International Journal of Pure and Applied Mathematics
Volume 64 No. 2 2010, 253-304

ON THE M/M/1 QUEUEING MODEL WITH COMPULSORY SERVER VACATIONS

Geni Gupur
College of Mathematics and Systems Science
Xinjiang University
Urumqi, 830046, P.R. CHINA
e-mails: genigupur@yahoo.cn, geni@xju.edu.cn

Abstract: By studying the spectrum of the operator corresponding to the M/M/1 queueing model with compulsory server vacations we discuss asymptotic behavior of the time-dependent solution of the model and asymptotic behavior of the time-dependent queueing length. First of all, through studying the resolvent set of the adjoint operator of the operator we obtain that all points on the imaginary axis except for zero belong to the resolvent set of the operator. Next, we prove that zero is an eigenvalue of the adjoint operator. In addition, we consider eigenvalue of the operator and get the explicit result when $\mathcal{M} = 1$. From the above results we deduce that the time-dependent solution of the model strongly converges to its steady-state solution and the time-dependent queueing length converges to a positive number when $\mathcal{M} = 1$.

AMS Subject Classification: 47A10, 47D03, 60K25
Key Words: M/M/1 queueing model with compulsory server vacations, resolvent set, eigenvalue

1. Introduction

Queueing models with server vacations have been investigated by many authors, see Doshi [1], Madan [9], Takagi [10] and Gupur et al [2, 3, 4, 5, 6, 8]. In 1992, Madan [9] considered the M/G/1 queueing system with compulsory server vacations. Firstly, by using supplementary variable technique he established the corresponding mathematical model, then he gave the Laplace transform

Received: August 2, 2010

of the probability generating function and obtained some steady-state results. Roughly speaking, he obtained the existence of the time-dependent solution of the model. In 2010, Lu and Gupur [8] have proved that the model has a unique positive time-dependent solution which satisfies probability condition. Until now, any other results about time-dependent solution of the model have not been found in the literatures. This paper is an effort on the subject. When the service rate is a constant, the M/G/1 queueing model with compulsory server vacations is called the M/M/1 queueing model with compulsory server vacations. In this paper, we study asymptotic behavior of the time-dependent solution of the M/M/1 queueing model with compulsory server vacations. First of all, we determine the expression of the adjoint operator of the operator corresponding to the model, next we study resolvent set of the adjoint operator and deduce the resolvent set of the operator. In addition, we prove that zero is an eigenvalue of the adjoint operator. Moreover, we discuss eigenvalue of the operator and verify that zero is an eigenvalue of the operator with geometric multiplicity one when $\mathcal{M} = 1$. Thus, in this case by Theorem 14 in Gupur et al [7] and the above results we obtain that the time-dependent solution of the model strongly converges to its steady-state solution. Finally, we prove that the time-dependent queueing length of the system converges to a positive number. We conclude this paper with some open problems.

According to Madan [9], the M/M/1 queueing model with compulsory server vacations can be described by the following equations.

$$\frac{dQ(t)}{dt} = -\lambda Q(t) + bV_0(t), \tag{1}$$

$$\frac{dV_0(t)}{dt} = -(\lambda + b)V_0(t) + \mu \int_0^\infty W_0(x,t)dx, \tag{2}$$

$$\frac{dV_n(t)}{dt} = -(\lambda + b)V_n(t) + \lambda V_{n-1}(t) + \mu \int_0^\infty W_n(x,t)dx, \quad n \geq 1 \tag{3}$$

$$\frac{\partial W_0(x,t)}{\partial t} + \frac{\partial W_0(x,t)}{\partial x} = -(\lambda + \mu)W_0(x,t), \tag{4}$$

$$\frac{\partial W_n(x,t)}{\partial t} + \frac{\partial W_n(x,t)}{\partial x} = -(\lambda + \mu)W_n(x,t) + \lambda W_{n-1}(x,t) \quad n \geq 1, \tag{5}$$

$$W_0(0,t) = \lambda Q(t) + b\sum_{r=1}^{\mathcal{M}} V_r(t), \tag{6}$$

$$W_n(0,t) = bV_{n+\mathcal{M}}(t), \quad n \geq 1, \tag{7}$$

$$Q(0) = 1,\ V_n(0) = 0, \quad n \geq 0; \quad W_n(x,0) = 0, \quad n \geq 0. \tag{8}$$

Here $(x,t) \in [0,\infty) \times [0,\infty)$; $W_n(x,t)dx$ $(n \geq 0)$ represents the probability that at time t the server is working and there are n customers in the queue with the elapsed service time of the undergoing service lying in $[x, x+dx)$; $V_n(t)$ $(n \geq 0)$ is the probability that at time t there are n customers in the queue and the server is on vacation; $Q(t)$ is the probability that at time t the server is idle but available; λ and $\frac{1}{b}$ represent the mean arrival rate of customers and mean vacation time of the server respectively. $\mathcal{M}$ is the maximum number of customers who can receive service at the same time, μ is the service rate of the server.

In this paper, we use the notations in Lu and Gupur [8]. For convenience, we introduce

$$\Gamma = \begin{pmatrix} \lambda & 0 & \overbrace{b & b & \cdots & b & b}^{\mathcal{M}} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Choose a state space

$$X = \left\{(V, W) \mid V \in l^1,\ W \in Y, \|(V, W)\| = \|V\|_{l^1} + \|W\|_Y\right\},$$

where

$$Y = \left\{ W \;\middle|\; \begin{array}{l} W \in L^1[0,\infty) \times L^1[0,\infty) \times \cdots, \\ \|W\| = \sum_{n=0}^{\infty} \|W_n\|_{L^1[0,\infty)} < \infty \end{array} \right\}.$$

It is obvious that X is a Banach space. In the following we introduce operators and their domains.

$$A\left(\begin{pmatrix} Q \\ V_0 \\ V_1 \\ V_2 \\ \vdots \end{pmatrix}, \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ \vdots \end{pmatrix}\right) = \left(\begin{pmatrix} -\lambda & 0 & 0 & 0 & \cdots \\ 0 & -(\lambda+b) & 0 & 0 & \cdots \\ 0 & 0 & -(\lambda+b) & 0 & \cdots \\ 0 & 0 & 0 & -(\lambda+b) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q \\ V_0 \\ V_1 \\ V_2 \\ \vdots \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} -\frac{d}{dx} & 0 & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ \vdots \end{pmatrix}\right),$$

$D(A) =$

$$\left\{(V,W)\in X \;\middle|\; \begin{array}{l} W_n(x)\ (n\geq 0) \text{ are absolutely continuous and} \\ W(0)=\Gamma V,\ \frac{dW_n}{dx}\in L^1[0,\infty),\ \sum_{n=0}^{\infty}\left\|\frac{dW_n}{dx}\right\|_{L^1[0,\infty)}<\infty \end{array}\right\};$$

$$U\left(\begin{pmatrix} Q\\ V_0\\ V_1\\ V_2\\ \vdots\end{pmatrix},\begin{pmatrix} W_0\\ W_1\\ W_2\\ W_3\\ \vdots\end{pmatrix}\right)=\left(\begin{pmatrix} 0 & b & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ 0 & \lambda & 0 & 0 & \cdots\\ 0 & 0 & \lambda & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix} Q\\ V_0\\ V_1\\ V_2\\ \vdots\end{pmatrix},\right.$$

$$\left.\begin{pmatrix} -(\lambda+\mu) & 0 & 0 & 0 & \cdots\\ \lambda & -(\lambda+\mu) & 0 & 0 & \cdots\\ 0 & \lambda & -(\lambda+\mu) & 0 & \cdots\\ 0 & 0 & \lambda & -(\lambda+\mu) & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix} W_0\\ W_1\\ W_2\\ W_3\\ \vdots\end{pmatrix}\right),$$

$$E\left(\begin{pmatrix} Q\\ V_0\\ V_1\\ V_2\\ \vdots\end{pmatrix},\begin{pmatrix} W_0\\ W_1\\ W_2\\ W_3\\ \vdots\end{pmatrix}\right)=\left(\begin{pmatrix} 0\\ \mu\int_0^\infty W_0(x)dx\\ \mu\int_0^\infty W_1(x)dx\\ \mu\int_0^\infty W_2(x)dx\\ \vdots\end{pmatrix},\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ \vdots\end{pmatrix}\right),\quad D(U)=D(E)=X,$$

then the above equations (1)-(8) can be rewritten as an abstract Cauchy problem which was given in Lu and Gupur [8]:

$$\begin{cases} \dfrac{d(V,W)(t)}{dt}=(A+U+E)(V,W)(t), \quad t\in(0,\infty), \\ (V,W)(0)=\left(\begin{pmatrix}1\\ 0\\ \vdots\end{pmatrix},\begin{pmatrix}0\\ 0\\ \vdots\end{pmatrix}\right). \end{cases} \tag{9}$$

Lu and Gupur [8] have proved the following result.

Theorem 1. *$A+U+E$ generates a positive contraction C_0-semigroup $T(t)$. The system (9) has a unique positive time-dependent solution $(V(t),W(x,t))=T(t)(V,W)(0)$ satisfying*

$$Q(t)+\sum_{n=0}^{\infty}V_n(t)+\sum_{n=0}^{\infty}\int_0^\infty W_n(x,t)dx=1, \quad t\in[0,\infty).$$

2. Asymptotic Behavior of the Time-Dependent Solution of the System (9)

It is easy to verify that X^*, the dual space of X, is

$$X^* = \left\{ (V^*, W^*) \,\middle|\, \begin{array}{l} V^* \in l^\infty,\ W^* \in Y^*, \\ (V^*, W^*) = \sup\{V^*, W^*\} \end{array} \right\},$$

$$l^\infty = \left\{ V^* \,\middle|\, \begin{array}{l} V^* = (Q^*, V_0^*, V_1^*, V_2^*, \cdots) \\ V^* = \sup\left\{ |Q^*|, \sup\limits_{n\ge 0} |V_n^*| \right\} < \infty \end{array} \right\},$$

$$Y^* = \left\{ W^* \,\middle|\, \begin{array}{l} W^* \in L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots \\ W^* = \sup\limits_{n\ge 0} \|W_n^*\|_{L^\infty[0,\infty)} < \infty \end{array} \right\}.$$

It is clear that X^* is a Banach space.

Lemma 1. *$(A+U+E)^*$, the adjoint operator of $A+U+E$, is as follows:*

$$(A+U+E)^*(V^*, W^*) = (H+J+S)(V^*, W^*), \quad (V^*, W^*) \in D(H),$$

where

$$H(V^*, W^*) = \left(\begin{pmatrix} -\lambda & 0 & 0 & 0 & \cdots \\ b & -(\lambda+b) & 0 & 0 & \cdots \\ 0 & 0 & -(\lambda+b) & 0 & \cdots \\ 0 & 0 & 0 & -(\lambda+b) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q^* \\ V_0^* \\ V_1^* \\ V_2^* \\ \vdots \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \frac{d}{dx} - (\lambda+\mu) & 0 & 0 & \cdots \\ 0 & \frac{d}{dx} - (\lambda+\mu) & 0 & \cdots \\ 0 & 0 & \frac{d}{dx} - (\lambda+\mu) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} W_0^*(x) \\ W_1^*(x) \\ W_2^*(x) \\ \vdots \end{pmatrix} \right),$$

$$D(H) = \left\{ (V^*, W^*) \in X^* \,\middle|\, \begin{array}{l} \frac{dW_n^*(x)}{dx}\ (n \ge 0) \text{ exists and} \\ \sup\limits_{n\ge 0} \left\| \frac{dW_n}{dx} \right\|_{L^\infty[0,\infty)} < \infty, \\ W_n^*(\infty) = \alpha,\ n \ge 0 \end{array} \right\}.$$

here α in $D(H)$ is a constant which is irrelevant to n.

$J(V^*, W^*) =$

$$\left(\begin{pmatrix} \lambda & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ b & 0 & 0 & 0 & \cdots \\ b & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b & 0 & 0 & 0 & \cdots \\ b & 0 & 0 & 0 & \cdots \\ 0 & b & 0 & 0 & \cdots \\ 0 & 0 & b & 0 & \cdots \\ 0 & 0 & 0 & b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} W_0^*(0) \\ W_1^*(0) \\ W_2^*(0) \\ W_3^*(0) \\ \vdots \\ W_{\mathcal{M}+1}^*(0) \\ W_{\mathcal{M}+2}^*(0) \\ W_{\mathcal{M}+3}^*(0) \\ W_{\mathcal{M}+4}^*(0) \\ W_{\mathcal{M}+5}^*(0) \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 & \mu & 0 & 0 & 0 & \cdots \\ 0 & 0 & \mu & 0 & 0 & \cdots \\ 0 & 0 & 0 & \mu & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mu & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q^* \\ V_0^* \\ V_1^* \\ V_2^* \\ \vdots \end{pmatrix}\right),$$

$$D(J) = X^*,$$

$$S(V^*, W^*) =$$

$$\left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & 0 & \cdots \\ 0 & 0 & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q^* \\ V_0^* \\ V_1^* \\ V_2^* \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 & \lambda & 0 & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & 0 & \cdots \\ 0 & 0 & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} W_0^*(x) \\ W_1^*(x) \\ W_2^*(x) \\ W_3^*(x) \\ \vdots \end{pmatrix}\right),$$

$$D(S) = X^*.$$

Proof. By the definition of adjoint operator, integrating by parts and the boundary conditions on $(V, W) \in D(A)$ we have, for $(V, W) \in D(A)$ and $(V^*, W^*) \in D(H)$

$$\langle (A + U + E)(V, W), (V^*, W^*)\rangle$$

$$= [-\lambda Q + bV_0]Q^* + \left[-(\lambda + b)V_0 + \mu \int_0^\infty W_0(x)dx\right] V_0^*$$

$$+ \sum_{n=1}^{\infty} \left[-(\lambda + b)V_n + \lambda V_{n-1} + \mu \int_0^\infty W_n(x)dx\right] V_n^*$$

$$+ \int_0^\infty \left[-\frac{dW_0(x)}{dx} - (\lambda + \mu)W_0(x)\right] W_0^*(x)dx$$

$$+ \sum_{n=1}^{\infty} \int_0^\infty \left[-\frac{dW_n(x)}{dx} - (\lambda + \mu)W_n(x) + \lambda W_{n-1}(x)\right] W_n^*(x)dx$$

$$= Q[-\lambda Q^*] + V_0[bQ^*] + \sum_{n=0}^{\infty} V_n[-(\lambda + b)V_n^*]$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\lambda V_{n-1}V_n^* + \sum_{n=0}^{\infty}\mu V_n^*\int_0^{\infty}W_n(x)dx \\
&-W_0(x)W_0^*(x)\big|_0^{\infty} + \int_0^{\infty}W_0(x)\frac{dW_0^*(x)}{dx}dx \\
&+\sum_{n=1}^{\infty}\left[-W_n(x)W_n^*(x)\big|_0^{\infty} + \int_0^{\infty}W_n(x)\frac{dW_n^*(x)}{dx}dx\right] \\
&+\sum_{n=0}^{\infty}\int_0^{\infty}W_n(x)[-(\lambda+\mu)W_n^*(x)]dx + \sum_{n=1}^{\infty}\int_0^{\infty}\lambda W_{n-1}(x)W_n^*(x)dx \\
=\ & Q[-\lambda Q^*] + V_0[bQ^*] + \sum_{n=0}^{\infty}V_n[-(\lambda+b)V_n^*] \\
&+\sum_{n=1}^{\infty}V_{n-1}[\lambda V_n^*] + \sum_{n=0}^{\infty}\int_0^{\infty}W_n(x)[\mu V_n^*]dx + W_0(0)W_0^*(0) + \sum_{n=1}^{\infty}W_n^*(0)W_n(0) \\
&+\sum_{n=0}^{\infty}\int_0^{\infty}W_n(x)\left[\frac{dW_n^*(x)}{dx} - (\lambda+\mu)W_n^*(x)\right]dx \\
&+\sum_{n=0}^{\infty}\int_0^{\infty}W_n(x)[\lambda W_{n+1}^*(x)]dx \\
=\ & Q[-\lambda Q^*] + V_0[bQ^*] + \sum_{n=0}^{\infty}V_n[-(\lambda+b)V_n^*] \\
&+\sum_{n=0}^{\infty}V_n[\lambda V_{n+1}^*] + \sum_{n=0}^{\infty}\int_0^{\infty}W_n(x)[\mu V_n^*]dx \\
&+W_0^*(0)\left[\lambda Q + b\sum_{r=1}^{\mathcal{M}}V_n\right] + \sum_{n=1}^{\infty}W_n^*(0)bV_{n+\mathcal{M}} \\
&+\sum_{n=0}^{\infty}\int_0^{\infty}W_n(x)\left[\frac{dW_n^*(x)}{dx} - (\lambda+\mu)W_n^*(x)\right]dx \\
&+\sum_{n=0}^{\infty}\int_0^{\infty}W_n(x)[\lambda W_{n+1}^*(x)]dx \\
=\ & \Bigg\{Q[-\lambda Q^*] + V_0[bQ^*] + \sum_{n=0}^{\infty}V_n[-(\lambda+b)V_n^*] \\
&+\sum_{n=0}^{\infty}\int_0^{\infty}W_n(x)\left[\frac{dW_n^*(x)}{dx} - (\lambda+\mu)W_n^*(x)\right]dx\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\{Q[\lambda W_0^*(0)]+\sum_{r=1}^{\mathcal{M}}V_n[bW_0^*(0)]+\sum_{l=\mathcal{M}+1}^{\infty}V_l[bW_{l-\mathcal{M}}^*(0)]\\
&+\sum_{n=0}^{\infty}\int_0^{\infty}W_n(x)[\mu V_n^*]dx\Big\}\\
&+\Big\{\sum_{n=0}^{\infty}V_n[\lambda V_{n+1}^*]+\sum_{n=0}^{\infty}\int_0^{\infty}W_n(x)[\lambda W_{n+1}^*(x)]dx\Big\}\\
=&\ \langle (V,W),(H+J+S)(V^*,W^*)\rangle.
\end{aligned}
\tag{10}
$$

(10) shows that the result of this lemma is right. □

Lemma 2.

$$
\left\{\gamma\in\mathbb{C}\;\middle|\;
\begin{aligned}
&Re\gamma+\lambda+\mu>0,\ |\gamma+\lambda|\geq\lambda,\\
&|\gamma+\lambda+\mu||\gamma+\lambda+b|>\mu b,\\
&\sup\Big\{\tfrac{\lambda^2\mu(Re\gamma+\lambda+\mu)+\lambda^2|\gamma+\lambda+\mu||\gamma+\lambda+b|}{(Re\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},\\
&\tfrac{\lambda(Re\gamma+\lambda+\mu)|\gamma+\lambda||\gamma+\lambda+\mu|+\lambda^2b|\gamma+\lambda+\mu|}{(Re\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},\\
&\tfrac{\lambda\mu b(Re\gamma+\lambda+\mu)|\gamma+\lambda|+\lambda b|\gamma+\lambda||\gamma+\lambda+\mu||\gamma+\lambda+b|}{(Re\gamma+\lambda+\mu)|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&\quad+\tfrac{\lambda}{|\gamma+\lambda+b|},\\
&\tfrac{\lambda|\gamma+\lambda+\mu|[Re\gamma+\lambda+\mu+b]}{(Re\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]},\\
&\tfrac{\lambda\mu|\gamma+\lambda|(Re\gamma+\lambda+\mu)+\lambda^2\mu b}{(Re\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&\quad+\tfrac{\lambda}{Re\gamma+\lambda+\mu},\\
&\quad\tfrac{\lambda\mu^2b|\gamma+\lambda|}{|\gamma+\lambda+\mu||\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&\quad+\tfrac{\lambda\mu}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}\\
&\quad+\tfrac{\lambda\mu b|\gamma+\lambda|}{(Re\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&\quad+\tfrac{\lambda}{Re\gamma+\lambda+\mu},\\
&\quad\tfrac{\lambda\mu(Re\gamma+\lambda+\mu+b)}{(Re\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]}\\
&\quad+\tfrac{\lambda}{Re\gamma+\lambda+\mu}\Big\}<1
\end{aligned}
\right\}
$$

belongs to the resolvent set of $(A+U+E)^*=H+J+S$. *Especially, all points on the imaginary axis except for zero belong to the resolvent set of* $(A+U+E)^*$ *and* $A+U+E$.

Proof. For any given $(y^*,z^*)\in X^*$ we consider the equation $(\gamma I-H-J)(V^*,W^*)=S(y^*,z^*)$ which is equivalent to

$$
(\gamma+\lambda)Q^*-\lambda W_0^*(0)=0, \tag{11}
$$

$$-bQ^* + (\gamma+\lambda+b)V_0^* = \lambda y_1^*, \tag{12}$$
$$(\gamma+\lambda+b)V_1^* - bW_0^*(0) = \lambda y_2^*, \tag{13}$$
$$(\gamma+\lambda+b)V_2^* - bW_0^*(0) = \lambda y_3^*, \tag{14}$$
$$\cdots\cdots$$
$$(\gamma+\lambda+b)V_{\mathcal{M}}^* - bW_0^*(0) = \lambda y_{\mathcal{M}+1}^*, \tag{15}$$
$$(\gamma+\lambda+b)V_{\mathcal{M}+1}^* - bW_1^*(0) = \lambda y_{\mathcal{M}+2}^*, \tag{16}$$
$$(\gamma+\lambda+b)V_{\mathcal{M}+2}^* - bW_2^*(0) = \lambda y_{\mathcal{M}+3}^*, \tag{17}$$
$$\cdots\cdots$$
$$(\gamma+\lambda+b)V_{\mathcal{M}+n}^* - bW_n^*(0) = \lambda y_{\mathcal{M}+n+1}^*, \quad n \geq 1, \tag{18}$$
$$\cdots\cdots$$
$$\frac{dW_n^*(x)}{dx} = (\gamma+\lambda+\mu)W_n^*(x) - \mu V_n^* - \lambda z_{n+1}^*(x), \quad n \geq 0, \tag{19}$$
$$W_n^*(\infty) = \alpha, \quad n \geq 0. \tag{20}$$

By solving (19) we have

$$W_n^*(x) = b_n e^{(\gamma+\lambda+\mu)x} - e^{(\gamma+\lambda+\mu)x}\int_0^x [\mu V_n^* + \lambda z_{n+1}^*(\tau)]e^{-(\gamma+\lambda+\mu)\tau}d\tau, \quad n \geq 0. \tag{21}$$

By multiplying with $e^{-(\gamma+\lambda+\mu)x}$ the both sides of (21) and using (20) we deduce

$$\begin{aligned} b_n &= \int_0^\infty [\mu V_n^* + \lambda z_{n+1}^*(\tau)]e^{-(\gamma+\lambda+\mu)\tau}d\tau \\ &= \frac{\mu}{\gamma+\lambda+\mu}V_n^* + \lambda\int_0^\infty z_{n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau = W_n^*(0), \quad n \geq 0. \end{aligned} \tag{22}$$

By inserting (22) into (21) we derive (assume Re$\gamma+\lambda+\mu > 0$)

$$\begin{aligned} W_n^*(x) &= e^{(\gamma+\lambda+\mu)x}\int_x^\infty [\mu V_n^* + \lambda z_{n+1}^*(\tau)]e^{-(\gamma+\lambda+\mu)\tau}d\tau \\ &= e^{(\gamma+\lambda+\mu)x}\Big[-V_n^*\frac{\mu}{\gamma+\lambda+\mu}e^{-(\gamma+\lambda+\mu)\tau}\Big|_{\tau=x}^{\tau=\infty} + \lambda\int_x^\infty z_{n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau\Big] \\ &= \frac{\mu}{\gamma+\lambda+\mu}V_n^* + \lambda e^{(\gamma+\lambda+\mu)x}\int_x^\infty z_{n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau, \quad n \geq 0. \end{aligned} \tag{23}$$

By combining (22) with (11) we calculate

$$\begin{aligned} Q^* &= \frac{\lambda}{\gamma+\lambda}W_0^*(0) = \frac{\lambda}{\gamma+\lambda}\Big[\frac{\mu}{\gamma+\lambda+\mu}V_0^* + \lambda\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau\Big] \\ &= \frac{\lambda\mu}{(\gamma+\lambda)(\gamma+\lambda+\mu)}V_0^* + \frac{\lambda^2}{\gamma+\lambda}\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau. \end{aligned} \tag{24}$$

(24) and (12) give

$$V_0^* = \frac{b}{\gamma+\lambda+b}Q^* + \frac{\lambda}{\gamma+\lambda+b}y_1^* = \frac{\lambda\mu b}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)}V_0^*$$
$$+ \frac{\lambda}{\gamma+\lambda+b}y_1^* + \frac{\lambda^2 b}{(\gamma+\lambda)(\gamma+\lambda+b)}\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$\Longrightarrow$$
$$V_0^* = \frac{\lambda(\gamma+\lambda)(\gamma+\lambda+\mu)}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}y_1^*$$
$$+ \frac{\lambda^2 b(\gamma+\lambda+\mu)}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau. \quad (25)$$

By inserting (25) into (22) and (25) into (24) it follows that

$$W_0^*(0) = \frac{\mu}{\gamma+\lambda+\mu}V_0^* + \lambda\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$= \frac{\lambda\mu(\gamma+\lambda)}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}y_1^*$$
$$+ \frac{\lambda^2\mu b}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+ \lambda\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$= \frac{\lambda\mu(\gamma+\lambda)}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}y_1^*$$
$$+ \frac{\lambda(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau, \quad (26)$$

$$Q^* = \frac{\lambda}{\gamma+\lambda}W_0^*(0)$$
$$= \frac{\lambda^2\mu}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}y_1^*$$
$$+ \frac{\lambda^2(\gamma+\lambda+\mu)(\gamma+\lambda+b)}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau. \quad (27)$$

From (13)-(15) and (26) it follows that

$$V_n^* = \frac{b}{\gamma+\lambda+b}W_0^*(0) + \frac{\lambda}{\gamma+\lambda+b}y_{n+1}^*$$
$$= \frac{\lambda\mu b(\gamma+\lambda)}{(\gamma+\lambda+b)[(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b]}y_1^* + \frac{\lambda}{\gamma+\lambda+b}y_{n+1}^*$$

$$+\frac{\lambda b(\gamma+\lambda)(\gamma+\lambda+\mu)}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau,$$
$$1\le n\le\mathcal{M} \quad (28)$$

$\Longrightarrow$

$$|V_n^*|\le\Bigg[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Bigg]$$
$$\times(y^*,z^*),\quad 1\le n\le\mathcal{M}. \quad (29)$$

By combining (18) with (22) we calculate

$$(\gamma+\lambda+b)V_{\mathcal{M}+n}^*-b\left[\frac{\mu}{\gamma+\lambda+\mu}V_n^*+\lambda\int_0^\infty z_{n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau\right]$$
$$=\lambda y_{\mathcal{M}+n+1}^*$$
$$\Longrightarrow$$
$$V_{\mathcal{M}+n}^*=\frac{\mu b}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)}V_n^*+\frac{\lambda b}{\gamma+\lambda+b}\int_0^\infty z_{n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+\frac{\lambda}{\gamma+\lambda+b}y_{\mathcal{M}+n+1}^*,\quad n\ge 1. \quad (30)$$

(30) implies

$$V_{\mathcal{M}+n}^*=\frac{\mu b}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)}V_n^*+\frac{\lambda b}{\gamma+\lambda+b}\int_0^\infty z_{n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+\frac{\lambda}{\gamma+\lambda+b}y_{\mathcal{M}+n+1}^*,\quad 1\le n\le\mathcal{M}, \quad (31)$$

$$V_{2\mathcal{M}+n}^*=\frac{\mu b}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)}V_{\mathcal{M}+n}^*+\frac{\lambda b}{\gamma+\lambda+b}\int_0^\infty z_{\mathcal{M}+n+1}^*(\tau)$$
$$\times e^{-(\gamma+\lambda+\mu)\tau}d\tau+\frac{\lambda}{\gamma+\lambda+b}y_{2\mathcal{M}+n+1}^*$$
$$=\frac{\mu b}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)}\Bigg[\frac{\mu b}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)}V_n^*$$
$$+\frac{\lambda b}{\gamma+\lambda+b}\int_0^\infty z_{n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau+\frac{\lambda}{\gamma+\lambda+b}y_{\mathcal{M}+n+1}^*\Bigg]$$
$$+\frac{\lambda b}{\gamma+\lambda+b}\int_0^\infty z_{\mathcal{M}+n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau+\frac{\lambda}{\gamma+\lambda+b}y_{2\mathcal{M}+n+1}^*$$
$$=\frac{(\mu b)^2}{(\gamma+\lambda+\mu)^2(\gamma+\lambda+b)^2}V_n^*$$

$$+\frac{\lambda\mu b^2}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)^2}\int_0^\infty z_{n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+\frac{\lambda b}{\gamma+\lambda+b}\int_0^\infty z_{\mathcal{M}+n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+\frac{\lambda\mu b}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)^2}y_{\mathcal{M}+n+1}^*+\frac{\lambda}{\gamma+\lambda+b}y_{2\mathcal{M}+n+1}^*,$$
$$1\le n\le\mathcal{M}, \tag{32}$$

$$V_{3\mathcal{M}+n}^*=\frac{\mu b}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)}V_{2\mathcal{M}+n}^*+\frac{\lambda b}{\gamma+\lambda+b}\int_0^\infty z_{2\mathcal{M}+n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+\frac{\lambda}{\gamma+\lambda+b}y_{3\mathcal{M}+n+1}^*$$
$$=\frac{(\mu b)^3}{(\gamma+\lambda+\mu)^3(\gamma+\lambda+b)^3}V_n^*$$
$$+\frac{\lambda(\mu b)^2 b}{(\gamma+\lambda+\mu)^2(\gamma+\lambda+b)^3}\int_0^\infty z_{n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+\frac{\lambda\mu b^2}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)^2}\int_0^\infty z_{\mathcal{M}+n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+\frac{\lambda(\mu b)^2}{(\gamma+\lambda+\mu)^2(\gamma+\lambda+b)^3}y_{\mathcal{M}+n+1}^*+\frac{\lambda\mu b}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)^2}y_{2\mathcal{M}+n+1}^*$$
$$+\frac{\lambda b}{\gamma+\lambda+b}\int_0^\infty z_{2\mathcal{M}+n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau+\frac{\lambda}{\gamma+\lambda+b}y_{3\mathcal{M}+n+1}^*$$
$$=\frac{(\mu b)^3}{(\gamma+\lambda+\mu)^3(\gamma+\lambda+b)^3}V_n^*$$
$$+\frac{\lambda(\mu b)^2 b}{(\gamma+\lambda+\mu)^2(\gamma+\lambda+b)^3}\int_0^\infty z_{n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+\frac{\lambda\mu b^2}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)^2}\int_0^\infty z_{\mathcal{M}+n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+\frac{\lambda b}{\gamma+\lambda+b}\int_0^\infty z_{2\mathcal{M}+n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+\frac{\lambda(\mu b)^2}{(\gamma+\lambda+\mu)^2(\gamma+\lambda+b)^3}y_{\mathcal{M}+n+1}^*+\frac{\lambda\mu b}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)^2}y_{2\mathcal{M}+n+1}^*$$
$$+\frac{\lambda}{\gamma+\lambda+b}y_{3\mathcal{M}+n+1}^*$$
$$=\frac{(\mu b)^3}{(\gamma+\lambda+\mu)^3(\gamma+\lambda+b)^3}V_n^*$$

$$+\sum_{k=1}^{3}\frac{\lambda(\mu b)^{3-k}b}{(\gamma+\lambda+\mu)^{3-k}(\gamma+\lambda+b)^{4-k}}\int_0^\infty z^*_{(k-1)\mathcal{M}+n+1}(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$+\sum_{k=1}^{3}\frac{\lambda(\mu b)^{3-k}}{(\gamma+\lambda+\mu)^{3-k}(\gamma+\lambda+b)^{4-k}}y^*_{k\mathcal{M}+n+1},\quad 1\le n\le\mathcal{M},\qquad(33)$$

$$\begin{aligned}
V^*_{4\mathcal{M}+n} &= \frac{\mu b}{(\gamma+\lambda+\mu)(\gamma+\lambda+b)}V^*_{3\mathcal{M}+n}\\
&\quad+\frac{\lambda b}{\gamma+\lambda+b}\int_0^\infty z^*_{3\mathcal{M}+n+1}(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau+\frac{\lambda}{\gamma+\lambda+b}y^*_{4\mathcal{M}+n+1}\\
&= \frac{(\mu b)^4}{(\gamma+\lambda+\mu)^4(\gamma+\lambda+b)^4}V^*_n\\
&\quad+\sum_{k=1}^{3}\frac{\lambda(\mu b)^{4-k}b}{(\gamma+\lambda+\mu)^{4-k}(\gamma+\lambda+b)^{5-k}}\int_0^\infty z^*_{(k-1)\mathcal{M}+n+1}(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau\\
&\quad+\sum_{k=1}^{3}\frac{\lambda(\mu b)^{4-k}}{(\gamma+\lambda+\mu)^{4-k}(\gamma+\lambda+b)^{5-k}}y^*_{k\mathcal{M}+n+1}\\
&\quad+\frac{\lambda b}{\gamma+\lambda+b}\int_0^\infty z^*_{3\mathcal{M}+n+1}(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau+\frac{\lambda}{\gamma+\lambda+b}y^*_{4\mathcal{M}+n+1}\\
&= \frac{(\mu b)^4}{(\gamma+\lambda+\mu)^4(\gamma+\lambda+b)^4}V^*_n\\
&\quad+\sum_{k=1}^{4}\frac{\lambda(\mu b)^{4-k}b}{(\gamma+\lambda+\mu)^{4-k}(\gamma+\lambda+b)^{5-k}}\int_0^\infty z^*_{(k-1)\mathcal{M}+n+1}(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau\\
&\quad+\sum_{k=1}^{4}\frac{\lambda(\mu b)^{4-k}}{(\gamma+\lambda+\mu)^{4-k}(\gamma+\lambda+b)^{5-k}}y^*_{k\mathcal{M}+n+1},\quad 1\le n\le\mathcal{M},\qquad(34)
\end{aligned}$$

$\cdots\cdots$

$$\begin{aligned}
V^*_{j\mathcal{M}+n} &= \frac{(\mu b)^j}{(\gamma+\lambda+\mu)^j(\gamma+\lambda+b)^j}V^*_n\\
&\quad+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}b}{(\gamma+\lambda+\mu)^{j-k}(\gamma+\lambda+b)^{j+1-k}}\int_0^\infty z^*_{(k-1)\mathcal{M}+n+1}(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau\\
&\quad+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}}{(\gamma+\lambda+\mu)^{j-k}(\gamma+\lambda+b)^{j+1-k}}y^*_{k\mathcal{M}+n+1},\ j\ge 1,\ 1\le n\le\mathcal{M}.
\end{aligned}\qquad(35)$$

By using (29) we estimate (35) as follows:

$$
\begin{aligned}
|V^*_{j\mathcal{M}+n}| \le{}& \frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}|V^*_n| \\
&+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}b}{|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}(\mathrm{Re}\gamma+\lambda+\mu)}\|z^*_{(k-1)\mathcal{M}+n+1}\|_{L^\infty[0,\infty)} \\
&+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}}{|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}}|y^*_{k\mathcal{M}+n+1}| \\
\le{}& \Bigg\{\frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}\Bigg[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|} \\
&+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Bigg] \\
&+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}b}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}} \\
&+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}}{|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}}\Bigg\}(y^*,z^*),\quad 1\le n\le\mathcal{M},\quad j\ge 1. \qquad (36)
\end{aligned}
$$

Let

$$
\begin{aligned}
c_j ={}& \Bigg\{\frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}\Bigg[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|} \\
&+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Bigg] \\
&+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}b}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}} \\
&+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}}{|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}}\Bigg\},\quad j\ge 1,
\end{aligned}
$$

then we deduce for $|\gamma+\lambda|\ge\lambda$ and $\mathrm{Re}\gamma+\lambda+\mu>0$

$$
\begin{aligned}
c_{j+1}-c_j ={}& \Bigg\{\frac{(\mu b)^{j+1}}{|\gamma+\lambda+\mu|^{j+1}|\gamma+\lambda+b|^{j+1}}\Bigg[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|} \\
&+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Bigg]
\end{aligned}
$$

$$+\sum_{k=1}^{j+1}\frac{\lambda(\mu b)^{j+1-k}b}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu|^{j+1-k}|\gamma+\lambda+b|^{j+2-k}}$$

$$+\sum_{k=1}^{j+1}\frac{\lambda(\mu b)^{j+1-k}}{|\gamma+\lambda+\mu|^{j+1-k}|\gamma+\lambda+b|^{j+2-k}}\Bigg\}$$

$$-\Bigg\{\frac{(\mu b)^{j}}{|\gamma+\lambda+\mu|^{j}|\gamma+\lambda+b|^{j}}\Bigg[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$

$$+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Bigg]$$

$$+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}b}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}}$$

$$+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}}{|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}}\Bigg\}$$

$$=\Bigg\{\frac{(\mu b)^{j+1}}{|\gamma+\lambda+\mu|^{j+1}|\gamma+\lambda+b|^{j+1}}\Bigg[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$

$$+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Bigg]$$

$$+\sum_{l=0}^{j}\frac{\lambda(\mu b)^{j-l}b}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu|^{j-l}|\gamma+\lambda+b|^{j+1-l}}$$

$$+\sum_{l=0}^{j}\frac{\lambda(\mu b)^{j-l}}{|\gamma+\lambda+\mu|^{j-l}|\gamma+\lambda+b|^{j+1-l}}\Bigg\}$$

$$-\Bigg\{\frac{(\mu b)^{j}}{|\gamma+\lambda+\mu|^{j}|\gamma+\lambda+b|^{j}}\Bigg[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$

$$+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Bigg]$$

$$+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}b}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}}$$

$$+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}}{|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}}\Bigg\}$$

$$=\frac{(\mu b)^{j+1}}{|\gamma+\lambda+\mu|^{j+1}|\gamma+\lambda+b|^{j+1}}\Bigg[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$

$$+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Bigg]$$

$$+\frac{\lambda(\mu b)^{j}b}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu|^{j}|\gamma+\lambda+b|^{j+1}}$$

$$
\begin{aligned}
&+\frac{\lambda(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^{j+1}}\\
&-\frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}\Big[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Big]\\
&=\frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}\\
&\times\Big\{\frac{\mu b}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}\Big[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Big]\\
&+\frac{\lambda b}{(\mathrm{Re}\gamma+\lambda+b)|\gamma+\lambda+b|}+\frac{\lambda}{|\gamma+\lambda+b|}\\
&-\Big[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Big]\Big\}\\
&=\frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}\\
&\times\Big\{\frac{\mu b}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}\Big[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Big]\\
&+\frac{\lambda b}{(\mathrm{Re}\gamma+\lambda+b)|\gamma+\lambda+b|}-\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&-\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Big\}\\
&=\frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}\\
&\times\Big\{\frac{\lambda(\mu b)^2|\gamma+\lambda|}{|\gamma+\lambda+\mu||\gamma+\lambda+b|^2|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&+\frac{\lambda\mu b}{|\gamma+\lambda+\mu||\gamma+\lambda+b|^2}\\
&+\frac{\lambda\mu b^2|\gamma+\lambda|}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&+\frac{\lambda b}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+b|}
\end{aligned}
$$

$$- \frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$\left. - \frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\right\}$$
$$= \frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}$$
$$\times \frac{1}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu||\gamma+\lambda+b|^2|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$\times \Big\{\lambda(\mu b)^2(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda|$$
$$+ \lambda\mu b(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|$$
$$+ \lambda\mu b^2|\gamma+\lambda||\gamma+\lambda+\mu||\gamma+\lambda+b|$$
$$+ \lambda b|\gamma+\lambda+\mu||\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|$$
$$- \lambda\mu b(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda||\gamma+\lambda+\mu||\gamma+\lambda+b| - \lambda b|\gamma+\lambda||\gamma+\lambda+\mu|^2|\gamma+\lambda+b|^2\Big\}$$
$$\geq \frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}$$
$$\times \frac{1}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu||\gamma+\lambda+b|^2|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$\times \Big\{\lambda(\mu b)^2(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda|$$
$$+ \lambda\mu b(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda||\gamma+\lambda+\mu||\gamma+\lambda+b|$$
$$-(\lambda\mu b)^2(\mathrm{Re}\gamma+\lambda+\mu)+\lambda\mu b^2|\gamma+\lambda||\gamma+\lambda+\mu||\gamma+\lambda+b|+\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|^2|\gamma+\lambda+b|^2$$
$$- \mu(\lambda b)^2|\gamma+\lambda+\mu||\gamma+\lambda+b| - \lambda\mu b(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda||\gamma+\lambda+\mu||\gamma+\lambda+b|$$
$$- \lambda b|\gamma+\lambda||\gamma+\lambda+\mu|^2|\gamma+\lambda+b|^2\Big\}$$
$$= \frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}$$
$$\times \frac{1}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu||\gamma+\lambda+b|^2|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$\times \Big\{\lambda(\mu b)^2(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda| - (\lambda\mu b)^2(\mathrm{Re}\gamma+\lambda+\mu)$$
$$+ \lambda\mu b^2|\gamma+\lambda||\gamma+\lambda+\mu||\gamma+\lambda+b| - \mu(\lambda b)^2|\gamma+\lambda+\mu||\gamma+\lambda+b|\Big\}$$
$$= \frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}$$
$$\times \frac{1}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu||\gamma+\lambda+b|^2|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$\times \Big\{\lambda(\mu b)^2(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda|-\lambda] + \lambda\mu b^2|\gamma+\lambda+\mu||\gamma+\lambda+b|[|\gamma+\lambda|-\lambda]\Big\}$$
$$\geq 0 \implies c_{j+1} \geq c_j, \quad j \geq 1. \quad (37)$$

(37) and (36) show

$$\begin{aligned}
\sup_{l\geq \mathcal{M}+1}|V_l^*| &= \sup_{\substack{j\geq 1\\ 1\leq n\leq \mathcal{M}}} |V_{j\mathcal{M}+n}^*| \leq \lim_{j\to\infty}\left\{\frac{(\mu b)^j}{|\gamma+\lambda+\mu|^j|\gamma+\lambda+b|^j}\right.\\
&\quad\times\left[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\right.\\
&\quad\left.+\frac{\lambda}{|\gamma+\lambda+b|}+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\right]\\
&\quad+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}b}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}}\\
&\quad\left.+\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}}{|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}}\right\}(y^*,z^*)\\
&=\left\{\lim_{j\to\infty}\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}b}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}}\right.\\
&\quad\left.+\lim_{j\to\infty}\sum_{k=1}^{j}\frac{\lambda(\mu b)^{j-k}}{|\gamma+\lambda+\mu|^{j-k}|\gamma+\lambda+b|^{j+1-k}}\right\}(y^*,z^*)\\
&=\left\{\lim_{j\to\infty}\frac{\lambda b}{(\mathrm{Re}\gamma+\lambda+\mu)}\frac{\frac{1}{|\gamma+\lambda+b|}}{1-\frac{\mu b}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}}\right.\\
&\quad\times\left[1-\left(\frac{\mu b}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}\right)^j\right]\\
&\quad\left.+\lim_{j\to\infty}\lambda\frac{\frac{1}{|\gamma+\lambda+b|}}{1-\frac{\mu b}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}}\left[1-\left(\frac{\mu b}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}\right)^j\right]\right\}(y^*,z^*)\\
&=\left[\frac{\lambda b|\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]}\right.\\
&\quad\left.+\frac{\lambda|\gamma+\lambda+\mu|}{|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b}\right](y^*,z^*). \qquad (38)
\end{aligned}$$

From (25), (27), (29) and (38) we have

$$V^*\leq\left\{|Q^*|,|V_0^*|,\sup_{1\leq n\leq M}|V_n^*|,\sup_{l\leq M+1}|V_l^*|\right\}$$

$$\leq \sup\left\{\frac{\lambda^2\mu}{|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\right.$$
$$+\frac{\lambda^2|\gamma+\lambda+\mu||\gamma+\lambda+b|}{(\text{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},$$
$$\frac{\lambda|\gamma+\lambda||\gamma+\lambda+\mu|}{|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$+\frac{\lambda^2 b|\gamma+\lambda+\mu|}{(\text{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},$$
$$\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$+\frac{\lambda}{|\gamma+\lambda+b|}$$
$$+\frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\text{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},$$
$$\frac{\lambda b|\gamma+\lambda+\mu|}{(\text{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]}$$
$$\left.+\frac{\lambda|\gamma+\lambda+\mu|}{|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b}\right\}(y^*,z^*)$$
$$=\sup\left\{\frac{\lambda^2\mu(\text{Re}\gamma+\lambda+\mu)+\lambda^2|\gamma+\lambda+\mu||\gamma+\lambda+b|}{(\text{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},\right.$$
$$\frac{\lambda(\text{Re}\gamma+\lambda+\mu)|\gamma+\lambda||\gamma+\lambda+\mu|+\lambda^2 b|\gamma+\lambda+\mu|}{(\text{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},$$
$$\frac{\lambda\mu b(\text{Re}\gamma+\lambda+\mu)|\gamma+\lambda|+\lambda b|\gamma+\lambda||\gamma+\lambda+\mu||\gamma+\lambda+b|}{(\text{Re}\gamma+\lambda+\mu)|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$+\frac{\lambda}{|\gamma+\lambda+b|},$$
$$\left.\frac{\lambda|\gamma+\lambda+\mu|[\text{Re}\gamma+\lambda+\mu+b]}{(\text{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]}\right\}(y^*,z^*). \tag{39}$$

By (23), (25), (29) and (38) we calculate and estimate

$$W_0^*(x)=\frac{\mu}{\gamma+\lambda+\mu}V_0^*+\lambda e^{(\gamma+\lambda+\mu)x}\int_x^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$
$$=\frac{\mu}{\gamma+\lambda+\mu}\left\{\frac{\lambda(\gamma+\lambda)(\gamma+\lambda+\mu)}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}y_1^*\right.$$

$$+ \frac{\lambda^2 b(\gamma+\lambda+\mu)}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau\Bigg\}$$

$$+ \lambda e^{(\gamma+\lambda+\mu)\tau}\int_x^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$

$$= \frac{\lambda\mu(\gamma+\lambda)}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}y_1^*$$

$$+ \frac{\lambda^2\mu b}{(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b}\int_0^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$

$$+ \lambda e^{(\gamma+\lambda+\mu)x}\int_x^\infty z_1^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$

$$\Longrightarrow \|W_0^*\|_{L^\infty[0,\infty)} \le \Bigg\{\frac{\lambda\mu|\gamma+\lambda|}{|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$

$$+ \frac{\lambda^2\mu b}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$

$$+ \frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu}\Bigg\}(y^*, z^*). \tag{40}$$

$$W_n^*(x) = \frac{\mu}{\gamma+\lambda+\mu}V_n^* + \lambda e^{(\gamma+\lambda+\mu)x}\int_x^\infty z_{n+1}^*(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$

$$\Longrightarrow \|W_n^*\|_{L^\infty[0,\infty)} \le \frac{\mu}{|\gamma+\lambda+\mu|}|V_n^*| + \frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu}\|z_{n+1}^*\|_{L^\infty[0,\infty)}$$

$$\le \Bigg\{\frac{\mu}{|\gamma+\lambda+\mu|}\Bigg[\frac{\lambda\mu b|\gamma+\lambda|}{|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$

$$+ \frac{\lambda}{|\gamma+\lambda+b|}$$

$$+ \frac{\lambda b|\gamma+\lambda||\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\Bigg]$$

$$+ \frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu}\Bigg\}(y^*, z^*)$$

$$= \Bigg\{\frac{\lambda\mu^2 b|\gamma+\lambda|}{|\gamma+\lambda+\mu||\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$

$$+ \frac{\lambda\mu}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}$$

$$+\frac{\lambda\mu b|\gamma+\lambda|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$

$$+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu}\Bigg\}(y^*,z^*),\ 1\le n\le\mathcal{M}. \quad (41)$$

$$W^*_{j\mathcal{M}+n}(x)=\frac{\mu}{\gamma+\lambda+\mu}V^*_{j\mathcal{M}+n}+\lambda e^{(\gamma+\lambda+\mu)x}\int_x^\infty z^*_{j\mathcal{M}+n+1}(\tau)e^{-(\gamma+\lambda+\mu)\tau}d\tau$$

$$\Longrightarrow \|W^*_{j\mathcal{M}+n}\|_{L^\infty[0,\infty)}\le\Bigg\{\frac{\mu}{|\gamma+\lambda+\mu|}$$

$$\times\left[\frac{\lambda b|\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]}\right.$$

$$\left.+\frac{\lambda|\gamma+\lambda+\mu|}{|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b}\right]$$

$$+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu}\Bigg\}(y^*,z^*)$$

$$=\Bigg\{\frac{\lambda\mu b}{(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]}$$

$$+\frac{\lambda\mu}{|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b}$$

$$+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu}\Bigg\}(y^*,z^*),\ j\ge1,\ 1\le n\le\mathcal{M}. \quad (42)$$

(40), (41) and (42) give

$$W^*=\sup_{n\ge1}\|W^*_n\|_{L^\infty[0,\infty)}$$

$$=\sup\left\{\|W^*_0\|_{L^\infty[0,\infty)},\ \sup_{1\le n\le M}\|W^*_n\|_{L^\infty[0,\infty)},\ \sup_{\substack{j\ge1\\1\le n\le M}}\|W^*_{jM+n}\|_{L^\infty[0,\infty)}\right\}$$

$$=\sup\Bigg\{\frac{\lambda\mu|\gamma+\lambda|}{|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$

$$+\frac{\lambda^2\mu b}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$

$$+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu},$$

$$
\begin{aligned}
&\frac{\lambda\mu^2 b|\gamma+\lambda|}{|\gamma+\lambda+\mu||\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&+\frac{\lambda\mu}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}\\
&+\frac{\lambda\mu b|\gamma+\lambda|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu},\\
&\frac{\lambda\mu b}{(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]}\\
&+\frac{\lambda\mu}{|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b}+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu}\Bigg\}(y^*,z^*)\\
=\sup\Bigg\{&\frac{\lambda\mu|\gamma+\lambda|(\mathrm{Re}\gamma+\lambda+\mu)+\lambda^2\mu b}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu},\\
&\frac{\lambda\mu^2 b|\gamma+\lambda|}{|\gamma+\lambda+\mu||\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&+\frac{\lambda\mu}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}\\
&+\frac{\lambda\mu b|\gamma+\lambda|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\
&+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu},\\
&\frac{\lambda\mu(\mathrm{Re}\gamma+\lambda+\mu+b)}{(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]}\\
&+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu}\Bigg\}(y^*,z^*). \qquad (43)
\end{aligned}
$$

From (39) and (43) we know

$$
\begin{aligned}
&(V^*,W^*)\\
&=\sup\{V^*,W^*\}\le\Bigg\{\frac{\lambda^2\mu(\mathrm{Re}\gamma+\lambda+\mu)+\lambda^2|\gamma+\lambda+\mu||\gamma+\lambda+b|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},\\
&\qquad\frac{\lambda(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda||\gamma+\lambda+\mu|+\lambda^2 b|\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},
\end{aligned}
$$

$$\frac{\lambda\mu b(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda|+\lambda b|\gamma+\lambda||\gamma+\lambda+\mu||\gamma+\lambda+b|}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$+\frac{\lambda}{|\gamma+\lambda+b|},\frac{\lambda|\gamma+\lambda+\mu|[\mathrm{Re}\gamma+\lambda+\mu+b]}{(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]},$$
$$\frac{\lambda\mu|\gamma+\lambda|(\mathrm{Re}\gamma+\lambda+\mu)+\lambda^2\mu b}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu},\frac{\lambda\mu^2 b|\gamma+\lambda|}{|\gamma+\lambda+\mu||\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$+\frac{\lambda\mu}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}+\frac{\lambda\mu b|\gamma+\lambda|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu},\frac{\lambda\mu(\mathrm{Re}\gamma+\lambda+\mu+b)}{(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]}$$
$$\left.+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu}\right\}(y^*,z^*). \quad (44)$$

(44) shows that $(\gamma I-H-J)^{-1}S: X\to D(H)$ exists when $\mathrm{Re}\gamma+\lambda+\mu>0$ and $|\gamma+\lambda+\mu||\gamma+\lambda+b|>\mu b$. Moreover, for such γ

$$\left\|(\gamma I-H-J)^{-1}S\right\|$$
$$\leq\sup\left\{\frac{\lambda^2\mu(\mathrm{Re}\gamma+\lambda+\mu)+\lambda^2|\gamma+\lambda+\mu||\gamma+\lambda+b|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},\right.$$
$$\frac{\lambda(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda||\gamma+\lambda+\mu|+\lambda^2 b|\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},$$
$$\frac{\lambda\mu b(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda|+\lambda b|\gamma+\lambda||\gamma+\lambda+\mu||\gamma+\lambda+b|}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$+\frac{\lambda}{|\gamma+\lambda+b|},\frac{\lambda|\gamma+\lambda+\mu|[\mathrm{Re}\gamma+\lambda+\mu+b]}{(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]},$$
$$\frac{\lambda\mu|\gamma+\lambda|(\mathrm{Re}\gamma+\lambda+\mu)+\lambda^2\mu b}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu},\frac{\lambda\mu^2 b|\gamma+\lambda|}{|\gamma+\lambda+\mu||\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}$$
$$+\frac{\lambda\mu}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}$$
$$+\frac{\lambda\mu b|\gamma+\lambda|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu},$$

$$\left.\frac{\lambda\mu(\mathrm{Re}\gamma+\lambda+\mu+b)}{(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]}+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu}\right\}. \tag{45}$$

Hence $\left[I-(\gamma I-H-J)^{-1}S\right]^{-1}$ exists and is bounded when γ belongs to the following set

$$\left\{\gamma\in\mathbb{C}\;\middle|\;\begin{array}{l}\mathrm{Re}\gamma+\lambda+\mu>0,\quad |\gamma+\lambda|\geq\lambda,\\ |\gamma+\lambda+\mu||\gamma+\lambda+b|>\mu b,\\ \sup\left\{\frac{\lambda^2\mu(\mathrm{Re}\gamma+\lambda+\mu)+\lambda^2|\gamma+\lambda+\mu||\gamma+\lambda+b|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},\right.\\ \frac{\lambda(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda||\gamma+\lambda+\mu|+\lambda^2 b|\gamma+\lambda+\mu|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|},\\ \frac{\lambda\mu b(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda|+\lambda b|\gamma+\lambda||\gamma+\lambda+\mu||\gamma+\lambda+b|}{(\mathrm{Re}\gamma+\lambda+\mu)|\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\ \quad+\frac{\lambda}{|\gamma+\lambda+b|},\\ \frac{\lambda|\gamma+\lambda+\mu|[\mathrm{Re}\gamma+\lambda+\mu+b]}{(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]},\\ \frac{\lambda\mu|\gamma+\lambda|(\mathrm{Re}\gamma+\lambda+\mu)+\lambda^2\mu b}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\ \quad+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu},\\ \quad\frac{\lambda\mu^2 b|\gamma+\lambda|}{|\gamma+\lambda+\mu||\gamma+\lambda+b||(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\ \quad+\frac{\lambda\mu}{|\gamma+\lambda+\mu||\gamma+\lambda+b|}\\ \quad+\frac{\lambda\mu b|\gamma+\lambda|}{(\mathrm{Re}\gamma+\lambda+\mu)|(\gamma+\lambda)(\gamma+\lambda+\mu)(\gamma+\lambda+b)-\lambda\mu b|}\\ \quad+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu},\\ \quad\frac{\lambda\mu(\mathrm{Re}\gamma+\lambda+\mu+b)}{(\mathrm{Re}\gamma+\lambda+\mu)[|\gamma+\lambda+\mu||\gamma+\lambda+b|-\mu b]}\\ \quad\left.+\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\mu}\right\}<1\end{array}\right\}. \tag{46}$$

It is the same as the above process, one can verify that $(\gamma I-H-J)^{-1}$ exists and is bounded when γ satisfies (46). Therefore, by the resolvent equation

$$[\gamma I-H-J-S]^{-1}=[I-(\gamma I-H-J)^{-1}S]^{-1}(\gamma I-H-J)^{-1}$$

we conclude that $[\gamma I-H-J-S]^{-1}$ exists and is bounded for all γ in (46). In other words, (46) belongs to the resolvent set of $(A+U+E)^*=H+J+S$.

Particularly, if $\gamma=ia,\ a\in\mathbb{R}\setminus\{0\},\ i^2=-1$, then all $\gamma=ia$ belong to (46). In fact,

$$\lambda+\mu>0,\quad \sqrt{a^2+\lambda^2}>\lambda, \tag{47}$$

$$\sqrt{a^2+(\lambda+\mu)^2}\sqrt{a^2+(\lambda+b)^2}>(\lambda+\mu)(\lambda+b)>\mu b. \tag{48}$$

To prove other inequalities we introduce several functions and discuss their properties. Let

$$\begin{aligned}
f_1(a) &= \frac{\lambda^2\mu(\lambda+\mu)+\lambda^2\sqrt{a^2+(\lambda+\mu)^2}\sqrt{a^2+(\lambda+b)^2}}{(\lambda+\mu)[\sqrt{(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]}-\lambda\mu b]} \\
&= \frac{\lambda^2}{\lambda+\mu}\Big\{\mu(\lambda+\mu)+[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\Big\} \\
&\quad\times\Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-1}.
\end{aligned}$$

It is easy to see that $f_1(a)$ is differentiable for all a. Moreover,

$$\begin{aligned}
\frac{df_1(a)}{da} &= \frac{\lambda^2}{\lambda+\mu}\Bigg\{\Big\{a[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&\quad+a[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\} \\
&\quad\times\Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-1} \\
&\quad-\Big\{\mu(\lambda+\mu)+[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\Big\} \\
&\quad\times\Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-2} \\
&\quad\times\Big\{a(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&\quad+a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&\quad+a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}\Bigg\} \\
&= \frac{a\lambda^2}{\lambda+\mu}\Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-2} \\
&\quad\times\Bigg\{\Big\{[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&\quad+[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\} \\
&\quad\times\Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\} \\
&\quad-[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&\quad\times\Big\{(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&\quad+(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&\quad+(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&- \mu(\lambda+\mu)\Big\{(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}\Bigg\} \\
=& \frac{a\lambda^2}{\lambda+\mu}\left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-2} \\
&\times\Bigg\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+b)^2] + (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2] \\
&- \lambda\mu b\Big\{[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ [a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\} \\
&- (a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2] \\
&- (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+b)^2] \\
&- (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2] \\
&- \mu(\lambda+\mu)\Big\{(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}\Bigg\} \\
=& -\frac{a\lambda^2}{\lambda+\mu}\left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-2} \\
&\times\Bigg\{\lambda\mu b\Big\{[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ [a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\} \\
&+ (a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2] \\
&+ \mu(\lambda+\mu)\Big\{(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}\Bigg\}
\end{aligned}
$$

$$= \begin{cases} > 0 & a < 0, \\ < 0 & a > 0. \end{cases}$$

From which together with

$$\begin{aligned} f_1(0) &= \frac{\lambda^2\mu(\lambda+\mu) + \lambda^2(\lambda+\mu)(\lambda+b)}{(\lambda+\mu)[\lambda(\lambda+\mu)(\lambda+b) - \lambda\mu b]} \\ &= \frac{\lambda\mu + \lambda(\lambda+b)}{\lambda(\lambda+\mu+b)} = 1 \end{aligned}$$

we know that

$$f_1(a) < f_1(0) = 1, \quad a \in \mathbb{R} \setminus \{0\}. \tag{49}$$

To prove the next inequality we consider

$$\begin{aligned} f_2(a) &= \frac{\lambda(\lambda+\mu)\sqrt{a^2+\lambda^2}\sqrt{a^2+(\lambda+\mu)^2} + \lambda^2 b\sqrt{a^2+(\lambda+\mu)^2}}{(\lambda+\mu)[\sqrt{(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]} - \lambda\mu b]} \\ &= \frac{\lambda}{\lambda+\mu}\Big\{(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}} \\ &\quad + \lambda b[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}\Big\} \\ &\quad \times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\}^{-1}. \end{aligned}$$

It is clear that $f_2(a)$ is continuously differentiable for all a. In addition,

$$\begin{aligned} \frac{df_2(a)}{da} &= \frac{\lambda}{\lambda+\mu}\Bigg\{\Big\{a(\lambda+\mu)(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}} \\ &\quad + a(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}} \\ &\quad + \lambda ba[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}\Big\} \\ &\quad \times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\}^{-1} \\ &\quad - \Big\{(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}} \\ &\quad + \lambda b[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}\Big\} \\ &\quad \times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\}^{-2} \\ &\quad \times \Big\{a(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\ &\quad + a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\quad + a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}\Bigg\}\\
&= \frac{\lambda a}{\lambda+\mu}\Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\}^{-2}\\
&\quad \times \Bigg\{(\lambda+\mu)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&\quad + (\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&\quad + \lambda b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&\quad - \lambda\mu b\Big\{(\lambda+\mu)(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}\\
&\quad + (\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}\\
&\quad + \lambda b[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}\Big\}\\
&\quad - (\lambda+\mu)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&\quad - (\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&\quad - (\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^{-\frac{1}{2}}\\
&\quad - \lambda b(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&\quad - \lambda b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&\quad - \lambda b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Bigg\}\\
&= -\frac{\lambda a}{\lambda+\mu}\Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\}^{-2}\\
&\quad \times \Bigg\{\lambda\mu b\Big\{(\lambda+\mu)(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}\\
&\quad + (\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}\\
&\quad + \lambda b[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}\Big\}\\
&\quad + (\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^{-\frac{1}{2}}\\
&\quad + \lambda b(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&\quad - \lambda b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Bigg\}
\end{aligned}
$$

$$= \begin{cases} > 0 & a < 0, \\ < 0 & a > 0. \end{cases}$$

$$\Longrightarrow$$

$$f_2(a) < f(0), \quad a \neq 0,$$

$$f_2(0) = \frac{\lambda(\lambda+\mu)\lambda(\lambda+\mu) + \lambda^2 b(\lambda+\mu)}{(\lambda+\mu)[\lambda(\lambda+\mu)(\lambda+b) - \lambda\mu b]} = \frac{\lambda(\lambda+\mu)+\lambda b}{\lambda(\lambda+\mu+b)} = 1.$$

Hence

$$f_2(a) < f_2(0) = 1, \quad a \in \mathbb{R} \setminus \{0\}. \tag{50}$$

We consider the following function

$$\begin{aligned} f_3(a) = & \frac{\lambda}{(\lambda+\mu)\sqrt{a^2+(\lambda+b)^2}\left[\sqrt{(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]} - \lambda\mu b\right]} \\ & \times \Big\{\mu b(\lambda+\mu)\sqrt{a^2+\lambda^2} \\ & + b\sqrt{(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]} \\ & + (\lambda+\mu)\left[\sqrt{(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]} - \lambda\mu b\right]\Big\} \\ = & \frac{\lambda}{\lambda+\mu}\Bigg\{\mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}} \\ & + b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\ & + (\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b(\lambda+\mu)\Bigg\} \\ & \times \left[a^2+(\lambda+b)^2\right]^{-\frac{1}{2}} \\ & \times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\}^{-1}. \end{aligned}$$

It is obvious that $f_3(a)$ is continuously differentiable for all a.

$$\begin{aligned} \frac{df_3(a)}{da} = & \frac{\lambda}{\lambda+\mu}\Bigg\{\Big\{a\mu b(\lambda+\mu)(a^2+\lambda^2)^{-\frac{1}{2}} \\ & + ab(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\ & + ab(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\ & + ab(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\ & + a(\lambda+\mu)(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&+ a(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ a(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\} \\
&\times \left[a^2+(\lambda+b)^2\right]^{-\frac{1}{2}} \\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\}^{-1} \\
&- a\Big\{\mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}} + b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b(\lambda+\mu)\Big\} \\
&\times \left[a^2+(\lambda+b)^2\right]^{-\frac{3}{2}} \\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\}^{-1} \\
&- a\Big\{\mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}} \\
&+ b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&- \lambda\mu b(\lambda+\mu)\Big\} \\
&\times \left[a^2+(\lambda+b)^2\right]^{-\frac{1}{2}} \\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\}^{-2} \\
&\times \Big\{a(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}\Bigg\} \\
= &\frac{a\lambda}{\lambda+\mu}\left[a^2+(\lambda+b)^2\right]^{-\frac{3}{2}} \\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\}^{-2} \\
&\times \Bigg\{\Big\{\mu b(\lambda+\mu)(a^2+\lambda^2)^{-\frac{1}{2}} \\
&+ b(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&+ b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\
&+ (\lambda+\mu)(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\} \\
&\times [a^2+(\lambda+b)^2] \\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\} \\
&- \Big\{\mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}} \\
&+ b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b(\lambda+\mu)\Big\} \\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\} \\
&- \Big\{\mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+b)^2] \\
&+ b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&+ (\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&- \lambda\mu b(\lambda+\mu)[a^2+(\lambda+b)^2]\Big\} \\
&\times \Big\{(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}\Bigg\} \\
= &\frac{a\lambda}{\lambda+\mu}\left[a^2+(\lambda+b)^2\right]^{-\frac{3}{2}} \\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\Big\}^{-2} \\
&\times \Bigg\{ - \lambda(\mu b)^2(\lambda+\mu)(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+b)^2] \\
&- \lambda\mu b^2(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&- \lambda\mu b^2(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&- \lambda\mu b^2(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda\mu b(\lambda+\mu)(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&- \lambda\mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&- \lambda\mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ \mu b(\lambda+\mu)[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&+ b[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^2 \\
&+ b(a^2+\lambda^2)[a^2+(\lambda+b)^2]^2 \\
&+ b(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2] \\
&+ (\lambda+\mu)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^2 \\
&+ (\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+b)^2]^2 \\
&+ (\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2] \\
&- \mu b(\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&- b(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2] \\
&- (\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2] \\
&+ \lambda\mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ \lambda(\mu b)^2(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}} \\
&+ \lambda\mu b^2(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ \lambda\mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&- (\lambda\mu b)^2(\lambda+\mu) \\
&- \mu b(\lambda+\mu)[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&- \mu b(\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&- \mu b(\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&- b[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^2 \\
&- b(a^2+\lambda^2)[a^2+(\lambda+b)^2]^2 \\
&- b(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2] \\
&- (\lambda+\mu)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]^2 \\
&- (\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+b)^2]^2 \\
&- (\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2] \\
&+ \lambda\mu b(\lambda+\mu)(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&+ \lambda\mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda\mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\Big\} \\
&= -\frac{a\lambda}{\lambda+\mu}\left[a^2+(\lambda+b)^2\right]^{-\frac{3}{2}} \\
&\quad \times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]\frac{1}{2}-\lambda\mu b\right\}^{-2} \\
&\quad \times \Big\{\lambda(\mu b)^2(\lambda+\mu)(a^2+\lambda^2)^{-\frac{1}{2}}[(\lambda+b)^2-\lambda^2] \\
&\quad + \lambda\mu b^2(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&\quad + \lambda\mu b^2(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&\quad + \mu b(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\left\{(a^2+\lambda^2)^{\frac{1}{2}}-\lambda\right\} \\
&\quad + b(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2] \\
&\quad + (\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&\qquad \times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\right\} \\
&\quad + (\lambda\mu b)^2(\lambda+\mu) \\
&\quad + \mu b(\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{3}{2}} \\
&\quad + \mu b(\lambda+\mu)(a^2+\lambda^2)[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\Big\} \\
&= \begin{cases} >0 & a<0 \\ <0 & a>0 \end{cases} \\
&\Longrightarrow
\end{aligned}
$$

$$f_3(a) < f_3(0), \quad a \neq 0. \tag{51}$$

$$
\begin{aligned}
f_3(0) &= \frac{\lambda^2\mu b(\lambda+\mu)+\lambda^2 b(\lambda+\mu)(\lambda+b)}{(\lambda+\mu)(\lambda+b)[\lambda(\lambda+\mu)(\lambda+b)-\lambda\mu b]} + \frac{\lambda}{\lambda+b} \\
&= \frac{\lambda\mu b+\lambda b(\lambda+b)}{(\lambda+b)[(\lambda+\mu)(\lambda+b)-\mu b]} + \frac{\lambda}{\lambda+b} = \frac{b}{\lambda+b}+\frac{\lambda}{\lambda+b} \\
&= 1.
\end{aligned} \tag{52}
$$

(51) and (52) show that

$$f_3(a) < 1, \quad a \in \mathbb{R}\setminus\{0\}. \tag{53}$$

We discuss the following function

$$\begin{aligned} f_4(a) &= \frac{\lambda(\lambda+\mu+b)\sqrt{a^2+(\lambda+\mu)^2}}{(\lambda+\mu)[\sqrt{[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]}-\mu b]} \\ &= \frac{\lambda(\lambda+\mu+b)}{\lambda+\mu}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}} \\ &\quad \times \left\{[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\mu b\right\}^{-1}, \end{aligned}$$

then $f_4(a)$ is differentiable for all $a \in \mathbb{R}$. And

$$\begin{aligned} \frac{df_4(a)}{da} &= \frac{\lambda(\lambda+\mu+b)}{\lambda+\mu} \\ &\quad \times \left\{ a[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}\left\{[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\mu b\right\}^{-1} \right. \\ &\quad - [a^2+(\lambda+\mu)^2]^{\frac{1}{2}}\left\{[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\mu b\right\}^{-2} \\ &\times \left.\left\{a[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}+a[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\right\}\right\} \\ &= \frac{a\lambda(\lambda+\mu+b)}{\lambda+\mu}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}} \\ &\times\left\{[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\mu b\right\}^{-2}\left\{[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\mu b \right. \\ &\quad - [a^2+(\lambda+\mu)^2]\left\{[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \right. \\ &\quad \left.\left. + [a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\right\}\right\} = -\frac{a\lambda(\lambda+\mu+b)}{\lambda+\mu}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}} \\ &\times\left\{[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\mu b\right\}^{-2}\left\{\mu b+[a^2+(\lambda+\mu)^2]^{\frac{3}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\right\} \\ &= \begin{cases} >0 & a<0 \\ <0 & a>0 \end{cases} \implies \quad f_4(a) < f_4(0), \quad a \neq 0, \end{aligned}$$

$$\begin{aligned} f_4(0) &= \frac{\lambda(\lambda+\mu)(\lambda+\mu+b)}{(\lambda+\mu)[(\lambda+\mu)(\lambda+b)-\mu b]} \\ &= \frac{\lambda(\lambda+\mu+b)}{\lambda(\lambda+\mu+b)} = 1. \end{aligned}$$

Therefore

$$f_4(a) < f_4(0) = 1, \quad a \in \mathbb{R} \setminus \{0\}. \tag{54}$$

To obtain the next inequality, we see the following differentiable function

$$\begin{aligned}
f_5(a) = {} & \frac{\lambda\mu(\lambda+\mu)\sqrt{a^2+\lambda^2}+\lambda^2\mu b}{(\lambda+\mu)[\sqrt{(a^2+\lambda^2)[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]}-\lambda\mu b]} \\
& + \frac{\lambda}{\lambda+\mu} \\
= {} & \frac{\lambda\mu}{\lambda+\mu}\left\{(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}+\lambda b\right\} \\
& \times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\right\}^{-1} \\
& + \frac{\lambda}{\lambda+\mu},
\end{aligned}$$

then

$$\begin{aligned}
\frac{df_5(a)}{da} = {} & \frac{\lambda\mu}{\lambda+\mu}\Bigg\{(\lambda+\mu)a(a^2+\lambda^2)^{-\frac{1}{2}}\Bigg\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2 \\
& + (\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Bigg\}^{-1} \\
& - \left[(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}+\lambda b\right]\Bigg\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2 \\
& + (\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Bigg\}^{-2} \\
& \times \Big[a(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
& + a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
& + a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big]\Bigg\} \\
= {} & \frac{a\lambda\mu}{\lambda+\mu}(a^2+\lambda^2)^{-\frac{1}{2}} \\
& \times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\right\}^{-2} \\
& \times \Bigg\{(\lambda+\mu)\left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\right\} \\
& - \left[(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}+\lambda b\right](a^2+\lambda^2)^{\frac{1}{2}} \\
& \times \Big[(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+ (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+ (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big]\Bigg\} \\
= &\frac{a\lambda\mu}{\lambda+\mu}(a^2+\lambda^2)^{-\frac{1}{2}} \\
&\times\Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-2} \\
&\times\Bigg\{(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&-\lambda\mu b(\lambda+\mu) \\
&-(\lambda+\mu)(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&-(\lambda+\mu)(a^2+\lambda^2)^{\frac{3}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&-(\lambda+\mu)(a^2+\lambda^2)^{\frac{3}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\
&-\lambda b[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&-\lambda b(a^2+\lambda^2)[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&-\lambda b(a^2+\lambda^2)[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Bigg\} \\
= &-\frac{a\lambda\mu}{\lambda+\mu}(a^2+\lambda^2)^{-\frac{1}{2}} \\
&\times\Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-2} \\
&\times\Bigg\{\lambda\mu b(\lambda+\mu) \\
&+(\lambda+\mu)(a^2+\lambda^2)^{\frac{3}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+(\lambda+\mu)(a^2+\lambda^2)^{\frac{3}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\
&+\lambda b[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+\lambda b(a^2+\lambda^2)[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} \\
&+\lambda b(a^2+\lambda^2)[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Bigg\}
\end{aligned}
$$

$$= \begin{cases} > 0 & a < 0, \\ < 0 & a > 0, \end{cases}$$
$$\Longrightarrow$$
$$f_5(a) < f_5(0), \quad a \neq 0,$$

$$\begin{aligned} f_5(0) &= \frac{\lambda^2\mu(\lambda+\mu)+\lambda^2\mu b}{(\lambda+\mu)[\lambda(\lambda+\mu)(\lambda+b)-\lambda\mu b]} + \frac{\lambda}{\lambda+\mu} \\ &= \frac{\lambda\mu(\lambda+\mu)+\lambda\mu b}{(\lambda+\mu)[(\lambda+\mu)(\lambda+b)-\mu b]} + \frac{\lambda}{\lambda+\mu} \\ &= \frac{\lambda\mu(\lambda+\mu+b)}{(\lambda+\mu)\lambda(\lambda+\mu+b)} + \frac{\lambda}{\lambda+\mu} \\ &= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} = 1. \end{aligned}$$

So

$$f_5(a) < f_5(0) = 1, \quad \forall a \in \mathbb{R} \setminus \{0\}. \tag{55}$$

Let

$$\begin{aligned} f_6(a) &= \lambda\mu^2 b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\ &\quad \times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-1} \\ &\quad + \lambda\mu[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\ &\quad + \frac{\lambda\mu b}{\lambda+\mu}(a^2+\lambda^2)^{\frac{1}{2}}\left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-1} \\ &\quad + \frac{\lambda}{\lambda+\mu}. \end{aligned}$$

It is easy to see that $f_6(a)$ is differentiable for all $a \in \mathbb{R}$. By calculating we have

$$\begin{aligned} \frac{df_6(a)}{da} &= \lambda\mu^2 ba(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\ &\quad \times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-1} \\ &\quad - \lambda\mu^2 ba(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{3}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\ &\quad \times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-1} \\ &\quad - \lambda\mu^2 ba(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{3}{2}} \\ &\quad \times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-1} \\ &\quad - \lambda\mu^2 b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-2}\\
&\times \Big\{a(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&+a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&+a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}\\
&+\lambda\mu\Big\{-a[a^2+(\lambda+\mu)^2]^{-\frac{3}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\\
&-a[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{3}{2}}\Big\}\\
&+\frac{\lambda\mu b}{\lambda+\mu}a(a^2+\lambda^2)^{-\frac{1}{2}}\\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-1}\\
&-\frac{\lambda\mu b}{\lambda+\mu}(a^2+\lambda^2)^{\frac{1}{2}}\\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-2}\\
&\times \Big\{a(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&+a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&+a(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}\\
=&\,\lambda\mu^2ba(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-2}\\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\\
&-(a^2+\lambda^2)(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\Big\}\\
&-\lambda\mu^2ba(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{3}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-1}\\
&-\lambda\mu^2ba(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{3}{2}}\\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda\mu^2 ba(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\
&\times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-2} \\
&\times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\right. \\
&\left. + (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\right\} \\
&- \lambda\mu a\Big\{[a^2+(\lambda+\mu)^2]^{-\frac{3}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\
&+ [a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{3}{2}}\Big\} \\
&+ \frac{\lambda\mu b}{\lambda+\mu} a(a^2+\lambda^2)^{-\frac{1}{2}} \\
&\times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-2} \\
&\times \Bigg\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b \\
&- (a^2+\lambda^2)(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\Bigg\} \\
&- \frac{\lambda\mu b}{\lambda+\mu} a(a^2+\lambda^2)^{\frac{1}{2}} \\
&\times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-2} \\
&\times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\right. \\
&\left. + (a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\right\} \\
= &-a\Bigg\{(\lambda\mu b)^2\mu(a^2+\lambda^2)^{-\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\
&\times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-2} \\
&+ \lambda\mu^2 b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{3}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}} \\
&\times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-1} \\
&+ \lambda\mu^2 b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{3}{2}} \\
&\times \left\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}} - \lambda\mu b\right\}^{-1} \\
&+ \lambda\mu^2 b(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-2}\\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&+(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}\\
&+\lambda\mu\Big\{[a^2+(\lambda+\mu)^2]^{-\frac{3}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\\
&+[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{3}{2}}\Big\}+\frac{(\lambda\mu b)^2}{\lambda+\mu}(a^2+\lambda^2)^{-\frac{1}{2}}\\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-2}\\
&+\frac{\lambda\mu b}{\lambda+\mu}(a^2+\lambda^2)^{\frac{1}{2}}\\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}-\lambda\mu b\Big\}^{-2}\\
&\times \Big\{(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{-\frac{1}{2}}[a^2+(\lambda+b)^2]^{\frac{1}{2}}\\
&+(a^2+\lambda^2)^{\frac{1}{2}}[a^2+(\lambda+\mu)^2]^{\frac{1}{2}}[a^2+(\lambda+b)^2]^{-\frac{1}{2}}\Big\}\Bigg\}=\begin{cases}>0 & a<0,\\ <0 & a>0,\end{cases}\\
&\Longrightarrow f_6(a)<f_6(0),\quad a\neq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
f_6(0)&=\frac{\lambda^2\mu^2 b}{(\lambda+\mu)(\lambda+b)[\lambda(\lambda+\mu)(\lambda+b)-\lambda\mu b]}+\frac{\lambda\mu}{(\lambda+\mu)(\lambda+b)}\\
&\quad+\frac{\lambda^2\mu b}{(\lambda+\mu)[\lambda(\lambda+\mu)(\lambda+b)-\lambda\mu b]}+\frac{\lambda}{\lambda+\mu}\\
&=\frac{\lambda\mu^2 b+\lambda\mu[(\lambda+\mu)(\lambda+b)-\mu b]}{(\lambda+\mu)(\lambda+b)[(\lambda+\mu)(\lambda+b)-\mu b]}+\frac{\lambda\mu b+\lambda[(\lambda+\mu)(\lambda+b)-\mu b]}{(\lambda+\mu)[(\lambda+\mu)(\lambda+b)-\mu b]}\\
&=\frac{\lambda\mu}{(\lambda+\mu)(\lambda+b)-\mu b}+\frac{\lambda(\lambda+b)}{(\lambda+\mu)(\lambda+b)-\mu b}\\
&=\frac{\lambda\mu+\lambda(\lambda+b)}{(\lambda+\mu)(\lambda+b)-\mu b}=\frac{\lambda(\lambda+\mu+b)}{\lambda(\lambda+\mu+b)}=1.
\end{aligned}
$$

The above formulas give

$$
f_6(a)<f_6(0)=1,\quad a\in\mathbb{R}\setminus\{0\}. \tag{56}
$$

To obtain the last inequality we investigate the following function

$$
f_7(a)=\frac{\lambda\mu(\lambda+\mu+b)}{(\lambda+\mu)[\sqrt{a^2+(\lambda+\mu)^2}\sqrt{a^2+(\lambda+b)^2}-\mu b]}+\frac{\lambda}{\lambda+\mu}.
$$

Since

$$\sqrt{[a^2+(\lambda+\mu)^2][a^2+(\lambda+b)^2]} > (\lambda+\mu)(\lambda+b), \quad a \neq 0,$$

$$\begin{aligned} f_7(a) &< \frac{\lambda\mu(\lambda+\mu+b)}{(\lambda+\mu)[(\lambda+\mu)(\lambda+b)-\mu b]} + \frac{\lambda}{\lambda+\mu} \\ &= \frac{\lambda\mu(\lambda+\mu+b)}{(\lambda+\mu)\lambda(\lambda+\mu+b)} + \frac{\lambda}{\lambda+\mu} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \\ &= 1, \quad a \in \mathbb{R}\setminus\{0\}. \end{aligned} \tag{57}$$

(47), (48), (49), (50), (53), (54), (55), (56) and (57) show that all points on the imaginary axis except for zero belong to the resolvent set of $(A+U+E)^*$. By using the relation between the spectrum of $A+U+E$ and the spectrum of $(A+U+E)^*$ we know that all points on the imaginary axis except for zero belong to the resolvent set of $A+U+E$. □

Lemma 3. *0 is an eigenvalue of* $(A+U+E)^*$.

Proof. We consider the equation $(A+U+E)^*(V^*,W^*) = (H+J+S)(V^*,W^*) = 0$, i.e.,

$$-\lambda Q^* + \lambda W_0^*(0) = 0, \tag{58}$$

$$bQ^* - (\lambda+b)V_0^* + \lambda V_1^* = 0, \tag{59}$$

$$-(\lambda+b)V_1^* + bW_0^*(0) + \lambda V_2^* = 0, \tag{60}$$

$$-(\lambda+b)V_2^* + bW_0^*(0) + \lambda V_{\mathcal{M}+1}^* = 0, \tag{61}$$

$$\cdots\cdots$$

$$-(\lambda+b)V_{\mathcal{M}}^* + bW_0^*(0) + \lambda V_{\mathcal{M}+1}^* = 0, \tag{62}$$

$$-(\lambda+b)V_{\mathcal{M}+n}^* + bW_n^*(0) + \lambda V_{\mathcal{M}+n+1}^* = 0, \quad n \geq 1, \tag{63}$$

$$\frac{dW_n^*(x)}{dx} - (\lambda+\mu)W_n^*(x) + \mu V_n^*(x) + \lambda W_{n+1}^*(x) = 0, \quad n \geq 1, \tag{64}$$

$$W_n^*(\infty) = \alpha, \quad n \geq 0. \tag{65}$$

It is easy to see that

$$(V^*,W^*) = \left(\begin{pmatrix}1\\1\\\vdots\end{pmatrix}, \begin{pmatrix}1\\1\\\vdots\end{pmatrix}\right) \in D(H) = D((A+U+E)^*)$$

is a solution of the above equations (58)–(65), that is to say, 0 is an eigenvalue of $(A+U+E)^* = H+J+S$. □

In the following we discuss eigenvalue of $A+U+E$. To do this, we consider the equation $(A+U+E)(V,W)=0$ which is equivalent to

$$\lambda Q = bV_0, \tag{66}$$

$$(\lambda+b)V_0 = \mu\int_0^\infty W_0(x)dx, \tag{67}$$

$$(\lambda+b)V_n = \lambda V_{n-1} + \mu\int_0^\infty W_n(x)dx, \quad n\geq 1, \tag{68}$$

$$\frac{dW_0(x)}{dx} = -(\lambda+\mu)W_0(x), \tag{69}$$

$$\frac{dW_n(x)}{dx} = -(\lambda+\mu)W_n(x) + \lambda W_{n-1}(x), \quad n\geq 1, \tag{70}$$

$$W_0(0) = \lambda Q + b\sum_{r=1}^{\mathcal{M}} V_r, \tag{71}$$

$$W_n(0) = bV_{n+\mathcal{M}}, \quad n\geq 1. \tag{72}$$

By solving (69) and (70) we have

$$W_0(x) = W_0(0)e^{-(\lambda+\mu)x}, \tag{73}$$

$$W_n(x) = W_n(0)e^{-(\lambda+\mu)x} + \lambda e^{-(\lambda+\mu)x}\int_0^x W_{n-1}(\tau)e^{(\lambda+\mu)\tau}d\tau, \quad n\geq 1. \tag{74}$$

By using (73) and (74) repeatedly we calculate

$$\begin{aligned} W_1(x) &= W_1(0)e^{-(\lambda+\mu)x} + \lambda e^{-(\lambda+\mu)x}\int_0^x W_0(\tau)e^{(\lambda+\mu)\tau}d\tau \\ &= W_1(0)e^{-(\lambda+\mu)x} + \lambda e^{-(\lambda+\mu)x}\int_0^x W_0(0)d\tau \\ &= [W_1(0)+\lambda x W_0(0)]e^{-(\lambda+\mu)x}, \end{aligned} \tag{75}$$

$$\begin{aligned} W_2(x) &= W_2(0)e^{-(\lambda+\mu)x} + \lambda e^{-(\lambda+\mu)x}\int_0^x W_1(\tau)e^{(\lambda+\mu)\tau}d\tau \\ &= W_2(0)e^{-(\lambda+\mu)x} + \lambda e^{-(\lambda+\mu)x}\left[W_1(0)x + \lambda\frac{x^2}{2}W_0(0)\right] \\ &= \left[W_2(0) + \lambda x W_1(0) + \frac{(\lambda x)^2}{2}W_0(0)\right]e^{-(\lambda+\mu)x}, \end{aligned} \tag{76}$$

$$\begin{aligned} W_3(x) &= W_3(0)e^{-(\lambda+\mu)x} + \lambda e^{-(\lambda+\mu)x}\int_0^x W_2(\tau)e^{(\lambda+\mu)\tau}d\tau \\ &= W_3(0)e^{-(\lambda+\mu)x} + \lambda e^{-(\lambda+\mu)x}\left[W_2(0) + \lambda\frac{x^2}{2}W_1(0) + \frac{\lambda^2}{2}\frac{x^3}{3}W_0(0)\right] \end{aligned}$$

$$= \left[W_3(0) + \lambda x W_2(0) + \frac{(\lambda x)^2}{2} W_1(0) + \frac{(\lambda x)^3}{3!} W_0(0) \right] e^{-(\lambda+\mu)x}$$

$$= e^{-(\lambda+\mu)x} \sum_{k=0}^{3} \frac{(\lambda x)^k}{k!} W_{3-k}(0), \tag{77}$$

$\cdots\cdots$

$$W_n(x) = e^{-(\lambda+\mu)x} \sum_{k=0}^{n} \frac{(\lambda x)^k}{k!} W_{n-k}(0), \quad n \geq 0. \tag{78}$$

Inserting (78) into (67) and (68) and using $\int_0^\infty x^k e^{-(\lambda+\mu)x} dx = \frac{k!}{(\lambda+\mu)^{k+1}}$ for $k \geq 0$ we determine

$$V_0 = \frac{\mu}{\lambda+b} \int_0^\infty W_0(x) dx = \frac{\mu}{\lambda+b} \int_0^\infty W_0(0) e^{-(\lambda+\mu)x} dx$$

$$= \frac{\mu}{(\lambda+b)(\lambda+\mu)} W_0(0), \tag{79}$$

$$V_n = \frac{\lambda}{\lambda+b} V_{n-1} + \frac{\mu}{\lambda+b} \int_0^\infty W_n(x) dx$$

$$= \frac{\lambda}{\lambda+b} V_{n-1} + \frac{\mu}{\lambda+b} \int_0^\infty e^{-(\lambda+\mu)x} \sum_{k=0}^{n} \frac{(\lambda x)^k}{k!} W_{n-k}(0) dx$$

$$= \frac{\lambda}{\lambda+b} V_{n-1} + \frac{\mu}{\lambda+b} \sum_{k=0}^{n} W_{n-k}(0) \frac{\lambda^k}{k!} \int_0^\infty x^k e^{-(\lambda+\mu)x} dx$$

$$= \frac{\lambda}{\lambda+b} V_{n-1} + \frac{\mu}{\lambda+b} \sum_{k=0}^{n} W_{n-k}(0) \frac{\lambda^k}{k!} \frac{k!}{(\lambda+\mu)^{k+1}}$$

$$= \frac{\lambda}{\lambda+b} V_{n-1} + \frac{\mu}{\lambda+b} \sum_{k=0}^{n} \frac{\lambda^k}{(\lambda+\mu)^{k+1}} W_{n-k}(0), \quad n \geq 1. \tag{80}$$

By applying (79) and (80) repeatedly we deduce

$$V_1 = \frac{\lambda}{\lambda+b} V_0 + \frac{\mu}{\lambda+b} \sum_{k=0}^{1} \frac{\lambda^k}{(\lambda+\mu)^{k+1}} W_{1-k}(0)$$

$$= \frac{\lambda}{\lambda+b} \frac{\mu}{(\lambda+b)(\lambda+\mu)} W_0(0) + \frac{\mu}{\lambda+b} \left\{ \frac{1}{\lambda+\mu} W_1(0) + \frac{\lambda}{(\lambda+\mu)^2} W_0(0) \right\}$$

$$= \frac{\mu}{(\lambda+b)(\lambda+\mu)} W_1(0) + \left\{ \frac{\lambda\mu}{(\lambda+b)^2(\lambda+\mu)} + \frac{\lambda\mu}{(\lambda+b)(\lambda+\mu)^2} \right\} W_0(0)$$

$$= \frac{\mu}{(\lambda+b)(\lambda+\mu)} W_1(0) + \sum_{k=0}^{1} \frac{\lambda\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{2-k}} W_0(0), \tag{81}$$

$$\begin{aligned}
V_2 &= \frac{\lambda}{\lambda+b}V_1 + \frac{\mu}{\lambda+b}\sum_{k=0}^{2}\frac{\lambda^k}{(\lambda+b)^{k+1}}W_{2-k}(0)\\
&= \frac{\lambda\mu}{(\lambda+b)^2(\lambda+\mu)}W_1(0) + \sum_{k=0}^{1}\frac{\lambda^2\mu}{(\lambda+b)^{k+2}(\lambda+\mu)^{2-k}}W_0(0)\\
&\quad + \frac{\mu}{\lambda+b}\left\{\frac{1}{\lambda+\mu}W_2(0) + \frac{\lambda}{(\lambda+\mu)^2}W_1(0) + \frac{\lambda^2}{(\lambda+\mu)^3}W_0(0)\right\}\\
&= \frac{\mu}{(\lambda+b)(\lambda+\mu)}W_2(0) + \sum_{k=0}^{1}\frac{\lambda\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{2-k}}W_1(0)\\
&\quad + \sum_{k=0}^{2}\frac{\lambda^2\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{3-k}}W_0(0),
\end{aligned} \tag{82}$$

$$\begin{aligned}
V_3 &= \frac{\lambda}{\lambda+b}V_2 + \frac{\mu}{\lambda+b}\sum_{k=0}^{3}\frac{\lambda^k}{(\lambda+\mu)^{k+1}}W_{3-k}(0)\\
&= \frac{\lambda}{\lambda+b}\Big\{\frac{\mu}{(\lambda+b)(\lambda+\mu)}W_2(0) + \sum_{k=0}^{1}\frac{\lambda\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{2-k}}W_1(0)\\
&\quad + \sum_{k=0}^{2}\frac{\lambda^2\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{3-k}}W_0(0)\Big\} + \frac{\mu}{\lambda+b}\Big\{\frac{1}{\lambda+\mu}W_3(0)\\
&\quad + \frac{\lambda}{(\lambda+\mu)^2}W_2(0) + \frac{\lambda^2}{(\lambda+\mu)^3}W_1(0) + \frac{\lambda^3}{(\lambda+\mu)^4}W_0(0)\Big\}\\
&= \frac{\mu}{(\lambda+b)(\lambda+\mu)}W_3(0) + \left\{\frac{\lambda\mu}{(\lambda+b)^2(\lambda+\mu)} + \frac{\lambda\mu}{(\lambda+b)(\lambda+\mu)^2}\right\}W_2(0)\\
&\quad + \left\{\sum_{k=0}^{1}\frac{\lambda^2\mu}{(\lambda+b)^{k+2}(\lambda+\mu)^{2-k}} + \frac{\lambda^2\mu}{(\lambda+b)(\lambda+\mu)^3}\right\}W_1(0)\\
&\quad + \left\{\sum_{k=0}^{2}\frac{\lambda^3\mu}{(\lambda+b)^{k+2}(\lambda+\mu)^{3-k}} + \frac{\lambda^3\mu}{(\lambda+b)(\lambda+\mu)^4}\right\}W_0(0)\\
&= \frac{\mu}{(\lambda+b)(\lambda+\mu)}W_3(0) + \sum_{k=0}^{1}\frac{\lambda\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{2-k}}W_2(0)\\
&\quad + \sum_{k=0}^{2}\frac{\lambda^2\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{3-k}}W_1(0) + \sum_{k=0}^{3}\frac{\lambda^3\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{4-k}}W_0(0),
\end{aligned} \tag{83}$$

$$
\begin{aligned}
V_4 &= \frac{\lambda}{\lambda+b}V_3 + \frac{\mu}{\lambda+b}\sum_{k=0}^{4}\frac{\lambda^k}{(\lambda+\mu)^{k+1}}W_{4-k}(0) \\
&= \frac{\lambda}{\lambda+b}\Big\{\frac{\mu}{(\lambda+b)(\lambda+\mu)}W_3(0) + \sum_{k=0}^{1}\frac{\lambda\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{2-k}}W_2(0) \\
&\quad + \sum_{k=0}^{2}\frac{\lambda^2\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{3-k}}W_1(0) + \sum_{k=0}^{3}\frac{\lambda^3\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{4-k}}W_0(0)\Big\} \\
&\quad + \frac{\mu}{\lambda+\mu}\Big\{\frac{1}{\lambda+\mu}W_4(0) + \frac{\lambda}{(\lambda+\mu)^2}W_3(0) \\
&\quad + \frac{\lambda^2}{(\lambda+\mu)^3}W_2(0) + \frac{\lambda^3}{(\lambda+\mu)^4}W_1(0) + \frac{\lambda^4}{(\lambda+\mu)^5}W_0(0)\Big\} \\
&= \frac{\mu}{(\lambda+b)(\lambda+\mu)}W_4(0) + \sum_{k=0}^{1}\frac{\lambda\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{2-k}}W_3(0) \\
&\quad + \sum_{k=0}^{2}\frac{\lambda^2\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{3-k}}W_2(0) + \sum_{k=0}^{3}\frac{\lambda^3\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{4-k}}W_1(0) \\
&\quad + \sum_{k=0}^{4}\frac{\lambda^4\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{5-k}}W_0(0), \qquad (84)
\end{aligned}
$$

$\cdots\cdots$

$$
\begin{aligned}
V_n &= \frac{\mu}{(\lambda+b)(\lambda+\mu)}W_n(0) + \sum_{k=0}^{1}\frac{\lambda\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{2-k}}W_{n-1}(0) \\
&\quad + \sum_{k=0}^{2}\frac{\lambda^2\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{3-k}}W_{n-2}(0) + \sum_{k=0}^{3}\frac{\lambda^3\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{4-k}}W_{n-3}(0) \\
&\quad + \cdots\cdots \\
&\quad + \sum_{k=0}^{n-2}\frac{\lambda^{n-2}\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{n-1-k}}W_2(0) + \sum_{k=0}^{n-1}\frac{\lambda^{n-1}\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{n-k}}W_1(0) \\
&\quad + \sum_{k=0}^{n}\frac{\lambda^n\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{n+1-k}}W_0(0), \quad n \geq 0. \qquad (85)
\end{aligned}
$$

By (66), (71), (72), (79), (81) and (85) we have

$$
V_0 = \frac{\mu}{(\lambda+b)(\lambda+\mu)}b\sum_{r=0}^{\mathcal{M}}V_r = \frac{b\mu}{(\lambda+b)(\lambda+\mu)}\left[V_0 + \sum_{r=1}^{\mathcal{M}}V_r\right], \qquad (86)
$$

$$V_1 = \frac{\mu}{(\lambda+b)(\lambda+\mu)} b V_{\mathcal{M}+1} + \sum_{k=0}^{1} \frac{\lambda\mu}{(\lambda+b)^{k+1}(\lambda+\mu)^{2-k}} b \sum_{r=0}^{\mathcal{M}} V_r$$
$$= \frac{b\mu}{(\lambda+b)(\lambda+\mu)} V_{\mathcal{M}+1} + \sum_{k=0}^{1} \frac{\lambda\mu b}{(\lambda+b)^{k+1}(\lambda+\mu)^{2-k}} \sum_{r=0}^{\mathcal{M}} V_r, \tag{87}$$

$\cdots\cdots$

$$V_n = \frac{\mu b}{(\lambda+b)(\lambda+\mu)} V_{\mathcal{M}+n} + \sum_{k=0}^{1} \frac{\lambda\mu b}{(\lambda+b)^{k+1}(\lambda+\mu)^{2-k}} V_{\mathcal{M}+n-1}$$
$$+ \sum_{k=0}^{2} \frac{\lambda^2\mu b}{(\lambda+b)^{k+1}(\lambda+\mu)^{3-k}} V_{\mathcal{M}+n-2} + \sum_{k=0}^{3} \frac{\lambda^3\mu b}{(\lambda+b)^{k+1}(\lambda+\mu)^{4-k}} V_{\mathcal{M}+n-3}$$
$$+ \cdots\cdots$$
$$+ \sum_{k=0}^{n-2} \frac{\lambda^{n-2}\mu b}{(\lambda+b)^{k+1}(\lambda+\mu)^{n-k-1}} V_{\mathcal{M}+2} + \sum_{k=0}^{n-1} \frac{\lambda^{n-1}\mu b}{(\lambda+b)^{k+1}(\lambda+\mu)^{n-k}} V_{\mathcal{M}+1}$$
$$+ \sum_{k=0}^{n} \frac{\lambda^{n}\mu b}{(\lambda+b)^{k+1}(\lambda+\mu)^{n+1-k}} \sum_{r=0}^{\mathcal{M}} V_r, \quad n \ge 0. \tag{88}$$

From (86), (87) and (88) we know that if $V_1, V_2, \cdots, V_{\mathcal{M}}$ are given, then we can determine all V_k $(k \ge 0)$ and all $W_n(0)$ $(n \ge 0)$ by (71) and (72). Thus, we calculate $W_n(x)$ $(n \ge 0)$ by (78). In other words, geometric multiplicity of 0 is $\mathcal{M}$ if $(A+U+E)(V,W) = 0$ has a nonzero solution in $D(A)$. Especially, we deduce an explicit result:

Lemma 4. *Let $\mathcal{M} = 1$. If $\lambda(\mu+b) < \mu b$, then 0 is an eigenvalue of $A+U+E$ with geometric multiplicity one.*

Proof. By (86) and (87) we determine

$$V_0 = \frac{\mu b}{(\lambda+\mu)(\lambda+b)}[V_0 + V_1]$$
$$\Longrightarrow \frac{\mu b}{(\lambda+\mu)(\lambda+b)} V_1 = \left[1 - \frac{\mu b}{(\lambda+\mu)(\lambda+b)}\right] V_0 = \frac{\lambda(\lambda+\mu+b)}{(\lambda+\mu)(\lambda+b)} V_0$$
$$\Longrightarrow V_1 = \frac{\lambda(\lambda+\mu+b)}{\mu b} V_0.$$

$$V_1 = \frac{\mu b}{(\lambda+b)(\lambda+\mu)} V_2 + \left\{ \frac{\lambda\mu b}{(\lambda+b)(\lambda+\mu)^2} + \frac{\lambda\mu b}{(\lambda+b)^2(\lambda+\mu)} \right\} [V_0 + V_1]$$
$$= \frac{\mu b}{(\lambda+b)(\lambda+\mu)} V_2 + \frac{\lambda\mu b(2\lambda+\mu+b)}{(\lambda+b)^2(\lambda+\mu)^2} [V_0 + V_1]$$

$$= \frac{\mu b}{(\lambda+b)(\lambda+\mu)} V_2 + \frac{\lambda(2\lambda+\mu+b)}{(\lambda+b)(\lambda+\mu)} V_0$$
$$\Longrightarrow \frac{\mu b}{(\lambda+b)(\lambda+\mu)} V_2 = V_1 - \frac{\lambda(2\lambda+\mu+b)}{(\lambda+b)(\lambda+\mu)} V_0$$
$$= \frac{\lambda(\lambda+\mu+b)}{\mu b} V_0 - \frac{\lambda(2\lambda+\mu+b)}{(\lambda+b)(\lambda+\mu)} V_0$$
$$= \frac{\lambda^2[(\lambda+\mu+b)^2 - \mu b]}{\mu b(\lambda+b)(\lambda+\mu)} V_0$$
$$\Longrightarrow V_2 = \frac{\lambda^2[(\lambda+\mu+b)^2 - \mu b]}{(\mu b)^2} V_0.$$

It is not easy to determine expressions of all V_n and $W_n(x)$. To overcome this difficulties, we use another method. We define probability generating functions for $|z| < 1$

$$V(z) = \sum_{n=0}^{\infty} V_n z^n, \quad W(x,z) = \sum_{n=0}^{\infty} W_n(x) z^n,$$

then Theorem 1 ensures that $V(z)$ and $W(x,z)$ are well-defined. (67) and (68) give

$$(\lambda+b)V_0 + \sum_{n=1}^{\infty}(\lambda+b)V_n z^n = \sum_{n=1}^{\infty} \lambda V_{n-1} z^n + \sum_{n=1}^{\infty} \mu \int_0^{\infty} W_n(x) z^n dx + \mu \int_0^{\infty} W_0(x) dx$$

$$\Longrightarrow (\lambda+b)\sum_{n=0}^{\infty} V_n z^n = \lambda z \sum_{n=1}^{\infty} V_{n-1} z^{n-1} + \mu \sum_{n=0}^{\infty} \int_0^{\infty} W_n(x) z^n dx$$
$$\Longrightarrow (\lambda+b)V(z) = \lambda z V(z) + \mu \int_0^{\infty} W(x,z) dx$$
$$\Longrightarrow (\lambda+b-\lambda z)V(z) = \mu \int_0^{\infty} W(x,z) dx. \tag{89}$$

(69) and (70) imply

$$\sum_{n=0}^{\infty} \frac{\partial W_n(x) z^n}{\partial x} = \sum_{n=0}^{\infty} -(\lambda+\mu) W_n(x) z^n + \sum_{n=1}^{\infty} \lambda W_{n-1}(x) z^n$$
$$\Longrightarrow \frac{\partial \sum_{n=0}^{\infty} W_n(x) z^n}{\partial x} = -(\lambda+\mu)\sum_{n=0}^{\infty} W_n(x) z^n + \lambda z \sum_{n=1}^{\infty} W_{n-1}(x) z^{n-1}$$

$$\Longrightarrow \frac{\partial W(x,z)}{\partial x} = -(\lambda+\mu)W(x,z) + \lambda z W(x,z) = (\lambda z - \lambda - \mu)W(x,z)$$

$$\Longrightarrow W(x,z) = W(0,z)e^{(\lambda z - \lambda - \mu)x}. \tag{90}$$

From (66), (71) and (72) we derive

$$W_0(0) + \sum_{n=1}^{\infty} W_n(0)z^n = b(V_0+V_1) + \sum_{n=1}^{\infty} bV_{n+1}z^n$$

$$\Longrightarrow$$

$$\sum_{n=0}^{\infty} W_n(0)z^n = b(V_0+V_1) + \frac{1}{z}\sum_{n=1}^{\infty} bV_{n+1}z^{n+1} = b(V_0+V_1) + \frac{b}{z}\sum_{n=2}^{\infty} V_n z^n$$

$$\Longrightarrow$$

$$W(0,z) = b(V_0+V_1) + \frac{b}{z}\left(\sum_{n=0}^{\infty} V_n z^n - V_0 - V_1 z\right) = \left(b - \frac{b}{z}\right)V_0 + \frac{b}{z}V(z)$$

$$\Longrightarrow$$

$$W(0,z) = \frac{(z-1)b}{z}V_0 + \frac{b}{z}V(z). \tag{91}$$

By combining (89) and (90) with (91) we calculate (assume $\mathrm{Re}(\lambda z - \lambda - \mu) < 0$)

$$\begin{aligned}
(\lambda + b - \lambda z)V(z) &= \mu \int_0^{\infty} W(0,z)e^{(\lambda z - \lambda - \mu)x}dx \\
&= \mu \left[\frac{(z-1)b}{z}V_0 + \frac{b}{z}V(z)\right]\int_0^{\infty} e^{(\lambda z - \lambda - \mu)x}dx \\
&= \left[\frac{(z-1)b}{z}V_0 + \frac{b}{z}V(z)\right]\frac{\mu}{\lambda z - \lambda - \mu}e^{(\lambda z - \lambda - \mu)x}\Big|_{x=0}^{x=\infty} \\
&= -\frac{\mu}{\lambda z - \lambda - \mu}\frac{(z-1)b}{z}V_0 - \frac{\mu b}{z(\lambda z - \lambda - \mu)}V(z) \\
&\Longrightarrow \\
&\left[\lambda + b - \lambda z + \frac{\mu b}{z(\lambda z - \lambda - \mu)}\right]V(z) = -\frac{\mu}{\lambda z - \lambda - \mu}\frac{(z-1)b}{z}V_0 \\
&\Longrightarrow \\
&\frac{z(\lambda z - \lambda - \mu)(\lambda + b - \lambda z) + \mu b}{z(\lambda z - \lambda - \mu)}V(z) = -\frac{b\mu(z-1)}{z(\lambda z - \lambda - \mu)}V_0 \\
&\Longrightarrow
\end{aligned}$$

$$V(z) = \frac{\mu b(1-z)}{z(\lambda z - \lambda - \mu)(\lambda + b - \lambda z) + \mu b}V_0. \tag{92}$$

By (92) and the L'Hospital rule we deduce

$$\begin{aligned}\lim_{z\to 1} V(z) &= \lim_{z\to 1} \frac{\mu b(1-z)}{z(\lambda z-\lambda-\mu)(\lambda+b-\lambda z)+\mu b} V_0\\ &= \lim_{z\to 1} \frac{-\mu b}{(\lambda z-\lambda-\mu)(\lambda+b-\lambda z)+\lambda z(\lambda+b-\lambda z)-\lambda z(\lambda z-\lambda-\mu)} V_0\\ &= \frac{-\mu b}{-\mu b+\lambda b+\lambda\mu} V_0 = \frac{\mu b}{\mu b-\lambda(b+\mu)} V_0 < \infty. \qquad (93)\end{aligned}$$

By inserting (92) into (91) and using (90) and (93) we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} W_n(x) &= \lim_{z\to 1} W(x,z) = \lim_{z\to 1}\left[\frac{(z-1)b}{z}V_0 + \frac{b}{z}V(z)\right] e^{(\lambda z-\lambda-\mu)x}\\ &= \frac{\mu b^2}{\mu b-\lambda(\mu+b)} V_0 e^{-\mu x}\\ &\Longrightarrow \sum_{n=0}^{\infty} W_n(x)dx = \frac{b^2}{\mu b-\lambda(\mu+b)} V_0 < \infty. \qquad (94)\end{aligned}$$

(93) and (94) give

$$\|(V,W)\| = Q + \sum_{n=0}^{\infty} V_n + \sum_{n=0}^{\infty}\int_0^{\infty} W_n(x)dx < \infty.$$

This shows that 0 is an eigenvalue of $A+U+E$. Moreover, (86), (87) and (88) show that geometric multiplicity of 0 is one. □

Since Theorem 1, Lemma 2, Lemma 3 and Lemma 4 are just the conditions of Theorem 14 in Gupur et al [7], the following result is conclusion of Theorem 14 in Gupur et al [7].

Theorem 2. *Let $\mathcal{M} = 1$. If $\lambda(\mu+b) < \mu b$, then the time-dependent solution of the system (9) strongly converges to its steady-state solution, that is,*

$$\lim_{t\to\infty} \|(V,W)(\cdot,t) - \beta(V,W)(\cdot)\| = 0.$$

Here $(V,W)(x)$ is the eigenvector in Lemma 4 and β is a positive constant which is decided by the eigenvector in Lemma 3 and the initial condition of the system (9).

3. Asymptotic Behavior of the Queueing Length

According to Madan [9], the queueing length is defined by

$$L(t)=\sum_{n=0}^{\infty} nV_n(t)+\sum_{n=0}^{\infty} n\int_0^{\infty} W_n(x,t)dx.$$

In the following, by using the idea in Gupur et al [7] we discuss the asymptotic behavior of $L(t)$. Let

$$Y=\left\{(V,W)\in X \,\middle|\, \begin{array}{l} V=(Q,V_0,V_1,V_2,\cdots),\ Q\geq 0,\ V_n\geq 0, \\ W(x)=(W_0(x),W_1(x),W_2(x),\cdots),\ W_n(x)\geq 0, \\ x\in[0,\infty),\ n\geq 0 \end{array}\right\},$$

then Y is a cone. In Y we introduce an order relation

$$\begin{aligned}&(V^{(1)},W^{(1)})\leq(V^{(2)},W^{(2)})\\ &\Longleftrightarrow\\ &Q^{(1)}\leq Q^{(2)},\ V_n^{(1)}\leq V_n^{(2)},\ W_n^{(1)}(x)\leq W_n^{(2)}(x),\quad n\geq 0,\ x\geq 0,\end{aligned}$$

then under this relation Y becomes a partial order set. Since $T(t)$ is a positive linear operator, $T(t)$ is monotone increasing. If we take $Q=1$ in Lemma 4, then the eigenvector $(V,W)(x)$ in Lemma 4 and the initial condition $(V,W)(0)$ of the system (9) satisfy $(V,W)(x)\geq(V,W)(0)$, and therefore

$$T(t)(V,W)(x)\geq T(t)(V,W)(0)\Rightarrow(V,W)(x)\geq(V,W)(x,t)\Rightarrow V_n(t)\leq V_n,\quad n\geq 0,$$

$$W_n(x,t)\leq W_n(x),\quad n\geq 0\Rightarrow\int_0^{\infty}W_n(x,t)dx\leq\int_0^{\infty}W_n(x)dx,\quad n\geq 0$$

$$\begin{aligned}&\Longrightarrow L(t)\\ &=\sum_{n=0}^{\infty}nV_n(t)+\sum_{n=0}^{\infty}n\int_0^{\infty}W_n(x,t)dx\leq\sum_{n=0}^{\infty}nV_n+\sum_{n=0}^{\infty}n\int_0^{\infty}W_n(x)dx<\infty.\end{aligned}$$

So, by the Lebesgue Control Theorem, Theorem 2 and Lemma 4 we deduce

$$\begin{aligned}\lim_{t\to\infty}L(t)&=\lim_{t\to\infty}\sum_{n=0}^{\infty}nV_n(t)+\lim_{t\to\infty}\sum_{n=0}^{\infty}n\int_0^{\infty}W_n(x,t)dx\\ &=\sum_{n=0}^{\infty}n\lim_{t\to\infty}V_n(t)+\sum_{n=0}^{\infty}n\lim_{t\to\infty}\int_0^{\infty}W_n(x,t)dx\\ &=\sum_{n=0}^{\infty}nV_n+\sum_{n=0}^{\infty}n\int_0^{\infty}W_n(x)dx=\lim_{z\to 1}\frac{dV(z)}{dz}+\int_0^{\infty}\lim_{z\to 1}\frac{\partial W(x,z)}{\partial z}dx<\infty.\end{aligned}$$

By summarizing we have:

Theorem 3. *Let $\mathcal{M} = 1$. If $\lambda(\mu + b) < \mu b$, then the queueing length of the $M/M/1$ queueing system with compulsory server vacations converges to a positive number.*

If $\mathcal{M} > 1$, then we are unable to prove that 0 is an eigenvalue of $A+U+E$. If 0 is an eigenvalue of $A + U + E$, then by the above discussion its geometric multiplicity equals to $\mathcal{M}$, that is to say, the equations (66)-(72) have $\mathcal{M}$ linear independent solutions. Because of Theorem 1 the system (9) has a unique time-dependent solution, it is impossible that the time-dependent solution of the system (9) converges to its $\mathcal{M}$ steady-state solutions. What does the time-dependent solution of the system (9) converge to?

Acknowledgments

This work was supported by the Natural Science Foundation of China (No: 10861011). The author completed this wok during his visit at Department of Mathematics and Statistics, York University.

References

[1] B.T. Doshi, Queueing systems with vacation – A survey, *Queueing Systems*, **1** (1986), 29-66.

[2] Geni Gupur, Well-posedness of M/G/1 queueing model with single vacations, *Computers and Mathematics with Applications*, **44** (2002), 1041-1056.

[3] Geni Gupur, Semigroup method for M/G/1 queueing system with exceptional service time for the first customer in each busy period, *Indian Journal of Mathematics*, **44** (2002), 125-146.

[4] Geni Gupur, An eigenvalue of exhaustive-service M/M/1 queueing model with single vacations, *International Journal of Differential Equations and Applications*, **7**, No. 2 (2003), 211-237.

[5] Geni Gupur, Asymptotic property of the solution of M/M/1 queueing model with exceptional service time for the first customer in each busy period, *International Journal of Differential Equations and Applications*, **8**, (2003), 23-94.

[6] Geni Gupur, Guo Bao-Zhu, Asymptotic property of the solution of exhaustive-service M/M/1 queueing model with single vacations, *International Journal of Differential Equations and Applications*, **6**, No. 1 (2002), 29-51.

[7] Geni Gupur, Xue-Zhi Li, Guang-Tian Zhu, *Functional Analysis Method in Queueing Theory*, Research Information Ltd., Hertfordshire (2001).

[8] Zhi-Jie Lu, Geni Gupur, Well-posedness of the M/G/1 queueing model with compulsory server vacations, *Mathematics in Theory and Practice*, **40**, No. 5 (2010), 139-148.

[9] K.C. Madan, An M/G/1 queueing system with compulsory server vacations, *Trabajos de Investigacion Operativa*, **7** (1992), 105-115.

[10] Hideaki Takagi, Time-dependent analysis of M/G/1 vacation models with exhaustive service, *Queueing Systems*, **6** (1990), 369-390.