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FUNCTIONS APPROXIMATED BY ANY SEQUENCE OF INTERPOLATING GENERALIZED POLYNOMIALS

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Abstract: In this note, we seek for functions f which are approximated by any sequence of interpolating generalized polynomials to f.

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1. Introduction

We seek for functions f which are approximated by any sequence of interpolating generalized polynomials. Before precisely stating the results, we have to explain some definitions and notations.

Let $C[a,b]$ be the space of all real-valued continuous functions on a nondegenerate compact interval $[a,b]$ of $\mathbf{R}$. $C[a,b]$ is endowed with the supremum norm $\|\cdot\|$. $\{u_0,\dots,u_n\}$ (resp. $\{u_0,\dots,u_n,\dots\}$) of $C[a,b]$ is called a *system* if $u_0,\dots,u_n$ (resp. each $u_0,\dots,u_k, k\in\{0\}\cup\mathbf{N}$) is linearly independent. For a subset U of $C[a,b]$, we denote by $\mathrm{Span}U$ the space spanned by U. A finite system $\{u_0,\dots,u_n\}$ of $C[a,b]$ is called a *Chebyshev system* if there exists a constant $\sigma=1$ or -1 and for any $n+1$ distinct points $(a\le)x_0<\cdots<x_n(\le b)$,

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the $n+1$-th order determinant

$$\sigma \cdot D \begin{pmatrix} u_0 & \dots & u_n \\ x_0 & \dots & x_n \end{pmatrix} := \sigma \cdot \det(u_i(x_j)) > 0.$$

In other words, a Chebyshev system $\{u_0, \dots, u_n\}$ is a system satisfying that for any nodes consisting of $n+1$ distinct points $X : (a \leq) x_0 < \dots < x_n (\leq b)$ and for any $n+1$ real values $y_0, \dots, y_n$ there exists a unique $p \in \mathrm{Span}\{u_0, \dots, u_n\}$ with $p(x_i) = y_i, i = 0, \dots, n$. If each $y_i, i = 0, \dots, n$ is a value of $f \in C[a,b]$ at x_i, then we call p the *interpolating generalized polynomial to f at the nodes* $X : x_0, \dots, x_n$ and denote $p_{f,X}$ instead of p. An infinite system $\{u_0, \dots, u_n, \dots\}$ of $C[a,b]$ is called an *infinite complete Chebyshev system* if each finite system $\{u_0, \dots, u_n\}, n \in \{0\} \cup \mathbf{N}$ is a Chebyshev system. The prototype of a Chebyshev system is $\{1, x, \dots, x^n\}$ on $[a,b]$. Useful properties of Chebyshev systems are found in [1], [2] and so on.

It is very well known that interpolation by polynomials is a basic and useful method to approximate functions. But in order to obtain better approximation by interpolating polynomials to an approximated function, it is of much importance to select good nodes for the approximated function. Before showing examples, we give a theorem.

Theorem 1. (see Theorem 2 in p. 337 in Kincaid and Cheney [2]) *Let f be an $n+1$ times continuously differentiable function on $[a,b]$ and let p be the algebraic polynomial of degree at most n that interpolates the function f at $n+1$ distinct nodes $x_0, \dots, x_n$ in $[a,b]$, i.e., $p(x_i) = f(x_i), i = 0, \dots, n$. To each $x \in [a,b]$ there exists a point $\xi_x \in (-1,1)$ such that*

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \Pi_{i=0}^{n} (x - x_i).$$

Here we give two examples.

Example A. Let $f(x) = \cos \pi x,\ x \in [-1,1]$. Let

$$X_n : (-1 \leq) x_0^{(n)} < x_1^{(n)} < \dots < x_{k_n}^{(n)} (\leq 1), \quad n \in \mathbf{N},$$

be any prescribed system of nodes, where $\lim_{n\to\infty} k_n = \infty$ and let $p_{f,X_n}(x)$, $n \in \mathbf{N}$ be the algebraic polynomials of degree at most k_n that interpolates the function f at the nodes of X_n. By Theorem 1, we have

$$|f(x) - p_{f,X_n}(x)| \leq \frac{1}{(k_n+1)!} \pi^{k_n+1} \left| \Pi_{i=0}^{k_n} (x - x_i^{(n)}) \right| \leq \frac{2^{k_n+1} \pi^{k_n+1}}{(k_n+1)!}$$

for all $x \in [-1,1]$. Hence, we have

$$\lim_{n\to\infty} \|f - p_{f,X_n}\| = 0.$$

Example B. Let $g(x) = \dfrac{1}{1+25x^2}$, $x \in [-1,1]$. Let

$$X_n : x_0^{(n)} = -1, x_1^{(n)} = -1 + \frac{1}{n}, \cdots, x_n^{(n)} = 0, \cdots, x_{2n}^{(n)} = 1, \quad n \in \mathbf{N},$$

be the system of equally spaced nodes and let $p_{g,X_n}(x), n \in \mathbf{N}$ be the algebraic polynomials of degree at most $2n$ that interpolates the function g at the nodes of X_n. Though g is an analytic function on $[-1,1]$, it is well known that

$$\overline{\lim}_{n\to\infty} \|f - p_{f,X_n}\| = +\infty.$$

This example is called Runge example.

In this note, we seek for functions which have the same property as the function of Example A. To do this, we make a difinition.

Definition 1. Let $\{u_0, \ldots, u_n, \ldots\}$ be an infinite complete Chebyshev system of $C[a,b]$. Let $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be the subsets of $C[a,b]$ consisting of functions f such that for any (resp. some) system $X_k, k \in \mathbf{N}$ of nodes, where each X_k consists of distinct points in $[a,b]$ and the numbers $|X_k|$ of nodes satisfy $\lim_{k\to\infty} |X_k| = \infty$,

$$\lim_{k\to\infty} \|f - p_{f,X_k}\| = 0.$$

From Definition 1, $\mathcal{F}_1$ is a subspace of $C[a,b]$ containing $\mathrm{Span}\{u_0, \ldots, u_n, \ldots\}$. We immediately obatin the following proposition.

Proposition 2. *Let $U = \{u_0, \ldots, u_n, \ldots\}$ be an infinite complete Chebyshev system of $C[a,b]$. Then*

$$\mathrm{Span}U \subset \mathcal{F}_1 \subset \mathcal{F}_2 = \overline{\mathrm{Span}U}$$

holds, where $\overline{\mathrm{Span}U}$ denotes the closure of $\mathrm{Span}U$.

Proof. Since $\mathrm{Span}U \subset \mathcal{F}_1 \subset \mathcal{F}_2$ is obvious by the definitions of a complete Chebyshev system, $\mathcal{F}_1$ and $\mathcal{F}_2$, we only show that $\mathcal{F}_2 = \overline{\mathrm{Span}U}$.

By the definition of $\mathcal{F}_2$, any function $f \in \mathcal{F}_2$ belongs to the closure of $\mathrm{Span}U$. Hence, $\mathcal{F}_2 \subset \overline{\mathrm{Span}U}$.

Conversely, let f be any function in $\overline{\mathrm{Span}U} - \mathrm{Span}U$ and p_k the best approximation to f from $\mathrm{Span}\{u_0, \ldots, u_k\}$ in $(C[a,b], \|\cdot\|)$. From Chebyshev's Alternation Theorem in p. 445 in [2], $f - p_k$ has $k+2$ points $(a \le)x_0 < \cdots < x_{k+1}(\le b)$ such that $f(x_{i-1})f(x_i) < 0, i = 1, \ldots, k+1$. As $f - p_k$ has at least $k+1$ distinct zeros, we put the $k+1$ zeros $X_k : z_1, \ldots, z_{k+1}$. Since $p_{f,X_k} = p_k, k \in \mathbf{N}$, $\{p_{f,X_k}\}$ converges to f in $(C[a,b], \|\cdot\|)$. Hence, f belongs to $\overline{\mathrm{Span}U}$. □

The purpose of this note is to consider the following problem:

Problem 1. Let $\{u_0, \ldots, u_n, \ldots\}$ be an infinite complete Chebyshev system of $C[a,b]$. Then, consider properties of functions in $\mathcal{F}_1$.

For functions $f, g \in \mathcal{F}_1$, if $f(x_i) = g(x_i)$ for distinct countable points $x_i, i \in \mathbf{N}$, then $f \equiv g$ on $[a,b]$. So roughly speaking, $\mathcal{F}_1$ is a subspace of $C[a,b]$ where identity theorem holds.

2. Some Results on Problem 1

First we give a proposition.

Proposition 3. *Let $\{u_0, \ldots, u_n\}$ be a system of $C[a,b]$. Then there exist positive numbers $L_i, i = 0, \ldots, n$ such that for any linear combination $\sum_{i=0}^n c_i u_i$ of $u_0, \ldots, u_n$ with $\|\sum_{i=0}^n c_i u_i\| \le 1$*

$$|c_i| \le L_i, \quad i = 0, \ldots, n, \tag{1}$$

hold.

Proof. Suppose on the contrary that (1) do not hold. Without loss of generality, we can suppose that there exists a sequence of linear combinations $\sum_{i=0}^n c_i^k u_i$ of $u_0, \ldots, u_n$ with $\|\sum_{i=0}^n c_i^k u_i\| \le 1, k \in \mathbf{N}$ which satisfy

$$|c_0^k| = \max\{|c_0^k|, \ldots, |c_n^k|\} > 0,\ k \in \mathbf{N} \quad \text{and} \quad \lim_{k\to\infty} |c_0^k| = \infty.$$

Then we consider a sequence of linear combinations $\{w_k\}$,

$$w_k := \frac{1}{c_0^k} \sum_{i=0}^n c_i^k u_i,\ k \in \mathbf{N},$$

of $u_0, \ldots, u_n$. Since $|c_i^k / c_0^k| \le 1, i = 0, \ldots, n,\ k \in \mathbf{N}$ and the coefficients of u_0 of $w_k, k \in \mathbf{N}$ are 1, a subsequence $\{w_{k_\ell}\}$ converges to a linear combination w^* of $u_0, \ldots, u_n$. Since the coefficient of u_0 of w^* is 1, w^* is nonzero by the linearly independence of $u_0, \ldots, u_n$. On the other hand, we obtain

$$\|w^*\| = \lim_{\ell\to\infty} \|w_{k_\ell}\| = \lim_{\ell\to\infty} \frac{1}{|c_0^{k_\ell}|} \left\| \sum_{i=0}^n c_i^{k_\ell} u_i \right\| = 0.$$

But this leads to a contradiction. □

From Proposition 3, we immediately have the following corollary.

Corollary 4. *Let k be a positive integer and let $\{u_0, \ldots, u_n \ldots\}$ be an infinite system of $C[a,b]$ which consists of k times continuously differentiable functions on $[a,b]$. Then there exist positive numbers $M_{n,j}, n \in \mathbf{N} \cup \{0\}, j = 0, \ldots, k$ such that for any linear combination $\sum_{i=0}^n c_i u_i$ of $u_0, \ldots, u_n$ with $\|\sum_{i=0}^n c_i u_i\| \le$*

1
$$\left\|\sum_{i=0}^{n} c_i u_i^{(j)}\right\| \le M_{n,j}, \quad j=0,\dots,k,$$

holds, where each $u_i^{(j)}$ *denotes the derivative of* j*-th order of* u_i.

Definition 2. Let f be a real valued function on an interval I and let $x_0,\dots,x_n$ be distinct points in I. Then, the n-th order divided difference $f[x_0,\dots,x_n]$ of f over $x_0,\dots,x_n$ is obtained by the equations

$$f[x_i]=f(x_i), \quad f[x_i,x_{i+1},\dots,x_{i+j}]=\frac{f[x_{i+1},\dots,x_{i+j}]-f[x_i,\dots,x_{i+j-1}]}{x_{i+j}-x_i}.$$

Hence, we see that $f[x_0,\dots,x_n]$ is determined only by the values $f(x_0)$, $f(x_1)$, …, $f(x_n)$ (see pp. 353-354 in [2]).

We need the following lemma to proceed our argument.

Lemma 5. (cf. Theorem 4 in p. 357 in Kincaid and Cheney [2]) *If* f *is* n *times continuously differentiable on an interval* I *and if* $x_0,\dots,x_n$ *are distinct points in* I*, then there exists a point* $\xi \in (a,b)$ *such that*

$$f[x_0,\dots,x_n]=\frac{1}{n!}f^{(n)}(\xi),$$

where $a=\min\{x_0,\dots,x_n\}, b=\max\{x_0,\dots,x_n\}$*. Furthermore, for any point* c *in* I *and any* $\varepsilon>0$ *there exists a* $\delta>0$ *such that for any* $n+1$ *distinct points* $z_0,\dots,z_n$ *in* $(c-\delta,c+\delta)\cap I$

$$\left|n!f[z_0,\dots,z_n]-f^{(n)}(c)\right|<\varepsilon.$$

Now we show a property of functions which belong to $\mathcal{F}_1$.

Theorem 6. *Let* k *be a positive integer more than* 1 *and let* $\{u_0,\dots,u_n,\dots\}$ *be an infinite complete Chebyshev system of* $C[a,b]$ *which consists of* k *times continuously differentiable functions on* $[a,b]$*. Then, every function which belongs to* $\mathcal{F}_1$ *is* $k-1$ *times continuously differentiable on* $[a,b]$.

Proof. We prove this by induction. We start with $k=2$. Let f be any function in $\mathcal{F}_1$. Since $\mathcal{F}_1$ is a subspace of $C[a,b]$, we can suppose that $\|f\|<\frac{1}{2}$.

First we show that f is differentiable on $[a,b]$. Suppose on the contrary that f is not differentiable at some $c\in[a,b]$. Then, there exists a sequence $\{x_n\}_{n\in\mathbf{N}}$ which converges to c and $x_n\ne c, n\in\mathbf{N}$ such that the sequence $\left\{\frac{f(x_n)-f(c)}{x_n-c}\right\}_{n\in\mathbf{N}}$ does not converge. Hence, there exist a positive number ε and subsequeces $\{y_n\}_{n\in\mathbf{N}}$, $\{z_n\}_{n\in\mathbf{N}}$ of $\{x_n\}_{n\in\mathbf{N}}$ (for simplicity, we use notations $\{y_n\}_{n\in\mathbf{N}}$, $\{z_n\}_{n\in\mathbf{N}}$ of subsequences of $\{x_n\}_{n\in\mathbf{N}}$ instead of $\{x_{n_i}\}$). Such that the points $c, y_1, \dots, y_n, \dots, z_1, \dots, z_n, \dots$ are distinct and

$$\left| \frac{f(y_n) - f(c)}{y_n - c} - \frac{f(z_n) - f(c)}{z_n - c} \right| > \varepsilon, \quad n = 1, 2, \dots, \tag{2}$$

$$\max\{c, y_n, z_n\} - \min\{c, y_n, z_n\} < \frac{\varepsilon}{M_{2n,2}}, \quad n = 1, 2, \dots, \tag{3}$$

where each $M_{2n,2}, n \in \mathbf{N}$ denotes the number stated in Corollary 4. We consider a system $X_n, n \in \mathbf{N}$ of nodes such that

$$X_n : c, y_1, \dots, y_n, z_1, \dots, z_n, \quad n \in \mathbf{N},$$

and a sequence of interpolating generalized polynomials p_{f,X_n} in Span$\{u_0, \dots, u_{2n}\}$, $n \in \mathbf{N}$. Since f is a function in $\mathcal{F}_1$, $\lim_{n\to\infty} \|f - p_{f,X_n}\| = 0$. Hence, there is an $N \in \mathbf{N}$ with $\|p_{f,X_N}\| \le \|f\| + \|f - p_{f,X_N}\| \le 1$. Then, by Mean Value Theorem we have

$$\begin{aligned} &\left| \frac{f(y_N) - f(c)}{y_N - c} - \frac{f(z_N) - f(c)}{z_N - c} \right| \\ &\qquad = \left| \frac{p_{f,X_N}(y_N) - p_{f,X_N}(c)}{y_N - c} - \frac{p_{f,X_N}(z_N) - p_{f,X_N}(c)}{z_N - c} \right| \\ &\qquad\qquad = |p'_{f,X_N}(\xi) - p'_{f,X_N}(\eta)| > \varepsilon. \end{aligned}$$

By (3) since $|\xi - \eta| < \max\{c, y_N, z_N\} - \min\{c, y_N, z_N\} < \frac{\varepsilon}{M_{2n,2}}$, from (2) we have

$$\left| \frac{p'_{f,X_N}(\xi) - p'_{f,X_N}(\eta)}{\xi - \eta} \right| = |p''_{f,X_N}(\tau)| > \frac{\varepsilon}{\frac{\varepsilon}{M_{2N,2}}} = M_{2N,2} \quad \text{for some} \quad \tau \in (\xi, \eta).$$

But this contradicts to Corollary 4. Hence, f is differentiable on $[a, b]$.

Next, we show that f' is continuous on $[a, b]$. Suppose on the contrary that f is not continuous at a point $d \in [a, b]$. Then, there exists a sequence $\{x_n\}_{n\in\mathbf{N}}$ which converges to d and $x_n \neq d, n \in \mathbf{N}$ such that the sequence $\{f'(x_n)\}_{n\in\mathbf{N}}$ does not converge to $f'(d)$. There exist a positive number ε, a subsequence $\{y_n\}_{n\in\mathbf{N}}$ of $\{x_n\}_{n\in\mathbf{N}}$ and two sequences $\{z_n\}_{n\in\mathbf{N}}, \{d_n\}_{n\in\mathbf{N}}$ satisfying that the points

$$d, d_1, \dots, d_n, \dots, y_1, \dots, y_n, \dots, z_1, \dots, z_n, \dots$$

are distinct and

$$|f'(y_n) - f'(d)| > \varepsilon, \quad n = 1, 2, \dots,$$

$$\left| \frac{f(z_n) - f(y_n)}{z_n - y_n} - \frac{f(d_n) - f(d)}{d_n - d} \right| > \varepsilon, \quad n = 1, 2, \dots, \tag{4}$$

$$\max\{d, d_n, y_n, z_n\} - \min\{d, d_n, y_n, z_n\} < \frac{\varepsilon}{M_{3n,2}}, \quad n = 1, 2, \ldots.$$

We consider a system $X_n, n \in \mathbf{N}$ of nodes such that

$$X_n : d, d_1, \ldots, d_n, y_1, \ldots, y_n, z_1, \ldots, z_n, \quad n \in \mathbf{N},$$

and a sequence of interpolating generalized polynomials p_{f,X_n} in Span$\{u_0, \ldots, u_{3n}\}$, $n \in \mathbf{N}$. Since f is a function in $\mathcal{F}_1$, $\lim_{n\to\infty} \|f - p_{f,X_n}\| = 0$. Hence, there is an $N \in \mathbf{N}$ with $\|p_{f,X_N}\| \leq 1$. In an analogous way to the proof stated above, we see that $\|p''_{f,X_N}\| > M_{3N,2}$ by (4) and (5), which contradicts Corollary 4. Consequently, f is continuously differentiable on $[a, b]$.

Under the condition that the proposition holds for $k = p(\geq 2)$, we consider the case $k = p+1$. Though a proof for $k = p+1$ is essentially analogous to the proof for $k = 2$, to make it clear, we give a proof that f is p times differentiable on $[a, b]$.

By the assumption, every $f \in \mathcal{F}_1$ is $p-1$ times continuously differentiable on $[a, b]$. Suppose on the contrary that $f^{(p-1)}$ is not differentiable at some $c \in [a, b]$. Then, there exists a sequence $\{x_n\}_{n\in\mathbf{N}}$ which converges to c and there exist a positive number ε and subsequences $\{y_n\}_{n\in\mathbf{N}}$, $\{z_n\}_{n\in\mathbf{N}}$ of $\{x_n\}_{n\in\mathbf{N}}$ such that

$$\left| \frac{f^{(p-1)}(y_n) - f^{(p-1)}(c)}{y_n - c} - \frac{f^{(p-1)}(z_n) - f^{(p-1)}(c)}{z_n - c} \right| > \varepsilon, \quad n = 1, 2, \ldots.$$

Moreover, from $p-1$ times continuously differentiablity of f and Lemma 5, for each $c, y_n, z_n, n \in \mathbf{N}$ we can take $3p$ distinct points $c_{(n,1)}, \ldots, c_{(n,p)}, y_{(n,1)}, \ldots, y_{(n,p)}$, $z_{(n,1)}, \ldots, z_{(n,p)}$ which satisfy the following conditions (6), (7) and (8). Intuitively, $c_{(n,1)}, \ldots, c_{(n,p)}$; $y_{(n,1)}, \ldots, y_{(n,p)}$ and $z_{(n,1)}, \ldots, z_{(n,p)}$ are sufficiently close to c, y_n and z_n, respectively. Put $C_n = \{c_{(n,1)}, \ldots, c_{(n,p)}\}$, $Y_n = \{y_{(n,1)}, \ldots, y_{(n,p)}\}$, $Z_n = \{z_{(n,1)}, \ldots, z_{(n,p)}\}$, $n \in \mathbf{N}$. Then,

$$\text{The points } c_{(i,j)}, y_{(i,j)}, z_{(i,j)}, i \in \mathbf{N}, j = 1, \ldots, p \text{ are distinct.} \tag{6}$$

$$\left| \frac{(p-1)!(f[y_{(n,1)}, \ldots, y_{(n,p)}] - f[c_{(n,1)}, \ldots, c_{(n,p)}])}{r - s} - \frac{(p-1)!(f[z_{(n,1)}, \ldots, z_{(n,p)}] - f[c_{(n,1)}, \ldots, c_{(n,p)}])}{t - u} \right| > \varepsilon, \tag{7}$$

for all $r \in (\min Y_n, \max Y_n)$, $t \in (\min Z_n, \max Z_n)$, $s, u \in (\min C_n, \max C_n)$.

$$\max(C_n \cup Y_n \cup Z_n) - \min(C_n \cup Y_n \cup Z_n) < \frac{\varepsilon}{M_{3pn-1,p+1}}. \tag{8}$$

Let $X_n, n \in \mathbf{N}$ be nodes which are $3pn$ distinct points $c_{(i,j)}, y_{(i,j)}, z_{(i,j)}, i = 1, \ldots, n, j = 1, \ldots, p$. Now we consider the system $X_n, n \in \mathbf{N}$ of nodes and a sequence of interpolating generalized polynomials p_{f,X_n} in Span$\{u_0, \ldots, u_{3pn-1}\}$,

$n \in \mathbf{N}$. Since f is a function in $\mathcal{F}_1$, $\lim_{n\to\infty} \|f - p_{f,X_n}\| = 0$ and there is an $N \in \mathbf{N}$ with $\|p_{f,X_N}\| \le 1$. As

$$(p-1)! p_{f,X_N}[y_{(N,1)}, \dots, y_{(N,p)}] = p^{(p-1)}_{f,X_N}(\xi),$$

$$(p-1)! p_{f,X_N}[z_{(N,1)}, \dots, z_{(N,p)}] = p^{(p-1)}_{f,X_N}(\tau),$$

and

$$(p-1)! p_{f,X_N}[c_{(N,1)}, \dots, c_{(N,p)}] = p^{(p-1)}_{f,X_N}(\eta),$$

from (7) and Mean Value Theorem we have

$$\left| \frac{p^{(p-1)}_{f,X_N}(\xi) - p^{(p-1)}_{f,X_N}(\eta)}{\xi - \eta} - \frac{p^{(p-1)}_{f,X_N}(\tau) - p^{(p-1)}_{f,X_N}(\eta)}{\tau - \eta} \right| = \left| p^{(p)}_{f,X_N}(\xi') - p^{(p)}_{f,X_N}(\tau') \right| > \varepsilon.$$

Since $|\xi' - \tau'| < \frac{\varepsilon}{M_{3pN-1,p+1}}$ by (8), we obtain

$$\left| \frac{p^{(p)}_{f,X_N}(\xi') - p^{(p)}_{f,X_N}(\tau')}{\xi' - \tau'} \right| = \left| p^{(p+1)}_{f,X_N}(\zeta) \right| > M_{3pN-1,p+1} \quad \text{for some} \quad \zeta \in (\xi', \tau')$$

which contradicts Corollary 4. The rest of this proof is to show that $f^{(p)}$ is continuous on $[a, b]$. But we can prove this in an analogous way to the proof that $f^{(p-1)}$ is differentiable on $[a, b]$. □

Corollary 7. *Let $\{u_0, \dots, u_n, \dots\}$ be an infinite complete Chebyshev system of $C[a, b]$ which consists of infinitely differentiable functions on $[a, b]$. Then, every function which belongs to $\mathcal{F}_1$ is infinitely differentiable on $[a, b]$.*

Corollary 8. (see Theorem in Kitahara [3]) *Let $\{u_0, \dots, u_n, \dots\}$ be $\{1, x, \dots, x^n, \dots\}$. Every function f of $\mathcal{F}_1$ is analytic on $[a, b]$ and the radiuses of convergence of the Taylor series about a, b for f are at least $b - a$.*

Proof. This is already shown in [3], but to make sure we give a proof.

From Corollary 7, every $f \in \mathcal{F}_1$ is infinitely differentiable on $[a, b]$. For any $f \in \mathcal{F}_1$ and any $c \in [a, b]$, we can consider a system $X_n : x^{(n)}_0, \dots, x^{(n)}_n, n \in \mathbf{N}$, of nodes such that $x^{(n)}_0, \dots, x^{(n)}_n$ are distinct and

$$\left\| p_{f,X_n} - \sum_{p=0}^{n} \frac{f^{(p)}(c)}{p!}(x - c)^p \right\| < \frac{1}{n}, \quad n \in \mathbf{N}. \tag{9}$$

If we take $n + 1$ distinct points $x^{(n)}_0, \dots, x^{(n)}_n$ which are sufficiently closed to c, since p_{f,X_n} is sufficiently closed to the Taylor polynomial of degree n about c for f, then p_{f,X_n} satisfies (9). Since $f \in \mathcal{F}_1$, $\lim_{n\to\infty} \|f - p_{f,X_n}\| = 0$. Hence, by (9) f is expressed as

$$f(x) = \sum_{p=0}^{\infty} \frac{f^{(p)}(c)}{p!}(x-c)^p, \quad x \in [a,b].$$

This means that f is analytic on $[a,b]$ and the radiuses of the Taylor series about a, b for f are at least $b-a$. □

We study only real-valued functions on an interval, but the readers can easily see that infinite complete Chebyshev systems of complex-valued functions on a domain D of the complex plane can be defined and the results obtained in this paper hold for complex-valued functions on D. Hence, we can state the following.

Corollary 9. *Let D be a bounded domain of the complex plane and let $\{u_0, \ldots, u_n, \ldots\}$ be $\{1, z, \ldots, z^n, \ldots\}$ on D. Every function f of $\mathcal{F}_1$ is regular on K, where*

$$K = \cup_{w \in D} U(w) \ , \ U(w) = \{z \in \mathbf{C} \mid |w - z| < \sup_{v \in D} |v - w|\}.$$

We think that there is still much room for investigating properties of functions of $\mathcal{F}_1$. To develop this topic, we give a problem.

Problem 2. For an infinite complete Chebyshev system $\{u_0, \ldots, u_n, \ldots\}$ of $C[a.b]$, find a sufficient condition that functions f belong to $\mathcal{F}_1-\mathrm{Span}\{u_0, \ldots, u_n, \ldots\}$.

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