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POSITIVE ADDITIVE REPRESENTATIONS

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Abstract: Axioms are presented which guarantee the positive additive numerical representability of an algebraic structure. Uniqueness of the representation up to a positive scalar multiple is shown as well. With insights derived from the literature on partial groupoids, an effort is made to limit restrictions on the domain of the concatenation operation, thus increasing the potential applicability of the system. The structure is compared to other similar measurement structures. An economic illustration in utility theory is presented.

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1. Introduction

Significant contributions to measurement theory have been made by Krantz, Luce, Suppes, and Tversky in their book, *Foundations of Measurement*, Volume I (1971). Given an empirical system, defined by a set of objects and an ordering over those objects, a measurement theorist might ask whether or not a real number assignment to the objects can be made which, along with the usual ordering on the real numbers, accurately reflects the ordering of the objects; equivalently, he/she might ask under what circumstances a utility function exists. Further, a measurement theorist might add to the system a concatenation (or binary) operation defined for some pairs of objects and then ask whether or not an image of the more complex system exists in the additive real numbers

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and even, more specifically, in the positive additive real numbers. Successful solutions have useful applications in economics and the social sciences, including measurement of cardinal utility.

The more complex system has been studied by Krantz et al [4]. At the heart of their book is a set of axioms describing the more complex system. This structure is the basis of representation results in the areas of expected utility and conjoint measurement, among others. Their axioms describe many types of empirical systems. However, their axiom system is not applicable to other empirical phenomena because at least a couple of its axioms place stringent requirements on the domain of the concatenation operation.

While there has been a volume of literature generated by the appearance of the text and its two subsequent volumes, there is still room to loosen restrictions on the domain of the concatenation operation. Here two axiom systems are presented to widen the applicability of this more complex system. These results rely on insights developed in the more recent partial groupoid literature. Incorporating approaches from this literature leads to axioms which are radically different in appearance from the axioms in previous measurement systems.

In the next section, we present preliminary definitions and notation. Then we present the main result and a corollary, introducing new measurement structures. We relate these results to comparable existing measurement results. A section follows indicating the results' usefulness in economics. Finally, we make concluding remarks.

2. Notation and Definitions

In this section, we present some definitions and axioms.

Definition 1. $\mathcal{A} = (A, \succsim, B, \circ)$ is an *ordered partial groupoid* if:

1. $(A, B, \circ)$ is a partial groupoid, i.e., A is a nonempty set, $B \subseteq A \times A$ a nonempty set, $\circ$ a function (or concatenation operation) from B into A, and
2. $\succsim$ is a simple order, i.e., $\succsim$ is:
   (a) complete, i.e., for all $a, b \in A$, $a \succsim b$ or $b \succsim a$.
   (b) transitive, i.e., for $a, b, c \in A$, if $a \succsim b$ and $b \succsim c$, then $a \succsim c$.
   (c) antisymmetric, i.e., for $a, b \in A$, if $a \succsim b$ and $b \succsim a$, then $a = b$.

We assume throughout that $\mathcal{A} = (A, \succsim, B, \circ)$ is an ordered partial groupoid.

Definition 2. Let $\mathcal{A} = (A, \succsim, B, \circ)$ be an ordered partial groupoid. If there exists a function $\varphi : A \to \mathbf{R}$ such that:

1. for every $a, b \in A$, $a \succsim b$ if and only if $\varphi(a) \geq \varphi(b)$, and
2. for every $(a, b) \in B$, $\varphi(a \circ b) = \varphi(a) + \varphi(b)$,

then $\mathcal{A}$ has an *additive representation.* The function φ is *unique up to a positive scalar multiple* if when $\varphi' : A \to \mathbf{R}$ is another additive representation of $\mathcal{A}$, then $\varphi'(a) = \alpha\varphi(a)$ for every $a \in A$, where $\alpha > 0$. The function φ is a *positive representation* if $\varphi(a) > 0$ for every $a \in A$.

Note that if the function φ is an additive representation, then so is $\varphi' = \alpha\varphi$, where $\alpha > 0$.

We determine axioms under which $\mathcal{A}$ has a positive additive representation, i.e., we investigate the embeddability of $\mathcal{A}$ in the positive additive real numbers. In addition, we seek a representation which is a ratio scale, i.e., one which is unique up to a positive scalar multiple.

We reserve the letters i, j, k, ℓ, m, n to denote positive integers only. The notation $\pi(a_1 \dots a_n)$ (with π often subscripted) represents a product formed by $a_1, \dots, a_n \in A$, in the given order, for a particular placement of parentheses, if it is defined. If $n = 1$, then $\pi(a_1) = a_1$.

The associativity axiom which follows, involving products of n terms, is necessarily satisfied by an $\mathcal{A}$ with an additive representation. While axioms involving n terms do not appear in the measurement literature, they do appear in the partial groupoid literature. This associativity axiom is a special case of an axiom that appears in Gensemer [1], Gensemer [2], Gensemer and Weinert [3], Ljapin [5], and Ljapin and Evseev [6]. Notice that it puts no restrictions on the domain of the concatenation operation other than what is necessary for additive representability of the structure.

Axiom 3. (A, Associativity) If $\pi_1(\underbrace{a \dots a}_{n \text{ times}})$ and $\pi_2(\underbrace{a \dots a}_{n \text{ times}})$ are defined in $(A, B, \circ)$, then $\pi_1(\underbrace{a \dots a}_{n \text{ times}}) = \pi_2(\underbrace{a \dots a}_{n \text{ times}})$.

The following axiom is necessarily satisfied by an $\mathcal{A}$ with a positive additive representation. We use $a \succ b$ to denote that $a \succsim b$ and not $b \succsim a$.

Axiom 4. (P, Positivity) If $\pi_1(\underbrace{a \dots a}_{m \text{ times}})$ and $\pi_2(\underbrace{a \dots a}_{n \text{ times}})$ are defined in

$(A, B, \circ)$, where $m > n$, then $\pi_1(\underbrace{a \ldots a}_{m \text{ times}}) \succ \pi_2(\underbrace{a \ldots a}_{n \text{ times}})$.

The commeasurability axiom which follows places restrictions on the domain of the concatenation operation.

Axiom 5. (C, Commeasurability) If $a, b \in A$, then there exist $c \in A$, m, and n such that $\pi_1(\underbrace{cc \ldots c}_{m \text{ times}}) = a$ and $\pi_2(\underbrace{cc \ldots c}_{n \text{ times}}) = b$.

In some situations, we make the assumption that there exists $a' \in A$ such that for every $a \in A$, $a \succsim a'$. In this case, we call a' the *minimal element* of A. We reserve the notation a' for such an element of A, if it exists.

Axiom 6. (M, Minimal Element) $(A, \succsim)$ contains a minimal element, a'.

The next two axioms will be assumed for a structure with a minimal element. The following axiom places some restriction on the domain of the concatenation operation.

Axiom 7. (D, Decomposition) If $a \in A$ and $a \neq a'$, then there exist $b, c \in A$, where $(b, c) \in B$, such that $a = b \circ c$.

Before introducing an Archimedean axiom, which must be satisfied by an ordered partial groupoid which has a minimal element and a positive additive representation, we present some preliminary notation. For $a \in A$, define

$$N(a) = \{n : \ a \succsim \pi_j (a_1 \ldots a_n), \text{ for some } \pi_j, \text{ where } a_i \in A, \ i = 1, \ldots, n\}.$$

Axiom 8. (Ar, Archimedean) For all $a \in A$, the set $N(a)$ is finite.

3. Results

We present a theorem which lists axioms on an ordered partial groupoid guaranteeing that it has a positive additive representation which is unique up to a positive scalar multiple. Then we present a corollary. First, we provide two lemmas.

Lemma 9. *Assume that $\mathcal{A}$ has an additive representation φ and that it satisfies Axioms (A), (P), and (C). Then the additive representation is positive.*

Proof. We show that for any $a \in A$, $\varphi(a) > 0$. Given $a \in A$ there exists $b \in A$ such that $a \neq b$. This follows since $B \neq \emptyset$. If a were the only element in A, then $a \circ a = a$, a violation of Axiom (P). By Axiom (C), for a and b we find that there exists $c \in A$ such that

$$\pi_1 \, (\underbrace{c \ldots c}_{m \text{ times}}) = a, \;\; \pi_2 \, (\underbrace{c \ldots c}_{n \text{ times}}) = b, \tag{1}$$

where $m \neq n$ by Axiom (A) since $a \neq b$. By (1) and since φ is an additive representation, we find that

$$\varphi(\underbrace{c \ldots c}_{m \text{ times}}) = m\varphi(c) = \varphi(a), \;\; \varphi(\underbrace{c \ldots c}_{n \text{ times}}) = n\varphi(c) = \varphi(b). \tag{2}$$

Given these equalities, if $\varphi(c) \leq 0$ and $m > n$ $(n > m)$, we find that $\varphi(a) = m\varphi(c) \leq (\geq) n\varphi(c) = \varphi(b)$, from which it follows that $b \succsim a$ $(a \succsim b)$. But by Axiom (P) and since $m > n$ $(n > m)$, we find that $a \succ b$ $(b \succ a)$, a contradiction. Therefore, $\varphi(c) > 0$; by this and (2), $\varphi(a) > 0$. □

Lemma 10. *Assume that $\mathcal{A}$ satisfies Axiom (C). If*

$$\pi_1(\underbrace{\hat{a} \ldots \hat{a}}_{n \text{ times}}) = \bar{a}, \tag{3}$$

$$\pi_2(\underbrace{\hat{a} \ldots \hat{a}}_{m \text{ times}}) = a, \tag{4}$$

$$\pi_3(\underbrace{\hat{b} \ldots \hat{b}}_{n' \text{ times}}) = \bar{a}, \tag{5}$$

and

$$\pi_4 \, (\underbrace{\hat{b} \ldots \hat{b}}_{m' \text{ times}}) = b, \tag{6}$$

where $a, \bar{a}, \hat{a}, b, \hat{b} \in A$, then there exists $c \in A$ such that

$$\pi_5(\underbrace{c \ldots c}_{kn \text{ times}}) = \bar{a} = \pi_6(\underbrace{c \ldots c}_{\ell n' \text{ times}}), \tag{7}$$

$$\pi_7(\underbrace{c \ldots c}_{km \text{ times}}) = a, \tag{8}$$

and

$$\pi_8 \, (\underbrace{c \ldots c}_{\ell m' \text{ times}}) = b, \tag{9}$$

where k and ℓ are such that

$$\pi_9(\underbrace{c \ldots c}_{k \text{ times}}) = \hat{a} \tag{10}$$

and

$$\pi_{10}(\underbrace{c \ldots c}_{\ell \text{ times}}) = \hat{b}. \tag{11}$$

Proof. From Axiom (C), (10) and (11) follow. From (3), (5), (10), and (11), (7) follows. From (4) and (10), (8) follows. From (6) and (11), (9) follows. □

Theorem 11. *If $\mathcal{A}$ satisfies Axioms (A), (P), and (C), then $\mathcal{A}$ has a positive additive representation which is unique up to a positive scalar multiple.*

Proof. Take any $\bar{a} \in A$ as an "anchor" on which to base the additive representation $\varphi_{\bar{a}}$. Let $a \in A$ be arbitrary. By Axiom (C), there exists $\hat{a} \in A$ such that (3) and (4) hold. Define $\varphi_{\bar{a}}(a) = \frac{m}{n}$. Notice that $\varphi_{\bar{a}}(a) > 0$.

We note for use throughout that given $a, b \in A$, by the way in which $\varphi_{\bar{a}}$ is defined, there exist m, n, m', n' such that $\varphi_{\bar{a}}(a) = \frac{m}{n}$, $\varphi_{\bar{a}}(b) = \frac{m'}{n'}$, and (3)-(6) hold. By Axiom (C) and Lemma 10, it follows that (7)-(11) hold. Given (7) and (P), it follows that

$$kn = \ell n'. \tag{12}$$

Now we show that $a \succsim b$ if and only if $\varphi_{\bar{a}}(a) = \frac{m}{n} \geq \frac{m'}{n'} = \varphi_{\bar{a}}(b)$; this shows that $\varphi_{\bar{a}}$ satisfies the first part of the definition of additive representation and it shows that $\varphi_{\bar{a}}$ is a function. Using (8), (9), (A), and (P), we find that $a \succsim b$ if and only if

$$km \geq \ell m'. \tag{13}$$

Given (12), (13) holds if and only if

$$\varphi_{\bar{a}}(a) = \frac{m}{n} = \frac{km}{kn} \geq \frac{\ell m'}{\ell n'} = \frac{m'}{n'} = \varphi_{\bar{a}}(b).$$

Now we show that $\varphi_{\bar{a}}$ is additive. By (8) and (9),

$$\pi_{11} \quad (\underbrace{c \ldots c}_{km+\ell m' \text{ times}}) \quad = a \circ b. \tag{14}$$

Use (7), (12), and (14) to find that

$$\varphi_{\bar{a}}(a \circ b) = \frac{km + \ell m'}{kn} = \frac{km}{kn} + \frac{\ell m'}{\ell n'} = \frac{m}{n} + \frac{m'}{n'} = \varphi_{\bar{a}}(a) + \varphi_{\bar{a}}(b).$$

Finally, we show that $\varphi_{\bar{a}}$ is unique up to a positive scalar multiple. Let $a \in A$ be arbitrary, and let φ' be another additive representation. By (C), (3) and (4) hold. By (3) and the additivity of $\varphi_{\bar{a}}$ and φ', $\varphi_{\bar{a}}(\bar{a}) = 1 = n\varphi_{\bar{a}}(\hat{a})$ and $\varphi'(\bar{a}) = n\varphi'(\hat{a})$; it follows that

$$\varphi'(\bar{a}) = \frac{\varphi'(\bar{a})}{\varphi_{\bar{a}}(\bar{a})} = \frac{n\varphi'(\hat{a})}{n\varphi_{\bar{a}}(\hat{a})} = \frac{\varphi'(\hat{a})}{\varphi_{\bar{a}}(\hat{a})}. \tag{15}$$

By (4) and the additivity of $\varphi_{\bar{a}}$ and φ', $\varphi_{\bar{a}}(a) = m\varphi_{\bar{a}}(\hat{a})$ and $\varphi'(a) = m\varphi'(\hat{a})$;

it follows that

$$\frac{\varphi'(a)}{\varphi_{\bar{a}}(a)} = \frac{m\varphi'(\hat{a})}{m\varphi_{\bar{a}}(\hat{a})} = \frac{\varphi'(\hat{a})}{\varphi_{\bar{a}}(\hat{a})} \Rightarrow \varphi'(a) = \frac{\varphi'(\hat{a})}{\varphi_{\bar{a}}(\hat{a})} \cdot \varphi_{\bar{a}}(a). \tag{16}$$

By (15) and (16), we find that

$$\varphi'(a) = \varphi'(\bar{a}) \cdot \varphi_{\bar{a}}(a),$$

where $\varphi'(\bar{a}) > 0$ by Lemma 9. □

Finally, we present a corollary. Note that in the corollary, the minimal element (M) and Archimedean (Ar) axioms can be replaced by the assumption that A is finite. In practical situations, if the empirical system has a minimal element, the axioms in the corollary are fairly weak given that Axiom (D) is the only non-necessary condition for a positive additive representation and seemingly not a very strong axiom.

Corollary 12. *If $\mathcal{A}$ satisfies Axioms (A), (P), (M), (D), and (Ar), then $\mathcal{A}$ has a positive additive representation which is unique up to a positive scalar multiple.*

Proof. Let $a \in A$. We show that for finite n,

$$a = \pi(\underbrace{a' \dots a'}_{n \text{ times}}), \tag{17}$$

where a' is the minimal element. If $a = a'$, then we are done. Otherwise, $a \succ a'$. As a first step in a process, by Axiom (D), we find a_1 and a_2 such that $a = \pi_1(a_1 a_2)$, where $a_1 \succsim a'$ and $a_2 \succsim a'$. If equality holds for both, then we are done. Otherwise, strict inequality holds for at least one, and as a second step, by Axiom (D), we decompose both a_1 and a_2, if necessary, so that $a = \pi_2(a_{11}a_{12}a_{21}a_{22})$, where $a_{ij} \succsim a'$ (if $a_1 = a'$ or $a_2 = a'$, we have only three a_{ij}'s). We continue in this way through the m-th step, where we decompose $a_{ij\cdots k}$ as long as any $a_{ij\cdots k} \succ a'$. By Axiom (Ar), the process must terminate in a finite number of steps, when we should then find that (17) holds, where n is finite and a' is the minimal element. It follows that the structure satisfies Axiom (C) and the corollary follows from the theorem. □

4. Relationships with Other Measurement Structures

There are existing measurement systems which satisfy Axioms (A), (P), (M), (D), and (Ar). One example is the Abelian positive ordered quasi-group (APOQG) with a minimal element; its axioms are given in the Appendix (Moore [7]). That such a structure satisfies Axioms (A), (P), and (Ar) follows from the fact that an APOQG with a minimal element has a positive additive representation (Moore [7], Theorem 1, p. 7). The APOQG satisfies Axiom (D) given APOQG Axiom 7. In general, the APOQG without a minimal element does not have a positive additive representation (see Gensemer [1]).

Another measurement structure which satisfies Axioms (A), (P), (M), (D), and (Ar) is the Archimedean, regular, positive, ordered local semigroup (AOSG) with a minimal element. This follows from the fact that it is an APOQG with a minimal element (see Moore [7], Proposition 9, p. 24). Alternatively, one could argue that the structure satisfies Axioms (A), (P), and (Ar) because the AOSG has a positive additive representation. The AOSG satisfies Axiom (D) given that any non-minimal element can be expressed as a series of concatenations of the minimal element with itself; this is shown on p. 48 of Krantz et al [4], vol. I.

Examples of structures which satisfy Axioms (A), (P), (M), (D), and (Ar), but not the axioms defining the APOQG with a minimal element (and therefore, not the axioms defining the AOSG system with a minimal element), are easy to generate. Such an example is given in the next section.

The positive additive rational numbers with the usual ordering form an ordered partial groupoid which satisfies the axioms of the theorem, but which is not an APOQG with a minimal element. The structure is an AOSG.

An ordered partial groupoid without a minimal element which satisfies the axioms of the theorem, but which is not an AOSG is the positive additive rational numbers with the restriction that $(1,1) \notin B$. Removing $(1,1)$ from B implies the structure is not an AOSG since Axiom 2 of that system is not satisfied.

The positive additive real numbers with the usual ordering form an AOSG. This ordered partial groupoid does not satisfy the axioms of the Theorem, nor is it an APOQG with a minimal element.

5. Economic Application

We present an example where a cardinal utility function is derived from the results in this paper that cannot be derived from other existing results. Let there be a set of alternatives $A^* = \{s, t, v, w, x, y, z\}$ on which there is a complete and transitive order $\succsim^*$, where it is assumed that $z \succ^* y \succ^* x \succ^* w \succ^* v \succ^* t \succ^* s$.

Furthermore, assume that the individual has ranked some changes (holding one alternative rather than another alternative) from more preferred changes to less preferred changes; let $\succsim^{**}$ denote the order over the changes. Specifically, assume that the ranking is given in the left hand column of the following table, where the more preferred the change, the higher it is in the table. The entry (s,t) represents the individual obtaining the alternative t in place of s. Since the change (v,z) is at the same level as (s,t) in the table, the individual is indifferent between (s,t) and (v,z), i.e., $(s,t) \succsim^{**} (v,z)$ and $(v,z) \succsim^{**} (s,t)$. But the individual prefers the change (s,t) over (v,y). The table gives a structure which is like a positive difference structure, although the same axioms are not satisfied (see Krantz et al [4], vol. I, p. 147).

D	A	φ
$(s,t),(v,z)$	$a_6 = \{(s,t),(v,z)\}$	8
(v,y)	$a_5 = \{(v,y)\}$	7
(v,x)	$a_4 = \{(v,x)\}$	6
$(t,v),(v,w),(w,z)$	$a_3 = \{(t,v),(v,w),(w,z)\}$	4
$(w,x),(x,z)$	$a_2 = \{(w,x),(x,z)\}$	2
$(x,y),(y,z)$	$a_1 = \{(x,y),(y,z)\}$	1

From the given information, we construct an ordered partial groupoid, $\mathcal{A}$. The elements are given in the second column above, where $a_{i+1} \succ a_i$, $i = 1,\ldots,5$. The concatenation operation on $B \subseteq A \times A$ is defined in the following way: $a_i \circ a_j = a_k$ if there exist $(\hat{x},\tilde{x}) \in a_i$, $(\tilde{x},x') \in a_j$, and $(\hat{x},x') \in a_k$. For example, $a_4 \circ a_1 = a_5$ since $(v,x) \in a_4$, $(x,y) \in a_1$, and $(v,y) \in a_5$. The results of these concatenations are given in the following table.

	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_2					
a_2		a_3				
a_3		a_4	a_6			
a_4	a_5	a_6				
a_5	a_6					
a_6						

$\mathcal{A}$ is not an APOQG because APOQG Axiom 8 does not hold. Note that $a_6 = a_5 \circ a_1 = a_3 \circ a_3$, where $a_5 \succ a_3 \succsim a_3 \succ a_1$; however, there exists no b such that $a_5 = b \circ a_3$ and $a_3 = b \circ a_1$. Since the ordered partial groupoid is not an APOQG with a minimal element, it is not an AOSG with a minimal element (also, AOSG Axiom 2 is not satisfied). However, this example does have a positive additive representation φ which is unique up to a positive scalar multiple, given by the third column in the first table; its existence follows from Theorem 11. Axioms (A) and (P) hold, and furthermore, a_2, a_3, a_4, a_5, and a_6 can be obtained as a result of the following number of concatenations of a_1 with itself: $2, 4, 6, 7$, and 8, respectively. The existence of φ can also be inferred from the corollary.

Finally, from the numerical assignments associated with the changes, we can derive a cardinal utility function Φ for the given set of alternatives.

A^*	z	y	x	w	v	t	s
Φ	20	19	18	16	12	8	0

The function Φ is a cardinal utility function in that not only does the function Φ represent the ordering on the original set of alternatives, but the difference in evaluations under Φ reflects the ordering of changes from one alternative to another. More precisely, Φ has the property that $\tilde{x} \succsim^* \hat{x}$ if and only if $\Phi(\tilde{x}) \geq \Phi(\hat{x})$. In addition, the function Φ also has the property that for $(\hat{x}, \tilde{x}), (x', \bar{x}) \in D$, $(\hat{x}, \tilde{x}) \succsim^{**} (x', \bar{x})$ if and only if $\Phi(\tilde{x}) - \Phi(\hat{x}) \geq \Phi(\bar{x}) - \Phi(x')$. The function Φ is unique up to a positive affine transformation, i.e., if Φ' is another function having the two properties previously indicated, then $\Phi' = r\Phi + r'$, where $r > 0$ and r' are real numbers.

6. Conclusion

We have presented axiom systems which generalize existing measurement results in the case where the underlying algebraic structure has a minimal element. The first axiom system presented here could be used to generate other measurement structures, much like the way in which the second axiom system here was generated from the first system.

Insights from the area of partial groupoids have been used to derive the axiom systems. This new approach allows construction of axiom systems with less stringent requirements on the domain of the concatenation operation.

A direction for future research would include developing axioms defining a positive difference structure, which would have a positive additive representation derived from the results in this paper.

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Appendix

We present two axiom systems.

Provided that the indicated products are defined, let

$$a^n \equiv \underbrace{(\cdots((a \circ a) \circ a) \cdots) \circ a}_{n \text{ times}}.$$

The following axioms define the *Archimedean, regular, positive, ordered local semigroup* (AOSG) (Krantz et al [4], p. 44).

1. $\succsim$ is a simple order, i.e., it is a complete, transitive, and antisymmetric binary relation on A.
2. If $(a, b) \in B$, $a \succsim c$, $b \succsim d$, then $(c, d) \in B$.
3. If $(c, a) \in B$ and $a \succsim b$, then $c \circ a \succsim c \circ b$.
4. If $(a, c) \in B$ and $a \succsim b$, then $a \circ c \succsim b \circ c$.
5. $(a, b), (a \circ b, c) \in B$ if and only if $(b, c), (a, b \circ c) \in B$; and when both conditions hold, $(a \circ b) \circ c = a \circ (b \circ c)$.
6. If $(a, b) \in B$, then $a \circ b \succ b$.
7. If $a \succ b$, then there exists $c \in A$ such that $(b, c) \in B$ and $a \succsim b \circ c$.
8. For all $a, b \in A$ such that $b \succ a$, the set $\{n : n \in N_a \text{ and } b \succ a^n\}$ is finite, where N_a is the set of consecutive positive integers such that a^n is defined.

We introduce a definition before listing the axioms defining an Abelian positive ordered quasi-group (APOQG) (Moore [7], pp. 4, 6-7). Let M be the set of positive integers or a set of the form $\{1, 2, \ldots, m\}$ for some m, a positive integer.

Definition 13. $\{a_n\}_{n \in M} \subseteq A$ is a *minimum-step monotone sequence* if there exist sequences, $\{b_n\}_{n \in M}$ and $\{c_n\}_{n \in M}$, in A and $d \in A$ such that:

1. for every $n \in M$, $a_n = b_n \circ c_n$,
2. for every $n \in M$, $c_n \succsim d$, and either
3. for every $n \in M \backslash \{1\}$, $b_n \succsim a_{n-1}$, or
4. for every $n \in M$, $n + 1 \in M$ implies that $b_n \succsim a_{n+1}$.

The following axioms define the *Abelian positive ordered quasi-group.*

1. $\succsim$ is a simple order.
2. If $a = b \circ c$, then $a \succ b$ and $a \succ c$.
3. If $(a, b), (c, d) \in B$, and $(a, b) \geq (c, d)$, then $a \circ b \succ c \circ d$.
4. If $\{a_n\}_{n \in M} \subseteq A$ is a minimum step monotone sequence which is bounded above and below, then M is finite.
5. If $(b, c), (a, b \circ c), (a, b), (a \circ b, c) \in B$, then $a \circ (b \circ c) = (a \circ b) \circ c$.
6. If $(a, b) \in B$, then $(b, a) \in B$, and $a \circ b = b \circ a$
7. If $a, b \in A$ are such that $a \succ b$, then either:
   (a) there exists $(a', a'') \in B$ such that $a = a' \circ a''$ and $a' \succsim b$, or

(b) there exist $(c, c'), (d, d') \in B$ such that $(c, c') > (d, d')$, $a = c \circ c'$, and $b = d \circ d'$.

8. If $a \succ c \succsim d \succ b$ and $a \circ b = c \circ d$, then either:
   (a) there exists $\hat{a}$ such that $a = \hat{a} \circ d$ and $c = \hat{a} \circ b$, or
   (b) there exists $\tilde{a}$ such that $a = \tilde{a} \circ c$ and $d = \tilde{a} \circ b$.