International Journal of Pure and Applied Mathematics

Volume 64 No. 3 2010, 331-338

SOME CLASSES OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS RELATED TO THE GAMMA FUNCTION

K. Nonlaopon[1 §], R. Kotnara[2]

Department of Mathematics
Khon Kaen University
Khon Kaen, 40002, THAILAND
[1]e-mail: nkamsi@kku.ac.th
[2]e-mail: adnapandy@gmail.com

Abstract: In this article, a sufficient condition for some classes of functions involving the gamma function which are logarithmically completely monotonic are established, and then some known results are extended.

AMS Subject Classification: 33B15, 26D07
Key Words: logarithmically complete monotonic function, gamma function

1. Introduction

Recall (see [2, 7, 11, 13, 14, 16, 17]) that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$(-1)^k[\ln f(x)]^{(k)} \geq 0 \tag{1.1}$$

for all $k \in \mathbb{N}$ on I. If the inequality (1.1) is strict, then f is called strictly logarithmically completely monotonic. The set of the logarithmically complete monotonic functions on I is denoted by $\mathcal{L}[I]$.

It is worthwhile to note that there have been a lot of literature on logarithmically completely monotonic functions related to the gamma function,

Received: March 5, 2009

§Correspondence author

psi function or polygamma function. Althought it is not practicable to read all of these papers, we still would like to offer some of them, for example [3, 4, 5, 9, 12, 15, 19] and the references therein, to the readers.

It is well known that the classical Eulers gamma function $\Gamma(x)$ is defined for $x > 0$ as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{1.2}$$

The logarithmic derivative of $\Gamma(x)$, denoted by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{1.3}$$

is called the psi or digamma function, and $\psi^{(n)}(x)$ for $n \in \mathbb{N}$ are known as the polygamma or multigamma functions. These functions play central roles in the theory of special functions and have lots of extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences.

In [6] and [8], while ones studied certain problems of traffic flow, the following double inequality was obtained for $n \in \mathbb{N}$

$$2\Gamma\left(n + \frac{1}{2}\right) \le \Gamma\left(\frac{1}{2}\right)\Gamma(n+1) \le 2^n \Gamma\left(n + \frac{1}{2}\right), \tag{1.4}$$

which can be rearranged for $n > 1$ as

$$1 \le \left[\frac{\Gamma(1/2)\Gamma(n+1)}{2\Gamma(n+1/2)}\right]^{1/(n-1)} \le 2. \tag{1.5}$$

In [20], by using the following double inequality due to J. Wendel in [21]:

$$\left(\frac{x}{x+a}\right)^{1-a} \le \frac{\Gamma(x+a)}{x^a \Gamma(x)} \le 1 \tag{1.6}$$

for $0 < a < 1$ and $x > 0$, inequality (1.4) was extended and refined as

$$\sqrt{x} \le \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \le \sqrt{x + \frac{1}{2}} \tag{1.7}$$

for $x > 0$.

The left hand side inequality in (1.6) reminds us to introduce

$$h_a(x) = \frac{(x+a)^{1-a}\Gamma(x+a)}{x\Gamma(x)} \tag{1.8}$$

for $x > 0$ and $a > 0$. F. Qi et al [18] have discussed its logarithmically complete monotonicity.

In order to obtain a refined upper bound in (1.6), F. Qi et al [18] have studied the logarithmically complete monotonicity of the function

$$f_a(x) = \frac{\Gamma(x+a)}{x^a\Gamma(x)} \tag{1.9}$$

the middle term in (1.6), for $x \in (0,\infty)$ and $a \in (0,\infty)$.

For a given number $b \geq 0$ and $a, r \in \mathbb{R}$ with $r \neq 0$, define

$$h_{a,r}(x) = \left[\frac{(x+a)^{1-a}\Gamma(x+a)}{x\Gamma(x)}\right]^r \tag{1.10}$$

and

$$f_{a,b,r}(x) = \left[\frac{\Gamma(x+a)}{x^a\Gamma(x)}\left(1+\frac{b}{x}\right)^{x+b}\right]^r \tag{1.11}$$

for $x \in (0,\infty)$.

In [18], the following conclusions were established :

(1) $h_{a,1}(x) \in \mathcal{L}[(0,\infty)]$ if $0 < a < 1$;

(2) $[h_{a,1}(x)]^{-1} \in \mathcal{L}[(0,\infty)]$ if $a > 1$;

(3) $f_{a,0,1}(x) \in \mathcal{L}[(0,\infty)]$ and $\lim_{x\to 0^+} f_{a,0,1}(x) = \infty$ if $a > 1$;

(4) $[f_{a,0,1}(x)]^{-1} \in \mathcal{L}[(0,\infty)]$ and $\lim_{x\to 0^+} f_{a,0,1}(x) = 0$ if $0 < a < 1$;

(5) $\lim_{x\to\infty} f_{a,0,1}(x) = 1$ if any $a \in (0,\infty)$.

In this article, a sufficient condition such that $h_{a,r}(x)$ and $f_{a,b,r}(x)$ are strictly logarithmically complete monotonic functions on $(0,\infty)$ will be established, which extends some known results mentioned above.

2. Main Results

Theorem 2.1. *If $0 < a < 1$ and $r > 0$ then the function $h_{a,r}(x)$ defined by (1.10) is strictly logarithmically complete monotonic function on $(0,\infty)$. Moreover,*

$$\lim_{x\to 0^+} h_{a,r}(x) = \left[\frac{\Gamma(a+1)}{a^a}\right]^r \quad \textit{and} \quad \lim_{x\to\infty} h_{a,r}(x) = 1,$$

for any $a > 0$.

Proof. It is clear that

$$\ln h_{a,r}(x) = r\left[(1-a)\ln(x+a) + \ln\Gamma(x+a) - \ln\Gamma(x+1)\right] \tag{2.1}$$

and

$$[\ln h_{a,r}(x)]' = \frac{1-a}{x+a} + \psi(x+a) - \psi(x+1). \tag{2.2}$$

The following formulas are known (see [10], p. 884): For $n \in \mathbb{N}$ and $x \in (0, \infty)$,

$$\frac{1}{x^n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} e^{-xt} dt \tag{2.3}$$

and

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1-e^{-t}} e^{-xt} dt. \tag{2.4}$$

Hence, for $n \in \mathbb{N}$, an easy computation yields

$$\begin{aligned}
[\ln h_{a,r}(x)]^{(n)} &= r\left[\frac{(-1)^{n-1}(n-1)!(1-a)}{(x+a)^n} + \psi^{(n-1)}(x+a) - \psi^{(n-1)}(x+1)\right] \\
&= (-1)^n r\left[\frac{(a-1)(n-1)!}{(x+a)^n} + (-1)^n\psi^{(n-1)}(x+a) - (-1)^n\psi^{(n-1)}(x+1)\right. \\
&\qquad \left. -(-1)^{2n}\int_0^\infty \frac{t^{n-1}}{1-e^{-t}} e^{-(x+1)t} dt\right] \\
&= (-1)^n r\left[\int_0^\infty \frac{e^{-(x+a)t} t^{n-1}}{1-e^{-t}} \left\{(a-1)(1-e^{-t}) + 1 - e^{(a-1)t}\right\} dt\right] \\
&\triangleq (-1)^n \int_0^\infty \frac{e^{-(x+a)t} t^{n-1}}{1-e^{-t}} g(t) dt,
\end{aligned} \tag{2.5}$$

where

$$g(t) := r\left[(a-1)(1-e^{-t}) + 1 - e^{(a-1)t}\right].$$

Direct computations show that

$$g'(t) = r\left[(a-1)e^{-t}(1-e^{at})\right].$$

It is easy to see that the function $g(t)$ is strictly increasing in $(0, \infty)$ and $\lim_{t\to 0^+} g(t) = 0$. Consequently, if $0 < a < 1$ and $r > 0$ then $g(t)$ is a positive in $(0, \infty)$. In view of (2.5), we have

$$(-1)^n [\ln h_{a,r}(x)]^{(n)} > 0$$

for $x \in (0, \infty)$.

Moreover, using the differences equation $\Gamma(x+1) = x\Gamma(x)$ and taking limit directly gives

$$\lim_{x\to 0^+} h_{a,r}(x) = \lim_{x\to 0^+}\left[\frac{(x+a)^{1-a}\Gamma(x+a)}{\Gamma(x+1)}\right]^r = \left[\frac{\Gamma(a+1)}{a^a}\right]^r.$$

It is well known that, as $x \to \infty$, the following asymptotic formula holds :

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x}\right) \qquad \text{(see [1], p. 257)}, \tag{2.6}$$

where a and b are two constants. Using the asymptotic expansion (2.6) yields

$$\lim_{x\to\infty} h_{a,r}(x) = \lim_{x\to\infty}\left[\frac{(x+a)^{1-a}\Gamma(x+a)}{\Gamma(x+1)}\right]^r$$
$$= \lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^{-a}\left(1+\frac{a(a+1)}{2x}+O\left(\frac{1}{x}\right)\right)\right]^r = 1,$$

for $a > 0$. This completes the proof. □

Corollary 2.1. *If $a > 1$ and $r < 0$ then $h_{a,r}(x)$ is strictly logarithmically completely monotonic function on $(0,\infty)$.*

Theorem 2.2. *If $a > 1, r > 0$ and $b \in [0,\infty)$ then the function $f_{a,b,r}(x)$ defined by (1.11) is strictly logarithmically complete monotonic function on $(0,\infty)$. Moreover,*

$$\lim_{x\to 0^+} f_{a,b,r}(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} f_{a,b,r}(x) = e^{br}.$$

Proof. Suppose that $a > 1, r > 0$ and $b \in (0,\infty)$. It is clear that

$$\ln f_{a,b,r}(x) = r\left[\ln\Gamma(x+a) - \ln\Gamma(x) - a\ln x + (x+b)\ln\left(1+\frac{b}{x}\right)\right] \tag{2.7}$$

and

$$[\ln f_{a,b,r}(x)]' = r\left[\psi(x+a) - \psi(x) - \frac{a}{x} + \ln(x+b) - \ln x - \frac{b}{x}\right]. \tag{2.8}$$

Using (2.3) and (2.4), we have

$$(-1)^n\left[\ln f_{a,b,r}(x)\right]^{(n)} = (-1)^n r\left[\psi^{(n-1)}(x+a) - \psi^{(n-1)}(x) + \frac{(-1)^n(n-1)!a}{x^n}\right.$$
$$\left.+\frac{(-1)^{n-2}(n-2)!}{(x+b)^{n-1}} - \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} - \frac{(-1)^{n-1}b(n-1)!}{x^n}\right]$$
$$= r\left[\int_0^\infty \frac{e^{-(x+a)t}t^{n-1}}{1-e^{-t}}dt - \int_0^\infty \frac{e^{-xt}t^{n-1}}{1-e^{-t}}dt + \int_0^\infty ae^{-xt}t^{n-1}dt\right.$$
$$\left.+\int_0^\infty t^{n-2}e^{-(x+b)t}dt - \int_0^\infty t^{n-2}e^{-xt}dt + b\int_0^\infty t^{n-1}e^{-xt}dt\right]$$
$$\triangleq \int_0^\infty \frac{t^{n-1}}{1-e^{-t}}s(t)e^{-xt}dt + \int_0^\infty t^{n-2}e^{-(x+b)t}g(t)dt, \tag{2.9}$$

where

$$s(t) := r\left[e^{-at} - 1 + a(1-e^{-t})\right]$$

and

$$g(t) := r\left[bte^{bt} - e^{bt} + 1\right].$$

It is clear that $s'(t) = ra\left[1 - e^{(1-a)t}\right]e^{-t}$ and $g'(t) = rb^2te^{bt}$. Consequently, if $a > 1, r > 0$ and $b > 0$ then $s'(t)$ and $g'(t)$ are positive on $(0,\infty)$, i.e., $s(t)$ and $g(t)$ are strictly increasing in $(0,\infty)$. It is easy to see that $\lim_{t\to 0^+} s(t) = 0$ and $\lim_{t\to 0^+} g(t) = 0$, then we have $s(t) > 0$ and $g(t) > 0$ for $t \in (0,\infty)$.

In view of (2.9), we have

$$(-1)^n \left[\ln f_{a,b,r}(x)\right]^{(n)} > 0$$

for $x \in (0,\infty)$.

In the case $b = 0$, we consider

$$f_{a,0,r}(x) = \left[\frac{\Gamma(x+a)}{x^a\Gamma(x)}\right]^r. \tag{2.10}$$

By the same process as above, we can show that

$$(-1)^n \left[\ln f_{a,0,r}(x)\right]^{(n)} > 0$$

for $x \in (0,\infty)$.

Moreover, from

$$f_{a,b,r}(x) = \left[\frac{\Gamma(x+a)}{x^a\Gamma(x)}\left(1+\frac{b}{x}\right)^{x+b}\right]^r,$$

it follows that

$$\lim_{x\to 0^+} f_{a,b,r}(x) = \begin{cases} 0, & 0 < a < 1; \\ \infty, & a > 1. \end{cases}$$

And applying (2.6) reveals

$$\frac{\Gamma(x+a)}{x^a\Gamma(x+b)} = 1 + \frac{a(a-1)}{2x} + O\left(\frac{1}{x}\right),$$

and we obtain $f_{a,b,r}(x) \to e^{br}$ as $x \to \infty$. This completes the proof. □

Corollary 2.2. *If $0 < a < 1, r < 0$ and $b \in [0,\infty)$ then $f_{a,b,r}(x)$ is strictly logarithmically completely monotonic functions on $(0,\infty)$.*

Acknowledgments

The first author would like to thank the Commission on Higher Education, the Thailand Research Fund (MRG5180058), and Khon Kaen University, Khon Kaen, Thailand, for financial support during the preparation of this paper.

References

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series, **55**, Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. (1965).

[2] R.D. Atanassov, U.V. Tsoukrovski, Some properties of a class of logarithmically completely monotonic functions, *C.R. Acal. Bulgare Sci.*, **41** (1988), 21-23.

[3] Ch.-P. Chen, F. Qi, Logarithmically completely monotonic functions relating the gamma functions, *J. Math. Anal. Appl.*, **321** (2006), 405-411.

[4] Ch.-P. Chen, F. Qi, Logarithmically completely monotonicity properties for the gamma functions, *Aust. J. Math. Anal. Appl.*, **2** (2005), Available online at: http://ajmaa.org/cgi-bin/paper.pl?string=v2n2/V2I2P8.tex.

[5] Ch.-P. Chen, F. Qi, Logarithmically completely monotonic ratios of mean values and an application, *Glob. J. Math. Math. Sci.*, **1** (2005), 71-76.

[6] M.J. Cloud, B.C. Drachman, *Inequalities with Applications to Engineering*, Springer-Verlag, New York (1998).

[7] A.Z. Grinshpan, M.E.H. Ismail, Completely monotonic functions involving the gamma and q-gamma functions, *Proc. Amer. Math. Soc.*, **134** (2006), 1135-1160.

[8] J. Lew, J. Frauenthal, N. Keyfitz, On the average distances in a circular disc, *SIAM Rev.*, **20** (1978), 584-592.

[9] A.-J. Li, W.-Zh. Zhao, Ch.-P. Chen, Logarithmically complete monotonicity and Shur-convexity for some ratios of gamma functions, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.*, **17** (2006), 88-92.

[10] W. Magnus, F. Oberhettinger, R.P. Soni, *Formulas and Theorems for Special Functions of Mathematical Physics*, Springer-Verlag, New York (1966).

[11] M.E. Muldoon, Certain logarithmically N-alternating monotonic functions involving the gamma and q-gamma functions, *Nonlinear Funct. Anal. Appl.*, **13** (2008), To Appear.

[12] F. Qi, A class of logarithmically completely monotonic functions and the best bounds in the first Kershaw's double inequality, *J. Comput. Appl. Math.*, **206** (2007), 1007-1014.

[13] F. Qi, Ch.-P. Chen, A completely monotonic property of the gamma functions, *J. Math. Anal. Appl.*, **296** (2004), 603-607.

[14] F. Qi, Ch.-P. Chen, Completely monotonic functions involving the gamma and polygamma functions, *RGMIA Res. Rep. Coll.*, **7**, No. 1 (2004), Art. 8, 63-72; Available online at http://rgmia.vu.edu.au/v7n1.html.

[15] F. Qi, Sh.-X. Chen, W.-S. Cheung, Logarithmically completely monotonic functions concerning gamma and digamma functions, *Integral Transforms Spec. Funct.*, **18** (2007), 435-443.

[16] F. Qi, B.-N. Guo, Ch.-P. Chen, Some completely monotonicities of functions involving the gamma and digamma functions, *RGMIA Res. Rep. Coll.*, **7**, No. 1 (2004), Art. 5, 31-36; Available online at: http://rgmia.vu.edu.au/v7n1.html.

[17] F. Qi, B.-N. Guo, Ch.-P. Chen, Some completely monotonicities of functions involving the gamma and digamma functions, *J. Aust. Math. Soc.*, **80** (2006), 81-88.

[18] F. Qi, D.-W. Nui, J. Cao, Sh.-X. Chen, Four logarithmically completely monotonic functions involving gamma function, *J. Korean Math. Soc.*, **45** (2008), 559-573.

[19] F. Qi, Q. Yang, W. Li, Two logarithmically completely monotonic functions connected with gamma function, *Integral Transforms Spec. Funct.*, **17** (2006), 539-542.

[20] J. Sándor, On certain inequalities for the gamma function, *RGMIA Res. Rep. Coll.*, **9** (2006), 115-117.

[21] J.G. Wendel, Note on the gamma function, *Amer. Math. Monthly*, **55** (1948), 563-564.