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PRODINGER'S ALGEBRAIC IDENTITIES AND THEIR APPLICATIONS

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Abstract: The aim of this paper is to present two fundamental applications of the, so called, Prodinger's algebraic identities. On the one hand we will receive the wide class of trigonometric identities, including identities for Chebyshev polynomials of the first and second kinds. On the second hand we will present the generalizations for Fibonacci and Lucas polynomials of some identities, proved by Prodinger for Fibonacci and Lucas numbers.

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1. Introduction

H. Prodinger in the paper [9] derived three following identities (the proof of the first one is given by induction in [9]):

$$\frac{(-1)^n}{2n+1}(x^{2n+1}+x^{-2n-1}) = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-1)^k}{2k+1}(x+x^{-1})^{2k+1}, \tag{1.1}$$

$$\frac{1}{2n+1}(x^{2n+1}-x^{-2n-1}) = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{1}{2k+1}(x-x^{-1})^{2k+1} \tag{1.2}$$

and

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$$x^{2n} - x^{-2n} = (x + x^{-1}) \sum_{k=0}^{n-1} \binom{n+k}{n-k-1} (x - x^{-1})^{2k+1}. \tag{1.3}$$

We can note, that after substituting $x := ix$ the identities (1.1) and (1.2) are equivalent.

The other version of identity (1.1) for the even exponents is the formula written below (which can be also proven by induction):

$$\frac{(-1)^n}{2n}(x^{2n} + x^{-2n}) = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-1)^k}{n+k} (x + x^{-1})^{2k}. \tag{1.4}$$

Assuming in here $x := ix$, we receive the identity

$$x^{2n} + x^{-2n} = \sum_{k=0}^{n} \frac{2n}{n+k} \binom{n+k}{n-k} (x - x^{-1})^{2k}, \tag{1.5}$$

equivalent to the identity (1.4).

One can also prove the following identity

$$x^{2n+1} - x^{-2n-1} = (x - x^{-1}) \sum_{k=0}^{n} (-1)^k \binom{n+k}{n-k} (x + x^{-1})^{2(n-k)}. \tag{1.6}$$

In paper [9] H. Prodinger used all of these identities, extremely effectively, for proving so called Melham's sums, for example

$$\sum_{k=0}^{n} F_{2k+1}^{2m+1} = \frac{1}{5^m} \sum_{j=0}^{m} \binom{2m+1}{j} \frac{F_{2(n+1)(2m+1-2j)}}{L_{2m+1-2j}},$$

$$\sum_{k=0}^{n} L_{2k}^{2m+1} = 4^m + \sum_{j=0}^{m} \binom{2m+1}{j} \frac{L_{(2n+1)(2m+1-2j)}}{L_{2m+1-2j}},$$

and so on.

In the current work we will apply identities (1.1)-(1.6) for proving two wide classes of identities. On the one hand there will be identities of trigonometric nature, including identities for Chebyshev polynomials of the first and second kinds (Sections 2, 3 and 4). On the other hand we will consider the identities for Fibonacci and Lucas polynomials (Section 5). For example some polynomial generalizations of Melham's sums for Fibonacci and Lucas numbers (see [9] and [7]) will be presented in here.

Remark 1.1. We note, that identities (1.1), (1.2), (1.4) and (1.5) are all special cases of the following identity, called the Kummer identity (see [6]):

$$x^n + y^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (x+y)^{n-2k} (x\,y)^k. \tag{1.7}$$

Among many other applications of the identity above we can note the following relations for Fibonacci and Lucas numbers:

$$5^n \big(F_{r-1}^{2n} + F_{r+1}^{2n}\big) \equiv 2\big(-5F_{r-1}F_{r+1}\big)^n \equiv 2(-1)^{(r-1)n} \ \big(\mathrm{mod}\, L_r\big), \tag{1.8}$$

since

$$5F_{r-1}F_{r+1} = L_r^2 + (-1)^r;$$

$$5^n \big(L_{r-1}^{2n} + L_{r+1}^{2n}\big) \equiv 2\big(-5L_{r-1}L_{r+1}\big)^n \equiv 2(-1)^{rn} 5^n \ \big(\mathrm{mod}\, 5F_r\big), \tag{1.9}$$

i.e.

$$5^{n-1} \big(L_{r-1}^{2n} + L_{r+1}^{2n}\big) \equiv 2(-1)^{rn} 5^{n-1} \ \big(\mathrm{mod}\, F_r\big), \tag{1.10}$$

since

$$5L_{r-1}L_{r+1} = 5L_{2r} - 15(-1)^r = 25F_r^2 - 5(-1)^r;$$

$$F_{r-2}^{2n} + F_{r+2}^{2n} \equiv 2\big(-F_{r-2}F_{r+2}\big)^n \equiv 2(-1)^{rn} \ \big(\mathrm{mod}\, F_r\big) \tag{1.11}$$

and

$$F_{r-2}^{2n} + F_{r+2}^{2n} \equiv 2\big(-F_{r-2}F_{r+2}\big)^n \ \big(\mathrm{mod}\, F_r\big), \tag{1.12}$$

since

$$F_{r-2} + F_{r+2} = 3F_r$$

and

$$F_{r-2}F_{r+2} = F_r^2 + (-1)^{r-1}.$$

Remark 1.2. In turn, equalities (1.3) and (1.6) are the special cases of the following version of Kummer identity:

$$x^n - y^n = (x-y) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-k-1}{k} (x\,y)^k (x+y)^{n-2k-1}. \tag{1.13}$$

Similarly like in the case of Kummer identity, from identity (1.13) one can also generate many of the congruence relations for Fibonacci and Lucas numbers, for example:

$$5^n \frac{F_{r+1}^{2n+1} - F_{r-1}^{2n+1}}{F_r} \equiv \big(-5F_{r-1}F_{r+1}\big)^n \equiv (-1)^{(r-1)n} \ \big(\mathrm{mod}\, L_r\big), \tag{1.14}$$

$$5^n \frac{L_{r+1}^{2n+1} - L_{r-1}^{2n+1}}{L_r} \equiv \big(-5L_{r-1}L_{r+1}\big)^n \equiv (-1)^{rn} 5^n \ \big(\mathrm{mod}\, 5F_r\big), \tag{1.15}$$

i.e.

$$5^{n-1}\frac{L_{r+1}^{2n+1}-L_{r-1}^{2n+1}}{L_r}\equiv(-1)^{rn}5^{n-1}\,\big(\operatorname{mod}F_r\big),\tag{1.16}$$

$$\frac{F_{r+2}^{2n+1}-F_{r-2}^{2n-1}}{L_r}\equiv\big(-F_{r-2}F_{r+2}\big)^n\equiv(-1)^{rn}\,\big(\operatorname{mod}F_r\big),\tag{1.17}$$

etc.

2. Trigonometric Identities

If we put $x := e^{ix}$ in (1.1), (1.2) and (1.3), then we get the formulae

$$\frac{(-1)^n}{2n+1}\cos((2n+1)x)=\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^k}{2k+1}\cos^{2k+1}(x),\tag{2.1}$$

$$\frac{1}{2n+1}\frac{\sin((2n+1)x)}{\sin x}=\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^k}{2k+1}\sin^{2k}(x),\tag{2.2}$$

and

$$\frac{\sin(2nx)}{\sin(2x)}=\sum_{k=0}^{n-1}\binom{n+k}{n-k-1}(-4)^k\sin^{2k}(x).\tag{2.3}$$

From (2.1), after differentiating, we obtain

$$(-1)^n\frac{\sin((2n+1)x)}{\sin(x)}=\sum_{k=0}^{n}\binom{n+k}{n-k}(-4)^k\cos^{2k}(x),\tag{2.4}$$

which implies, when x converges to 0:

$$(-1)^n(2n+1)=\sum_{k=0}^{n}\binom{n+k}{n-k}(-4)^k.\tag{2.5}$$

For the particular values of x in (2.4) we receive the following equalities:
for $x=\frac{\pi}{6}$:

$$2(-1)^n\sin((2n+1)\frac{\pi}{6})=\sum_{k=0}^{n}\binom{n+k}{n-k}(-3)^k,\tag{2.6}$$

for $x=\frac{\pi}{4}$:

$$\sqrt{2}(-1)^n\sin((2n+1)\frac{\pi}{4})=\sum_{k=0}^{n}\binom{n+k}{n-k}(-2)^k,\tag{2.7}$$

for $x = \frac{\pi}{3}$:

$$\frac{2}{\sqrt{3}}(-1)^n \sin((2n+1)\frac{\pi}{3}) = \sum_{k=0}^{n} \binom{n+k}{n-k}(-1)^k. \tag{2.8}$$

Subsequently, from (2.1) we obtain the following identities:
for $x = 0$:

$$\frac{(-1)^n}{2n+1} = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-4)^k}{2k+1}, \tag{2.9}$$

for $x = \frac{\pi}{6}$:

$$\frac{2}{\sqrt{3}} \cdot \frac{(-1)^n}{2n+1} \cos((2n+1)\frac{\pi}{6}) = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-3)^k}{2k+1}, \tag{2.10}$$

for $x = \frac{\pi}{4}$:

$$\sqrt{2}\frac{(-1)^n}{2n+1} \cos((2n+1)\frac{\pi}{4}) = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-2)^k}{2k+1}, \tag{2.11}$$

for $x = \frac{\pi}{3}$:

$$2\frac{(-1)^n}{2n+1} \cos((2n+1)\frac{\pi}{3}) = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-1)^k}{2k+1}. \tag{2.12}$$

Next, by differentiating identity (2.2) we get

$$\frac{\cos((2n+1)x)}{\cos(x)} = \sum_{k=0}^{n} \binom{n+k}{n-k} (-4)^k \sin^{2k}(x). \tag{2.13}$$

Hence and from (2.4) we obtain the relations

$$\frac{\cos((2n+1)x)}{\cos(x)} + (-1)^n \frac{\sin((2n+1)x)}{\sin(x)} = \tag{2.14}$$

$$= \sum_{k=0}^{n} \binom{n+k}{n-k} (-4)^k \big(\cos^{2k}(x) + \sin^{2k}(x)\big),$$

$$\cos\left((2n+1)\frac{\pi}{3}\right) = (-1)^n \sin\left((2n+1)\frac{\pi}{6}\right) \tag{2.15}$$

and

$$\cos\left((2n+1)\frac{\pi}{6}\right) = (-1)^n \sin\left((2n+1)\frac{\pi}{3}\right). \tag{2.16}$$

Finally, from (2.3) we receive the following identities for the particular values of x:

for $x = \frac{\pi}{6}$:

$$\frac{2}{\sqrt{3}}\sin(n\frac{\pi}{3}) = \sum_{k=0}^{n-1}\binom{n+k}{n-k-1}(-1)^k, \tag{2.17}$$

for $x = \frac{\pi}{4}$:

$$\sin(n\frac{\pi}{2}) = \sum_{k=0}^{n-1}\binom{n+k}{n-k-1}(-2)^k, \tag{2.18}$$

for $x = \frac{\pi}{3}$:

$$\frac{2}{\sqrt{3}}\sin(2n\frac{\pi}{3}) = \sum_{k=0}^{n-1}\binom{n+k}{n-k-1}(-3)^k \tag{2.19}$$

and for x converging to $\frac{\pi}{2}$:

$$(-1)^{n-1}n = \sum_{k=0}^{n-1}\binom{n+k}{n-k-1}(-4)^k. \tag{2.20}$$

Moreover let us notice, that for every $k \in \mathbb{N}$ the following development of function $\cos^k(x)$ to the power series holds:

$$\cos^k(x) = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots\right)^k = \tag{2.21}$$
$$= 1 - \frac{k}{2}x^2 + \frac{1}{24}k(3k-2)x^4 + \left(-\frac{k}{720} - \frac{1}{8}\binom{k}{3} - \frac{1}{48}k(k-1)\right)x^6 + \ldots.$$

Hence and from (2.1), after comparing the coefficients occuring by the corresponding powers of x, the infinite sequence of identities can be generated. It will be presented below three of these identities, corresponding to the powers x^2, x^4 and x^6, respectively. Thus we have

$$(-1)^n(2n+1) = \sum_{k=0}^{n}\binom{n+k}{n-k}(-4)^k, \tag{2.22}$$

$$(-1)^n(2n+1)^3 = \sum_{k=0}^{n}\binom{n+k}{n-k}(-4)^k(6k+1), \tag{2.23}$$

$$(-1)^n(2n+1)^5 = \sum_{k=0}^{n}\binom{n+k}{n-k}(-4)^k k(1+15(k-1)^2). \tag{2.24}$$

Besides, from the known formula of the form (see [4] and [11]):

$$4^n\cos^{2n+1}(x) = \sum_{k=0}^{n}\binom{2n+1}{n-k}\cos((2k+1)x) \tag{2.25}$$

one can imply, according to (2.1), the relations written below

$$\begin{bmatrix} \cos x \\ 4\cos^3 x \\ 4^2\cos^5 x \\ 4^3\cos^7 x \\ 4^4\cos^9 x \\ \dots \\ 4^n \cos^{2n+1} x \\ \dots \end{bmatrix} \tag{2.26}$$

$$= \begin{bmatrix} 1 & 0 & \dots \\ 3 & 1 & 0 & \dots \\ 10 & 5 & 1 & 0 & \dots \\ 35 & 21 & 7 & 1 & 0 & \dots \\ 126 & 84 & 36 & 9 & 1 & 0 & \dots \\ \dots \\ \binom{2n+1}{n} & \binom{2n+1}{n-1} & \binom{2n+1}{n-2} & \dots & \binom{2n+1}{1} & 1 & 0 & \dots \\ \dots \end{bmatrix} \begin{bmatrix} \cos x \\ \cos 3x \\ \cos 5x \\ \cos 7x \\ \cos 9x \\ \dots \\ \cos(2n+1)x \\ \dots \end{bmatrix}$$

and

$$\begin{bmatrix} \cos x \\ \cos 3x \\ \cos 5x \\ \cos 7x \\ \cos 9x \\ \dots \\ \cos(2n+1)x \\ \dots \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots \\ -3 & 4 & 0 & \dots \\ 5 & -20 & 16 & 0 & \dots \\ -7 & 56 & -112 & 64 & 0 & \dots \\ 9 & -120 & \dots \\ \dots \end{bmatrix} \begin{bmatrix} \cos x \\ \cos^3 x \\ \cos^5 x \\ \cos^7 x \\ \cos^9 x \\ \dots \\ \cos^{2n+1} x \\ \dots \end{bmatrix}$$

The nth row of the infinite matrix above has the form

$$(-1)^n(2n+1), \ \binom{n+1}{2}\frac{(-1)^{n-1}4}{3}(2n+1), \ \binom{n+2}{4}\frac{(-1)^{n-2}4^2}{5}(2n+1),$$
$$\dots, \ -4^{n-1}(2n+1), \ 4^n, \ 0, \ \dots .$$

Let us point the fact, that every element in this row is an integer number. Indeed, it should be noticed, that for every $k \in \mathbb{N}$, $1 \le k \le n$, the number

$$\frac{(n+k)(n+k-1)\dots(n-k+1)(2n+1)}{(2k+1)!}$$

is a positive integer. If the number $2k+1$ would not be a divisor of $(n+k)(n+k-1)\dots(n-k+1)$, then the remainder of dividing the number $n-k+1$ by $2k+1$ would have to equal to 1. In consequence, the remainders of dividing the numbers n and $n+1$ by $2k+1$ are equal to k and $k+1$, respectively, which means, that $2n+1$ is divisible by $2k+1$.

Now let us substitute in formulas (1.4), (1.5) and (1.6) the expression e^{ix} for the element x. We receive the following relations

$$\frac{(-1)^n}{n}\cos(2nx) = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^k}{n+k}\cos^{2k}(x), \tag{2.27}$$

$$\frac{1}{n}\cos(2nx) = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^k}{n+k}\sin^{2k}(x) \tag{2.28}$$

and

$$\frac{\sin((2n+1)x)}{\sin(x)} = \sum_{k=0}^{n}\binom{n+k}{n-k}(-4)^k\cos^{2(n-k)}(x). \tag{2.29}$$

From (2.27), after differentiating, we get

$$(-1)^n\frac{\sin(2nx)}{\sin(x)} = \sum_{k=1}^{n}\binom{n+k}{n-k}\frac{k(-4)^k}{n+k}\cos^{2k-1}(x), \tag{2.30}$$

hence, when x converges to 0, we receive

$$(-1)^n 2n = \sum_{k=1}^{n}\binom{n+k}{n-k}\frac{k(-4)^k}{n+k} \tag{2.31}$$

and for particular values of x we calculate:
for $x = \frac{\pi}{6}$:

$$(-1)^n\sqrt{3}\sin\left(n\frac{\pi}{3}\right) = \sum_{k=1}^{n}\binom{n+k}{n-k}\frac{k(-3)^k}{n+k}, \tag{2.32}$$

for $x = \frac{\pi}{4}$:

$$(-1)^n\sin\left(n\frac{\pi}{2}\right) = \sum_{k=1}^{n}\binom{n+k}{n-k}\frac{k(-2)^k}{n+k}, \tag{2.33}$$

for $x = \frac{\pi}{3}$:

$$(-1)^n\sin\left(n\frac{2\pi}{3}\right) = \sqrt{3}\sum_{k=1}^{n}\binom{n+k}{n-k}\frac{k(-1)^k}{n+k}. \tag{2.34}$$

Whereas, by using identity (2.27) we obtain, respectively:
for $x = 0$:

$$\frac{(-1)^n}{n} = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^k}{n+k}, \tag{2.35}$$

for $x = \frac{\pi}{6}$:

$$\frac{(-1)^n}{n}\cos\left(n\frac{\pi}{3}\right) = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-3)^k}{n+k}, \tag{2.36}$$

for $x = \frac{\pi}{4}$:

$$\frac{(-1)^n}{n}\cos\left(n\frac{\pi}{2}\right) = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-2)^k}{n+k}, \tag{2.37}$$

for $x = \frac{\pi}{3}$:

$$\frac{(-1)^n}{n}\cos\left(n\frac{2\pi}{3}\right) = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{n+k}. \tag{2.38}$$

3. Identities for Chebyshev Polynomials

Equalities (2.1) and (2.27) imply also the following algebraic formulas for Chebyshev polynomials of the first kind $T_n(x)$ (see [8] and [10]):

$$\frac{(-1)^n}{2n+1}T_{2n+1}(x) = \frac{1}{2}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{2k+1}(2x)^{2k+1} \tag{3.1}$$

and

$$\frac{(-1)^n}{2n}T_{2n}(x) = \frac{1}{2}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{n+k}(2x)^{2k}. \tag{3.2}$$

Moreover, by using equalities (2.3), (2.4) and (2.30) we receive the algebraic formulas for Chebyshev polynomials of the second kind $U_n(x)$ (see [8] and [10]), written below

$$U_{n-1}(2x^2-1) = \sum_{k=0}^{n-1}\binom{n+k}{n-k-1}(-4)^k(1-x^2)^k, \tag{3.3}$$

$$U_{n-1}(2x^2-1) = \frac{1}{2}(-1)^n\sum_{k=1}^{n}\binom{n+k}{n-k}\frac{(-4)^k k}{n+k}x^{2(k-1)} \tag{3.4}$$

and

$$(-1)^n U_{2n}(x) = \sum_{k=0}^{n}\binom{n+k}{n-k}(-4x^2)^k. \tag{3.5}$$

4. Tangent-Type Identities

From identity (1.1), by using some proper substitutions, we can obtain the following relations (see also [12] and [11]):
for $x \mapsto \tan x$:

$$\frac{(-1)^n}{2n+1}\left(\cot^{2n+1}(x)+\tan^{2n+1}(x)\right)=2\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^k}{2k+1}\csc^{2k+1}(2x), \quad (4.1)$$

for $x \mapsto \tanh x$:

$$\frac{(-1)^n}{2n+1}\left(\coth^{2n+1}(x)+\tanh^{2n+1}(x)\right)=2\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^k}{2k+1}\coth^{2k+1}(2x). \quad (4.2)$$

Analogically, from identity (1.4) we have:
for $x \mapsto \tan x$:

$$\frac{(-1)^n}{2n}\left(\cot^{2n}(x)+\tan^{2n}(x)\right)=\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^k}{n+k}\csc^{2k}(2x), \quad (4.3)$$

for $x \mapsto \tanh x$:

$$\frac{(-1)^n}{2n}\left(\coth^{2n}(x)+\tanh^{2n}(x)\right)=2\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^k}{n+k}\coth^{2k+1}(2x). \quad (4.4)$$

Furthermore, from identity (1.2) we get:
for $x \mapsto \cot x$:

$$\frac{2}{2n+1}\left(\cot^{2n+1}(x)-\tan^{2n+1}(x)\right)=\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{4^k}{2k+1}\cot^{2k+1}(2x). \quad (4.5)$$

5. Applications to Fibonacci and Lucas Polynomials

S. Falcon and A. Plaza in [1] and [2] have defined the, so called, $k-$Fibonacci numbers $F_{x,n}$, for $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$, which correspond to the known Fibonacci polynomials $F_n(x)$ [5]. The definition has the form

$$F_{x,n}=F_n(x)=xF_n(x)+F_{n-1}(x), \qquad n \in \mathbb{N} \quad (5.1)$$

and

$$F_{x,0}=F_n(0)=1, \qquad F_{x,1}=F_1(x)=x.$$

We note, that the following Binet's formula holds (see [1] and [2]):

$$F_{x,n}=F_n(x)=\frac{\varphi_x^n-(-\varphi_x)^{-n}}{\varphi_x-(-\varphi_x)}=\frac{\varphi_x^n-(-\varphi_x)^{-n}}{\sqrt{x^2+4}}, \quad (5.2)$$

where

$$\varphi_x := \frac{1}{2}\big(x + \sqrt{x^2+4}\big),$$

which implies

$$-\varphi_x^{-1} = \frac{1}{2}\big(x - \sqrt{x^2+4}\big).$$

Functions of the form

$$L_n(x) := \varphi_x^n + (-\varphi_x)^{-n}, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N} \tag{5.3}$$

are called the Lucas polynomials.

Substituting φ_x for x in the formulae (1.1)-(1.6) we can derive the following polynomial representations of the Fibonacci and Lucas functions:

From identity (1.2):

$$L_{x,2n+1} = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{2n+1}{2k+1} x^{2k+1}, \tag{5.4}$$

from identity (1.5):

$$L_{x,2n} = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{2n}{n+k} x^{2k}, \tag{5.5}$$

from identity (1.3):

$$F_{x,2n} = \sum_{k=0}^{n} \binom{n+k}{n-k-1} x^{2k+1}, \tag{5.6}$$

from identity (1.1):

$$\frac{(-1)^n}{2n+1} F_{x,2n+1} = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-1)^k}{2k+1} (x^2+4)^k \tag{5.7}$$

and finally from identity (1.6):

$$L_{x,2n+1} = x \sum_{k=0}^{n} (-1)^k \binom{n+k}{n-k} (x^2+4)^{n-k}. \tag{5.8}$$

Moreover, from equalities (1.1-1.5) one can obtain the reduction formulae of the form (some of them are already known, but only for the case of $x = 1$, see [9]):

$$\frac{(-1)^n}{2n+1} F_{x,(2m+1)(2n+1)} = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-1)^k}{2k+1} (x^2+4)^k F_{x,2m+1}^{2k+1}, \tag{5.9}$$

$$\frac{(-1)^n}{2n+1} L_{x,2m(2n+1)} = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{(-1)^k}{2k+1} L_{x,2m}^{2k+1}, \tag{5.10}$$

$$\frac{1}{2n+1}F_{x,2m(2n+1)} = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{1}{2k+1}(x^2+4)^k F_{x,2m}^{2k+1}, \tag{5.11}$$

$$\frac{1}{2n+1}L_{x,(2m+1)(2n+1)} = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{1}{2k+1}L_{x,2m+1}^{2k+1}, \tag{5.12}$$

$$F_{x,(2m+1)2n} = F_{x,2m+1}\sum_{k=0}^{n-1}\binom{n+k}{n-k-1}L_{x,2m+1}^{2k+1}, \tag{5.13}$$

$$F_{x,4mn} = L_{x,2m}\sum_{k=0}^{n-1}\binom{n+k}{n-k-1}(x^2+4)^k F_{x,2m}^{2k+1}, \tag{5.14}$$

$$\frac{(-1)^n}{2n}L_{x,4mn} = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{n+k}L_{x,2m}^{2k}, \tag{5.15}$$

$$\frac{(-1)^n}{2n}L_{x,(2m+1)2n} = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{n+k}(x^2+4)^k F_{x,2m+1}^{2k}. \tag{5.16}$$

For $x := i\,\varphi_x^{2m+1}$ we have

$$x^{-1} = i\left(\frac{x-\sqrt{x^2+4}}{2}\right)^{2m+1},$$

which implies, from identity (1.4):

$$\frac{1}{2n}L_{x,(2m+1)2n} = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{1}{n+k}L_{x,2m+1}^{2k}. \tag{5.17}$$

On the other hand, for $x := i\,\varphi_x^{2m}$ we get

$$x^{-1} = -i\left(\frac{x-\sqrt{x^2+4}}{2}\right)^{2m},$$

which follows, from identity (1.3):

$$(-1)^n F_{x,4mn} = F_{x,2m}\sum_{k=0}^{n-1}\binom{n+k}{n-k-1}(-1)^{k+1}L_{x,2m}^{2k+1} \tag{5.18}$$

and also from identity (1.4):

$$\frac{1}{2n}L_{x,4mn} = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-1)^k}{n+k}(x^2+4)^k F_{x,2m}^{2k}. \tag{5.19}$$

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