $N = 7, 8, \ldots$ the polynomials of variable π and degree $7, 8, \ldots$, respectively, are obtained. For example we have

$$J_7 = 5887\pi/23040,$$
$$S_7 = -\frac{1}{2} + \frac{43141}{15360}\pi - \frac{16807}{3840}\pi^2 + \frac{2401}{768}\pi^3 - \frac{343}{288}\pi^4 + \frac{49}{192}\pi^5 - \frac{7}{240}\pi^6 + \frac{1}{720}\pi^7.$$

Furthermore, Mordell in paper [21] applied formula (3.26) for proving the following theorem.

If r is a natural even number, then there exists a rational number k_r, such that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^r n^r (m+n)^r} = \pi^{3r} k_r. \tag{3.34}$$

In particular we have $k_2 = 1/2835$.

Remark 3.9. K. Matthies and D. Mazkewitsch [20] determined the Fourier sine series expansion and the Fourier cosine series expansion of the function $\sin^a x$, where $a > -1$. They deduced the following formulae, for $0 < x < \pi$:

$$\sin^a x = \\ =\frac{1}{\sqrt{\pi}} \frac{\Gamma\big(1+\frac{a}{2}\big)}{\Gamma\big(1-\frac{1}{2}(a+1)\big)} \sum_{n=0}^{\infty} \frac{\Gamma\big(n+1-\frac{1}{2}(a+1)\big)}{\Gamma\big(n+1+\frac{1}{2}(a+1)\big)} \sin\big((2n+1)x\big), \tag{3.35}$$

$$\sin^a x = \frac{1}{\sqrt{\pi}} \frac{\Gamma\big(\frac{a}{2}+\frac{1}{2}\big)}{\Gamma\big(-\frac{a}{2}\big)} \left(\frac{\Gamma\big(-\frac{a}{2}\big)}{2\,\Gamma\big(1+\frac{a}{2}\big)} + \sum_{n=1}^{\infty} \frac{\Gamma\big(n-\frac{a}{2}\big)}{\Gamma\big(n+1+\frac{a}{2}\big)} \right) \cos\big(2nx\big). \tag{3.36}$$

We note, that for all x in the interval $k\pi < x < (k+1)\pi$, where $k = 0, \pm1, \pm2, \ldots$, the right sides of (3.35) and (3.36) are equal to $(-1)^k |\sin x|^a$ and $|\sin x|^a$, respectively.

3.1. Applications

I. Let us start by applying Parseval's Formula to the Fourier expansion (3.18), which implies

$$\frac{\pi^3}{12} = \int_0^{\pi} \ln^2\left(2\sin\frac{y}{2}\right) dy. \tag{3.37}$$

As we will show below, after nontrivial rescaling in (3.37) we can obtain the following Bremekamp integral (1957, see [5]):

$$\int_0^{\pi} (\ln\sin y)^2 dy = \frac{1}{12}\pi^3 + \pi \ln^2 2. \tag{3.38}$$

First, let us notice, that

$$\int_0^{\pi} \ln^2\left(2\sin\frac{y}{2}\right) dy \overset{2x:=y}{=} 2\int_0^{\frac{\pi}{2}} \ln^2(2\sin x)dx = \int_0^{\pi} \ln^2(2\sin x)dx =$$

$$= \int_0^{\pi} \big(\ln 2 + \ln\sin x\big)^2 dx = \pi\ln^2 2 + 2\ln 2\int_0^{\pi} \ln\sin x dx + \int_0^{\pi} \ln^2 \sin x dx. \tag{3.39}$$

Furthermore

$$\int_0^\pi \ln\sin x dx = \int_0^\pi \big(\ln(2\sin x) - \ln 2\big) dx = \int_0^\pi \ln(2\sin x) dx - \pi\ln 2 \overset{2x:=y}{=}$$

$$= \frac{1}{2}\int_0^{2\pi} \ln\left(2\sin\frac{y}{2}\right) dy - \pi\ln 2 = \int_0^\pi \ln\left(2\sin\frac{y}{2}\right) dy - \pi\ln 2 = -\pi\ln 2, \quad (3.40)$$

since, according to (3.18), we have

$$\int_0^\pi \ln\left(2\sin\frac{y}{2}\right) dy = -\sum_{n=1}^\infty \frac{1}{n}\int_0^\pi \cos(ny)dy = 0. \quad (3.41)$$

Summarizing, we get

$$\int_0^\pi \ln^2\left(2\sin\frac{y}{2}\right) dy = \int_0^\pi \ln^2\sin x dx - \pi\ln^2 2.$$

Thus, because of (3.37), we receive (3.38).

Remark 3.10. M.G. Beumer in paper [5] discussed also more general integral of the form

$$D(n) = \frac{(-1)^{n-1}}{(n-1)!}\int_0^{\frac{\pi}{2}} \ln^{n-1}\sin x dx, \quad n\in\mathbb{N}, \quad (3.42)$$

deriving for them the following recursive relation

$$D(1)D(2n-1) - D(2)D(2n-2) + \ldots + D(2n-1)D(1) = \frac{2^{2n}-1}{(2n)!}\pi^{2n}B_n \quad (3.43)$$

where B_n denotes the nth Bernoulli number. C.B. Colins [10] show that the values of integrals (3.42) and more general integrals

$$I_{n,\alpha,\beta} := \int_0^{\frac{\pi}{2}} \ln^n\left(\sin^\alpha x\,\cos^\beta x\right) dx, \qquad n\in\mathbb{N}, \quad \alpha,\beta\in\mathbb{R}$$

are a simple multiple of a Bell polynomials.

II. Another application of the Parseval's Formula (based on the idea of Borwein and Bradley [6]) concerns the following Fourier series (below we will use the formulae (3.18) and (3.19) for $y\in(0,2\pi)$). We derive ($x\in(0,2\pi)$):

$$\sum_{n=1}^\infty \frac{\sin(n\,x)}{n}\sum_{k=1}^{n-1}\frac{1}{k} = \frac{1}{2}\sum_{n=1}^\infty \frac{\sin(n\,x)}{n}\sum_{k=1}^{n-1}\left(\frac{1}{k}+\frac{1}{n-k}\right) = \frac{1}{2}\sum_{n=1}^\infty\sum_{k=1}^{n-1}\frac{\sin(n\,x)}{k\,(n-k)}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\big((m+n)\,x\big)}{2m\,n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(\frac{\sin(m\,x)\,\cos(n\,x)}{2m\,n}+\frac{\cos(m\,x)\,\sin(n\,x)}{2m\,n}\Big)$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin(m\,x)\,\cos(n\,x)}{m\,n} = \left(\sum_{m=1}^{\infty}\frac{\sin(m\,x)}{m}\right)\left(\sum_{n=1}^{\infty}\frac{\cos(n\,x)}{n}\right)$$
$$= -\frac{\pi-x}{2}\,\ln\left(2\,\sin\left(\frac{x}{2}\right)\right). \quad (3.44)$$

By integrating this relation, term by term, we receive

$$\sum_{n=1}^{\infty}\frac{\cos(n\,\theta)}{n^2}\sum_{k=1}^{n-1}\frac{1}{k} = \zeta(2,1)+\frac{1}{2}\int_0^{\theta}\big(\pi-t\big)\,\ln\left(2\,\sin\left(\frac{t}{2}\right)\right)dt, \quad (3.45)$$

for every $0 \leqslant \theta \leqslant 2\,\pi$, where $\zeta(2,1) := \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{k}$. Furthermore, by double integrating the equality (3.18), we obtain

$$\sum_{n=1}^{\infty}\frac{\sin(n\,y)}{n^2} = -\int_0^{y}\ln\left(2\,\sin\left(\frac{x}{2}\right)\right)dx,$$

for $0 \leqslant y \leqslant 2\,\pi$, and next

$$\sum_{n=1}^{\infty}\frac{\cos(n\,\theta)}{n^3} = \zeta(3)+\int_0^{\theta}\left(\int_0^{y}\ln\left(2\,\sin\left(\frac{x}{2}\right)\right)dx\right)dy =$$
$$= \zeta(3)+\left[y\int_0^{y}\ln\left(2\,\sin\left(\frac{x}{2}\right)\right)dx\right]_0^{\theta}-\int_0^{\theta}y\,\ln\left(2\,\sin\left(\frac{y}{2}\right)\right)dy =$$
$$= \zeta(3)+\int_0^{\theta}\big(\theta-y\big)\,\ln\left(2\,\sin\left(\frac{y}{2}\right)\right)dy \quad (3.46)$$

for $0 \leq \theta \leq 2\pi$, where $\zeta(3) := \sum_{n=1}^{\infty}\frac{1}{n^3}$. Taking $\theta = \pi$ in identities (3.45) and (3.46), we generate the formula

$$\zeta(2,1)-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sum_{k=1}^{n-1}\frac{1}{k} = -\frac{1}{2}\int_0^{\pi}\big(\pi-t\big)\,\ln\left(2\,\sin\left(\frac{t}{2}\right)\right)dt =$$
$$= \frac{1}{2}\left(\zeta(3)-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\right),$$

hence, according to the equality $\zeta(2,1)=\zeta(3)$ (see [6]), we get

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sum_{k=1}^{n-1}\frac{1}{k}=\frac{1}{2}\left(\zeta(3)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\right)=$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1+(-1)^n}{n^3}=\sum_{m=1}^{\infty}\frac{1}{(2\,m)^3}=\frac{1}{8}\,\zeta(3).$$

By applying Parseval's Formula to the Fourier series (3.44), we obtain one more interesting equality, having the form (the last equality is derived by Mathematica assistance):

$$\frac{1}{4\,\pi}\int\limits_0^{2\,\pi}\left(\pi-t\right)^2\,\ln^2\left(2\,\sin\left(\frac{t}{2}\right)\right)dt=\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)^2}=\frac{11}{4}\,\zeta(4).$$

Remark 3.11. Since we have [17]:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2}\cos(kx)=\frac{\pi^2-3x^2}{12}, \tag{3.47}$$

so from (1.1) and (3.18) (and (3.20) respectively) it could be easily deduced the formulae

$$\sum_{k=1}^{\infty}(-1)^k\frac{1}{k^3}=\frac{1}{6\pi}\int\limits_0^{\pi}(\pi^2-3x^2)\ln\left(2\sin\frac{x}{2}\right)dx, \tag{3.48}$$

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}=\frac{1}{12\pi}\int\limits_0^{\pi}(\pi^2-3x^2)\ln\left(\cot\frac{x}{2}\right)dx. \tag{3.49}$$

Moreover, since we have ([17], [18] and [25]):

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\big((2k-1)x\big)=\frac{1}{8}\pi(\pi-2x), \tag{3.50}$$

by (1.1) and (3.20) we get the equality

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}=\frac{1}{8}\int\limits_0^{\pi}(\pi-2x)\ln\left(\cot\frac{x}{2}\right)dx. \tag{3.51}$$

From (3.49) and (3.51) we receive

$$\int\limits_0^{\pi}(6x^2-6\pi x+\pi^2)\ln\left(\cot\frac{x}{2}\right)dx=0. \tag{3.52}$$

Remark 3.12. Let us consider now the periodic function $f\colon\mathbb{R}\to\mathbb{R}$,

$f(x+2\pi)=f(x)$, defined by the formula [3]:

$$f(x)=\begin{cases} -\frac{1}{2}(\pi-|x|) & \text{for} \quad x\in[-\pi,-1],\\ \frac{\pi-1}{2}x & \text{for} \quad x\in[-1,1],\\ \frac{1}{2}(\pi-x) & \text{for} \quad x\in[1,\pi]. \end{cases} \tag{3.53}$$

Obviously f is an odd function. It can be shown, that the trigonometric Fourier series of this function has the form

$$f(x)=\sum_{n=1}^{\infty}\frac{\sin n}{n^2}\sin(nx), \quad x\in\mathbb{R}. \tag{3.54}$$

It can be also noticed, that

$$f(1)=\sum_{n=1}^{\infty}\left(\frac{\sin n}{n}\right)^2=\frac{\pi-1}{2}=f'(0)=\sum_{n=1}^{\infty}\frac{\sin n}{n},$$

$$f\big(\frac{\pi}{2}\big)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(2n-1)}{(2n-1)^2}=\frac{\pi}{4},$$

$$\frac{2\sqrt{3}}{3}f\big(\frac{\pi}{3}\big)=\frac{\sin 1}{1^2}+\frac{\sin 2}{2^2}-\frac{\sin 4}{4^2}-\frac{\sin 5}{5^2}+\frac{\sin 7}{7^2}+\frac{\sin 8}{8^2}+\ldots=$$
$$=\sum_{n=0}^{\infty}(-1)^k\left(\frac{\sin(3k+1)}{(3k+1)^2}+\frac{\sin(3k+2)}{(3k+2)^2}\right)=\frac{2\sqrt{3}}{9}\pi.$$

Furthermore, aplying the Parseval's Formula to (3.54) we obtain

$$\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^4}=\frac{(\pi-1)^2}{6}.$$

Let us also notice, that from (3.53) and (3.54) the formula follow

$$f'(x)=\sum_{n=1}^{\infty}\frac{\sin n}{n}\cos(nx)=\begin{cases} \frac{\pi-1}{2} & \text{gdy } |x|<1,\\ -\frac{1}{2} & \text{gdy } 1<|x|<\pi,\\ \frac{\pi-2}{4} & \text{gdy } |x|=1, \end{cases} \tag{3.55}$$

which implies the relations:

$$f'(1)=\sum_{n=1}^{\infty}\frac{\sin 2n}{2n}=\frac{\pi-2}{4} \quad\Longrightarrow\quad \sum_{n=1}^{\infty}\frac{\sin(2n-1)}{2n-1}=\frac{\pi}{4},$$

$$f'\big(\frac{\pi}{2}\big)=\sum_{n=1}^{\infty}(-1)^n\frac{\sin 2n}{2n}=-\frac{1}{2},$$

$$\frac{1}{2}\left(f'(1)+f'\big(\frac{\pi}{2}\big)\right)=\sum_{n=1}^{\infty}\frac{\sin 4n}{4n}=\frac{\pi-4}{8}.$$

Moreover, by applying Parseval's Formula (1.1) to Fourier series expansions (3.18) and (3.55) we get

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^2}=\frac{1-\pi}{\pi}\int_0^1\ln\left(2\sin\frac{x}{2}\right)dx+\frac{1}{\pi}\int_1^{\pi}\ln\left(2\sin\frac{x}{2}\right)dx=$$

$$=\frac{1}{\pi}\int_0^{\pi}\ln\left(2\sin\frac{x}{2}\right)dx-\int_0^1\ln\left(2\sin\frac{x}{2}\right)dx\overset{(3.41)}{=}-\int_0^1\ln\left(2\sin\frac{x}{2}\right)dx,\quad(3.56)$$

while from (3.20) and (3.55) we obtain

$$\sum_{n=1}^{\infty}\frac{\sin(2n-1)}{(2n-1)^2}=\frac{\pi-1}{2\pi}\int_0^1\ln\left(\cot\frac{x}{2}\right)dx-\frac{1}{2\pi}\int_1^{\pi}\ln\left(\cot\frac{x}{2}\right)dx=$$

$$=\frac{1}{2}\int_0^1\ln\left(\cot\frac{x}{2}\right)dx-\frac{1}{2\pi}\int_0^{\pi}\ln\left(\cot\frac{x}{2}\right)dx\overset{(3.41)}{=}$$

$$=\frac{1}{2}\int_0^1\ln\left(\cot\frac{x}{2}\right)dx-\frac{1}{2\pi}\int_0^{\pi}\ln\left(2\cos\frac{x}{2}\right)dx\overset{(y=\pi-x)}{=}$$

$$=\frac{1}{2}\int_0^1\ln\left(\cot\frac{x}{2}\right)dx-\frac{1}{2\pi}\int_0^{\pi}\ln\left(2\sin\frac{y}{2}\right)dy\overset{(3.41)}{=}\frac{1}{2}\int_0^1\ln\left(\cot\frac{x}{2}\right)dx.\quad(3.57)$$

4. The Sine Integral Function

Let $\mathrm{Si}(x)=\int_0^x\frac{\sin t}{t}dt$ denote the sine integral function. Putting $f(t)=-\frac{\pi}{2}\ln t$ we have

$$\int_0^1(\ln t)^n dt=t(\ln t)^n\Big|_0^1-n\int_0^1(\ln t)^{n-1}dt=-n\int_0^1(\ln t)^{n-1}dt=(-1)^n n!,$$

because of the limit $\lim_{t\to 0^+} t(-\ln t)^\alpha = \lim_{t\to 0^+} \frac{(-\ln t)^\alpha}{\frac{1}{t}} \overset{[\frac{\infty}{\infty}]}{=} \ldots = 0$, which is satisfied for every $\alpha > 0$. So, $f(t) \in L^2(0,\pi)$ and we obtain

$$\int_0^\pi \ln t dt = t\ln t\Big|_0^\pi - \int_0^\pi dt = \pi(\ln\pi - 1),$$

$$\int_0^\pi \ln t\cos(nt)dt = \ln t\frac{\sin(nt)}{n}\Big|_0^\pi - \int_0^\pi \frac{\sin(nt)}{nt}dt \overset{u=nt}{=} -\int_0^{n\pi} \frac{\sin u}{nu}du = -\frac{\mathrm{Si}(n\pi)}{n},$$

which implies $\hat{f}(0) = \pi(1-\ln\pi)$ and $\hat{f}(n) = \frac{\mathrm{Si}(n\pi)}{n}, n = 1,2,\ldots$. Hence, by (3.18) and at last by (1.1) we obtain

$$\sum_{n=1}^\infty \frac{\mathrm{Si}(n\pi)}{n^2} = \int_0^\pi \ln t\ln\left(2\sin\frac{t}{2}\right) dt.$$

On the other hand, by (3.21) we get

$$\sum_{n=1}^\infty \frac{\mathrm{Si}\left((2n-1)\pi\right)}{(2n-1)^2} = -\frac{1}{2}\int_0^\pi \ln t\ln\cot\frac{t}{2}dt,$$

Additionally in [9], by using Parseval's Formula the following equalities are derived

$$\sum_{n=1}^\infty \left(\frac{\mathrm{Si}(n\pi)}{n}\right)^2 = \frac{\pi^2}{2}, \quad \sum_{n=1}^\infty \frac{\mathrm{Si}(n\pi)}{n^3} = \pi^3\left(\frac{1}{8} - \frac{1}{18}\right), \quad \sum_{n=1}^\infty \frac{\mathrm{Si}(n\pi)}{n^3} = -\frac{\pi^3}{18}.$$

5. Certain Salaev's Trigonometric Series

B. W. Salaev in paper [24] found the following Fourier series expansion

$$\sum_{n=2}^\infty \left(\ln\frac{n^2}{n^2-1}\right)\cos(nx) =$$
$$= \ln 2 - \frac{\pi}{2}\sin x + (1-\cos x)\left(\ln\pi + \gamma + \psi\left(\frac{2\pi - x}{2\pi}\right)\right), \quad (5.1)$$

for $x \in [0, 2\pi)$, where γ denotes the Euler constant, $\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \ln n\right)$ and ψ denotes the psi function (see [1]).

From (2.1) and (5.1) by applying (1.1) we get the following identities

$$\pi 2^{1-2n} \sum_{k=1}^{n} \binom{2n}{n+k} \cos(2k\varphi) \ln \frac{4k^2}{4k^2-1} = \int_0^\pi C_{2n}^{+}(x,\varphi) \Big[\ln 2 - \frac{\pi}{2} \sin x + \\ + (1-\cos x) \left(\ln \pi + \gamma + \psi \left(\frac{2\pi - x}{2\pi} \right) \right) \Big] dx, \quad (5.2)$$

for $n \in \mathbb{N}$ and $\varphi \in \mathbb{R}$. Moreover, in [24] some generalization of (5.1) is presented and the following Tveritin's generalizations of the Fourier expansions (3.18) and (3.19) are deduced (see also [30]):

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n-a} = -\frac{\pi}{2} \cdot \frac{\cos\big(a(\pi - x)\big)}{\sin(a\pi)} - \sum_{j=0}^{p-1} P_j(x) \cos\big(a(2\pi j - x)\big),$$

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n+a} = \frac{\pi}{2} \cdot \frac{\cos\big(a(\pi - x)\big)}{\sin(a\pi)} - \frac{1}{a} - \sum_{j=0}^{p-1} P_j(x) \cos\big(a(2\pi j - x)\big),$$

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n \pm a} = \frac{\pi}{2} \cdot \frac{\sin\big(a(\pi - x)\big)}{\sin(a\pi)} \mp \sum_{j=0}^{p-1} P_j(x) \sin\big(a(2\pi j - x)\big),$$

for $a \in (0,1), a = \frac{p}{q}, p, q \in \mathbb{N}, (p,q) = 1$, where

$$P_j(x) := \ln \left| 2 \sin \left(\frac{x - 2\pi j}{2p} \right) \right|, \quad j = 0, 1, \ldots, p-1.$$

6. The Fourier Sine Series Expansion of the Exponent Function

C. R. Edstrom in [13] discussed the Fourier sine series expansion of $f(x) = \exp(a\,x)$, $0 \leq x < \pi$, where $a \in \mathbb{R}$ is fixed

$$\exp(a\,x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n\big(1 - (-1)^n \exp(a\,\pi)\big)}{n^2 + a^2} \sin(n\,x). \quad (6.1)$$

From (6.1) for $x = \pi/2$ we can find

$$\exp\left(\frac{a\,\pi}{2}\right) = \frac{2\big(1 + \exp(a\,\pi)\big)}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n\,(2\,n+1)}{(2\,n+1)^2 + a^2}, \quad (6.2)$$

which implies

$$\mathrm{Sech}\left(\frac{a\,\pi}{2}\right) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n\,(2\,n+1)}{(2\,n+1)^2 + a^2}. \quad (6.3)$$

We note, that Edstrom generated from (6.3) the following nice identity of Rainville

$$\sum_{n=0}^{\infty} \frac{(-1)^n \operatorname{Sech}\left(\frac{(2n+1)\pi}{2}\right)}{2\,n+1} = \frac{\pi}{8}. \tag{6.4}$$

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