International Journal of Pure and Applied Mathematics

Volume 64 No. 3 2010, 421-432

OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS

S. Lourdu Marian[1], M. Reni Sagaya Raj[2], A. George Maria Selvam[3 §]

[1]Department of Master of Computer Applications
Saveetha Engineering College
Thandalam, Chennai, 602 105, INDINA
[1]e-mail: lm_jerrome @ rediffmail.com

[2,3]Department of Mathematics
Sacred Heart College
Tirupattur, 635 601, Vellore Dist., INDIA
[2]e-mail: reni.sagaya @ gmail.com
[3]e-mail: agm_shc @ rediffmail.com

Abstract: In this paper, we are concerned with second order nonlinear difference equations and give sufficient conditions for their solutions to be oscillatory. Oscillation theorems are established involving comparison with linear equations.

AMS Subject Classification: 39A10, 39A11
Key Words: oscillation, nonlinear, second order, difference equations

1. Introduction

In this paper, we consider the second order nonlinear delay difference equations of the form

$$\Delta[a(n)\Delta x(n)] + \delta_1 p(n)\Delta x(n) + \delta_2 q(n) f(x(\sigma(n))) = 0, \tag{1}$$

where $n \geq n_0 \geq 0$ and $\delta_1 = \pm 1, \delta_2 = \pm 1$. Throughout this paper we shall assume the following conditions hold:

(H_1) $a(n) > 0, p(n) \geq 0, q(n) \geq 0$ are real sequences, $q(n)$ is not identically zero for $n \geq n_0$;

Received: September 8, 2010

§Correspondence author

(H_2) $a(n) - p(n) > 0$ for large values of n;

(H_3) $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\frac{f(u)}{u} \geq \lambda > 0$ for $u \neq 0$;

(H_4) $\sigma(n) : \mathbb{N} \to \mathbb{Z}$ is a nondecreasing sequence such that $\sigma(n) \leq n - 1$ for $n \in \mathbb{N}$ and $\sigma(n) \geq 0$ for $n \geq n_0$ and $\lim_{n\to\infty} \sigma(n) = \infty$.

By a solution of (1), we mean a sequence $\{x(n)\}$ which is defined for $n \geq n_0$ and satisfies (1) for $n \geq n_0$. The solution $\{x(n)\}$ of equation (1) is said to be oscillatory if the terms $x(n)$ of the solution are neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory.

In recent years, the study of the oscillation and nonoscillation of solutions of difference equations has drawn extensive attention, see, for example, the monographs [1, 2] and the references cited therein. Several papers have been devoted to the study of oscillatory and asymptotic properties of damped difference equations, see [8, 9, 10].

Throughout this paper we shall use the following notations:

$$\tau(n) = \max\{\min\{k, \sigma(k)\}; 0 \leq k \leq n\} : \rho(n) = \min\{\max\{k, \sigma(k)\}; k \geq n\}.$$

We notice that the following inequalities hold.

$$\sigma(k) \leq \tau(n) \text{ for } \tau(n) < k < n \text{ and } \sigma(k) \geq \rho(n) \text{ for } n < k < \rho(n).$$

In this paper, we discuss the oscillatory and nonoscillatory behavior of solutions of (1) when $\delta_1 = \pm 1$ and $\delta_2 = \pm 1$.

2. Preliminary Lemmas

In this section we provide some lemmas and results which play crucial role in proving the results in Section 3.

Theorem A. (see [1], Discrete Taylor's Formula) *Let $u(k)$ be defined on $\mathbb{N}(a)$. Then for all $k \in \mathbb{N}(a)$ and $n \geq 1$,*

$$u(k) = \sum_{i=0}^{n-1} \frac{(k-1)^{(i)}}{i!} \Delta^i u(a) + \frac{1}{(n-1)!} \sum_{\ell=a}^{k-n} (k - \ell - 1)^{(n-1)} \Delta^n u(\ell).$$

We consider the following inequalities

$$\Delta[a(n)\Delta x(n)] + q(n) f(x(\sigma(n))) \leq 0, \tag{E_1}$$

$$\Delta[a(n)\Delta x(n)] + q(n) f(x(\sigma(n))) \geq 0, \tag{E_2}$$

and the equation

$$\Delta[a(n)\Delta x(n)] + q(n)f(x(\sigma(n))) = 0. \qquad (E_3)$$

The following lemmas are taken from [3].

Lemma B. (see [3], Lemma 2.1) *If inequality* (E_1) *(inequality* (E_2)*) has an eventually positive (negative) solution, then equation* (E_3) *also has eventually positive (negative) solution.*

Lemma C. (see [3], Lemma 2.3) *If the inequality*

$$\Delta x(n) + q(n)f(x[\sigma(n)]) \leq 0 \text{ or } \Delta x(n) - q(n)f(x[\sigma(n)]) \geq 0,$$

where $\sigma(n), q(n)$ *and* $f(x)$ *satisfy conditions given in* $(H_1)-(H_4)$ *has eventually positive solution, then the equation*

$$\Delta x(n) + q(n)f(x[\sigma(n)]) = 0 \text{ or } \Delta x(n) - q(n)f(x[\sigma(n)]) = 0$$

also has eventually positive solution.

Theorem D. (see [4, 6, 5, 11]) *Assume that* (H_4) *holds and* $p(n) \geq 0$ *eventually. Furthermore, assume that*

$$\lim_{n\to\infty} \sup p(n) > 0,$$

$$\lim_{n\to\infty} \inf \sum_{i=\sigma(n)}^{n-1} p(i) > \frac{1}{e}.$$

Then every solution of

$$x(n+1) - x(n) + p(n)x(\sigma(n)) = 0, \quad n = 0, 1, 2, \ldots$$

is oscillatory.

Lemma E. *If*

$$\lim_{n\to\infty} \inf \left[\frac{1}{a(n)-1} \sum_{j=n+1}^{n+a(n)-1} q(j) \right] > \lim_{n\to\infty} \sup \frac{(a(n)-1)^{a(n)-1}}{(a(n))^{a(n)}}$$

then the following statements hold:

(i) Inequality $\Delta x(n) - q(n)x(n+a(n)) \geq 0$ *has no eventually positive solution.*

(ii) Inequality $\Delta x(n) - q(n)x(n+a(n)) \leq 0$ *has no eventually negative solution.*

(iii) Every solution of equation $\Delta x(n) - q(n)x(n+a(n)) = 0$ *oscillates.*

3. Main Results

In this section we establish conditions under which the equation (1) oscillates for $\delta_1 = \pm 1$ and $\delta_2 = \pm 1$.

Theorem 1. *Assume that the following conditions hold.*

(C_1) $\sigma(n) \le n, \Delta\sigma(n) \ge 0$ *and* $\Delta p(n) \le 0$ *for* $n \ge n_0$.

(C_2) *The difference equation*

$$\Delta[a(n)\Delta x(n)] + \lambda q(n)x(\sigma(n)) = 0, n \ge n_0 \tag{2}$$

is oscillatory.

(C_3) *The condition*

$$\lim_{n\to\infty} \inf \left\{ \sum_{j=\sigma(n)}^{n-1} \left[\frac{\lambda}{a(j)} \sum_{i=\sigma(j)}^{j-1} q(i) - \frac{p(\sigma(j))}{a(j)} \right] \right\} > \frac{1}{e} \tag{3}$$

holds.

Then the equation (1) is oscillatory when $\delta_1 = 1$ *and* $\delta_2 = 1$.

Proof. Suppose that (1) has a nonoscillatory solution when $\delta_1 = \delta_2 = 1$. Without loss of generality, we may assume that $x(n)$ is positive for $n \ge n_0$. Hence $x(n) > 0$ and $x(\sigma(n)) > 0$ for all $n \ge n_0$ (the proof for the case $x(n) < 0$ for $n \ge n_0$ is similar). First we claim that $\Delta x(n)$ is eventually of the same sign. To show this we assume that, on the contrary $\Delta x(n)$ is oscillatory. Then there exits an integer $n_1 \ge n_0$ such that $\Delta x(n_1) < 0$ or $\Delta x(n_1) = 0$. First consider $\Delta x(n_1) < 0$.

Hence equation (1) becomes

$$\begin{aligned} \Delta\left[a(n_1)\Delta x(n_1)\right] &= -\, p(n_1)\Delta x(n_1) - q(n_1)f(x(\sigma(n_1))) \\ &\le -\, p(n_1)\Delta x(n_1), \\ a(n_1+1)\Delta x(n_1+1) &\le \left[a(n_1) - p(n_1)\right]\Delta x(n_1). \end{aligned}$$

Thus we obtain $\Delta x(n_1 + 1) < 0$.

By induction, we get $\Delta x(n) < 0$ for all $n \ge n_1$.

Now consider $\Delta x(n_1) = 0$. From equation (1), we have

$$\begin{aligned} \Delta\left[a(n_1)\Delta x(n_1)\right] &= -\, q(n_1)f(x(\sigma(n_1))), \\ \Delta\left[a(n_1)\Delta x(n_1)\right] &< 0, \\ a(n_1+1)\Delta x(n_1+1) &< a(n_1)\Delta x(n_1). \end{aligned}$$

Thus $\Delta x(n)$ is decreasing for $n \geq n_1$. In both cases, we obtain contradiction to the assumption that $\{\Delta x(n)\}$ oscillates. Thus $\{\Delta x(n)\}$ is eventually of fixed sign. Now we consider the following two cases.

(i) $\Delta x(n) > 0$ eventually.

(ii) $\Delta x(n) < 0$ eventually.

If (i) holds, then (H_3) in (1) yields $\Delta[a(n)\Delta x(n)] + \lambda q(n)x(\sigma(n) \leq 0$ eventually. Hence by Lemma B, we conclude that the equation

$$\Delta[a(n)\Delta x(n)] + \lambda q(n)x(\sigma(n)) = 0$$

has an eventually positive solution, which is a contradiction to the condition (2).

If (ii) holds, then by using (H_3) in (1), we obtain

$$\Delta[a(n)\Delta x(n)] + p(n)\Delta x(n) + \lambda q(n)x(\sigma(n)) \leq 0, \quad n \geq n_2. \tag{4}$$

Note that $\sigma(s) \leq \sigma(n)$ for $s < n$.

By summing (4) from $\sigma(n)$ to $n-1$ for $\sigma(n) < s < n_2 \leq n_1$ we have

$$\sum_{i=\sigma(n)}^{n-1} \Delta\left[a(i)\Delta x(i)\right] + \sum_{i=\sigma(n)}^{n-1} \left[p(i)\Delta x(i)\right] + \lambda \sum_{i=\sigma(n)}^{n-1} q(i)x(\sigma(i)) \leq 0,$$

$$a(n)\Delta x(n) - a(\sigma(n))x(\sigma(n)) + \lambda x(\sigma(n)) \sum_{i=\sigma(n)}^{n-1} q(i) \leq 0,$$

$$\Delta x(n) - \frac{p(\sigma(n))}{a(n)}x(\sigma(n)) + \frac{\lambda}{a(n)}x(\sigma(n)) \sum_{i=\sigma(n)}^{n-1} q(i) \leq 0,$$

$$\Delta x(n) + \left[\frac{\lambda}{a(n)} \sum_{i=\sigma(n)}^{n-1} q(i) - \frac{p(\sigma(n))}{a(n)}\right] x(\sigma(n)) \leq 0. \tag{5}$$

From Lemma C and Theorem D, we conclude that the inequality (5) has no eventually positive solution. This is a contradiction. Thus $x(n)$ is an oscillatory solution for equation (1). Therefore equation (1) is oscillatory when $\delta_1 = 1$ and $\delta_2 = 1$. □

Theorem 2. *If $\sigma(n)$ is of mixed type, $\Delta\sigma(n) \geq 0$, $\Delta p(n) \geq 0$ for $n \geq n_0$,*

$$\limsup_{n\to\infty} \left\{ \lambda \sum_{\ell=n_2}^{\rho(n)-1} \sum_{j=u}^{\rho(u)-1} \frac{1}{a(j)} \prod_{i=s}^{\rho(n)-1} \left(1 - \frac{p(i)}{a(i)}\right) q(\ell) \right\} > 1 \tag{6}$$

and

$$\lim_{n\to\infty} \sup \left\{ \frac{\lambda}{a(\tau(n))} \sum_{j=\tau(n)}^{n-1} q(j)\left[\sigma(j) - \tau(n)\right] \right\} > 1 \tag{7}$$

hold, then the equation (1) is oscillatory when $\delta_1 = 1$ and $\delta_2 = -1$.

Proof. Suppose that (1) has a nonoscillatory solution $x(n)$ when $\delta_1 = 1$ and $\delta_2 = -1$. Without loss of generality assume that $x(n)$ is positive for $n \geq n_1$. Let $n_2 \geq n_1$ be such that $x(\sigma(n)) > 0$ for $n \geq n_2$. First we show that $\Delta x(n)$ is eventually of the same sign. To show this, we assume that it is oscillatory. Then there exists an integer $n_1 \geq n_0$ such that $\Delta x(n_1) > 0$ or $\Delta x(n_1) = 0$.

When $\Delta x(n_1) > 0$, we obtain from (1),

$$\begin{aligned} \Delta\left[a(n_1)\Delta x(n_1)\right] &= -\, p(n_1)\Delta x(n_1) + q(n_1) f(x(\sigma(n_1))) \\ &\geq -\, p(n_1)\Delta x(n_1), \\ a(n_1+1)\Delta x(n_1+1) &\geq \left[a(n_1) - p(n_1)\right]\Delta x(n_1), \\ \Delta x(n_1+1) &> 0. \end{aligned}$$

By induction, we obtain $\Delta x(n) > 0$ for $n \geq n_1$.

Now consider $\Delta x(n_1) = 0$. From equation (1), it follows that

$$\begin{aligned} \Delta\left[a(n_1)\Delta x(n_1)\right] &= q(n_1) f(x(\sigma(n_1)) > 0, \\ a(n_1+1)\Delta x(n_1+1) &> 0, \\ \Delta x(n) &> 0 \text{ for } n \geq n_1, \end{aligned}$$

which contradicts the assumption that $\{\Delta x(n)\}$ is oscillatory. Hence $\{\Delta x(n)\}$ is not oscillatory. Now we consider the following two cases.

(i) $\Delta x(n) > 0$ eventually.

(ii) $\Delta x(n) < 0$ eventually.

If (i) holds, then equation (1) becomes

$$\Delta[a(n)\Delta x(n)] + p(n)\Delta x(n) - \lambda q(n) x(\sigma(n)) \geq 0, \quad n \geq n_2. \tag{8}$$

Summing the inequality (8) from n_2 to $n-1$, we obtain

$$\sum_{i=n_2}^{n-1} \Delta\left[a(i)\Delta x(i)\right] + \sum_{i=n_2}^{n-1} \left[p(i)\Delta x(i)\right] - \lambda \sum_{i=n_2}^{n-1} q(i) x(\sigma(i)) \geq 0,$$

$$a(n)\Delta x(n) + p(n)x(n) - \lambda x(\rho(n)) \sum_{i=n_2}^{n-1} q(i) \geq 0,$$

$$\Delta x(n)+\frac{p(n)}{a(n)}x(n)-\frac{\lambda}{a(n)}x(\rho(n))\sum_{i=n_2}^{n-1}q(i)\geq 0,$$

where $\sigma(u)<\sigma(s), x(\sigma(u))\geq x(\rho(n))$ for $u\leq\rho(n)\leq\sigma(u)$.

We have

$$\Delta\left\{x(n)\prod_{j=n_2}^{n-1}\left[1-\frac{p(j)}{a(j)}\right]^{-1}\right\}\geq\frac{\lambda}{a(n)}\left\{\sum_{i=n_2}^{n-1}q(i)\prod_{j=n_2}^{n}\left[1-\frac{p(j)}{a(j)}\right]^{-1}\right\}x(\rho(n)).$$

Summing the above inequality from n_2 to $\rho(n)-1$, we obtain

$$x(\rho(n))\prod_{j=n_2}^{\rho(n)-1}\left[1-\frac{p(j)}{a(j)}\right]^{-1}\geq\sum_{\ell=n_2}^{\rho(n)-1}\left\{\frac{\lambda}{a(\ell)}\sum_{i=n_2}^{\ell-1}q(i)\prod_{j=n_2}^{\ell}\left[1-\frac{p(j)}{a(j)}\right]^{-1}\right\}x(\rho(n))$$

$$\prod_{j=n_2}^{\rho(n)-1}\left[1-\frac{p(j)}{a(j)}\right]^{-1}\geq\sum_{\ell=n_2}^{\rho(n)-1}\left\{\frac{\lambda}{a(\ell)}\sum_{i=n_2}^{\ell-1}q(i)\prod_{j=n_2}^{\ell}\left[1-\frac{p(j)}{a(j)}\right]^{-1}\right\}.$$

Hence

$$\sum_{\ell=n_2}^{\rho(n)-1}\frac{\lambda}{a(\ell)}\sum_{i=n_2}^{\ell-1}q(i)\prod_{j=\ell+1}^{\rho(n)-1}\left[1-\frac{p(j)}{a(j)}\right]\leq 1. \tag{9}$$

The inequality (9) can be rewritten for $n_2\leq u\leq s\leq\rho(n)$ in the following form.

$$\lambda\sum_{\ell=n_2}^{\rho(n)-1}\sum_{i=n_2}^{\rho(n)-1}\frac{1}{a(i)}\prod_{j=\ell+1}^{\rho(n)-1}\left[1-\frac{p(j)}{a(j)}\right]q(\ell)\leq 1. \tag{10}$$

Taking limit superior of both sides of the inequality (10) as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\sup\left[\lambda\sum_{\ell=n_2}^{\rho(n)-1}\sum_{i=n_2}^{\rho(n)-1}\frac{1}{a(i)}\prod_{j=\ell+1}^{\rho(n)-1}\left[1-\frac{p(j)}{a(j)}\right]q(\ell)\right]\leq 1.$$

This contradicts the hypothesis (6).

If (ii) holds, then by using (H_3) in equation (1) we get

$$\Delta[a(n)\Delta x(n)]\geq q(n)f(x(\sigma(n)))\geq q(n)\lambda x(\sigma(n)) \tag{11}$$

eventually, where $\sigma(n)\geq\sigma(s), x(\sigma(s))\geq x(\sigma(n))$ for $n_2\leq\sigma(s)\leq\tau(n)\leq s\leq n$.

By the discrete Taylor formula for $x(\sigma(n))$, we can deduce the following inequality

$$x(\sigma(n))\geq[\sigma(s)-\tau(n)]\Delta x(\tau(n)). \tag{12}$$

Summing the inequality (11) from $\tau(n)$ to $n-1$ and using (12), we obtain

$$\sum_{j=\tau(n)}^{n-1} \Delta\left[a(j)\Delta x(j)\right] \geq \lambda \sum_{j=\tau(n)}^{n-1} q(j)x(\sigma(n)).$$

Thus

$$\frac{\lambda}{a(\tau(n))}\sum_{j=\tau(n)}^{n-1} q(j)[\sigma(j)-\tau(n)] \leq 1 \tag{13}$$

Taking limit superior of both sides of the inequality (13) as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\sup\left[\frac{\lambda}{a(\tau(n))}\sum_{j=\tau(n)}^{n-1} q(j)[\sigma(j)-\tau(n)]\right] \leq 1$$

This contradicts the condition (7). Thus $x(n)$ is an oscillatory solution of equation (1). Hence equation (1) is oscillatory when $\delta_1 = 1$ and $\delta_2 = -1$. □

Theorem 3. *Assume that the following conditions hold.*

(I) $\sigma(n) = n + \beta(n)$, $\sigma(n) \geq n$, $\Delta\sigma(n) \geq 0$ *and* $\Delta p(n) \geq 0$ *for* $n \geq n_0$.

(II) The equation $\Delta[a(n)\Delta x(n)] + \lambda q(n)x(\sigma(n)) = 0, n \geq n_0$ *is oscillatory.*

(III) $\lim_{n\to\infty}\sup\left[\frac{1}{\beta(n)-1}\sum_{i=n+1}^{\sigma(n)-1}\frac{\lambda}{a(i)}\sum_{j=s}^{\sigma(s)-1} q(j) - \frac{p(\sigma(i))}{a(i)}\right] > \lim_{n\to\infty}\sup\frac{(\beta(n)-1)^{\beta(n)-1}}{(\beta(n))^{\beta(n)}}$

Then the equation (1) is oscillatory when $\delta_1 = -1$ *and* $\delta_2 = 1$.

Proof. Let $x(n)$ be a nonoscillatory solution of equation (1) when $\delta_1 = -1$ and $\delta_2 = 1$. There is no loss of generality in assuming that there exists $n_1 \geq n_0$ such that $x(n) > 0$. Let $n_2 \geq n_1$ be chosen such that $x(\sigma(n)) > 0$ for $n \geq n_2$. From the proof of Theorem 1, we see that $\Delta x(n)$ is eventually of the same sign. Now we consider the following two cases.

(i) $\Delta x(n) < 0$ eventually.

(ii) $\Delta x(n) > 0$ eventually.

If (i) holds, then from equation (1) and using (H_3), we have

$$\Delta[a(n)\Delta x(n)] + \lambda q(n)x(\sigma(n)) \leq 0 \tag{14}$$

eventually.

Therefore by Lemma C, the equation $\Delta[a(n)\Delta x(n)]+\lambda q(n)x(\sigma(n)) = 0, n \geq n_0$ has a positive solution. This contradicts the given condition (ii).

If (ii) holds, then using (H_3) in equation (1) we obtain,

$$\Delta[a(n)\Delta x(n)] - p(n)\Delta x(n) + \lambda q(n)x(\sigma(n)) \le 0, \quad n \ge n_2. \tag{15}$$

Summing the above inequality (15) from n to $\sigma(n)-1$, we obtain

$$\sum_{i=n}^{\sigma(n)-1} \Delta\left[a(i)\Delta x(i)\right] - \sum_{i=n}^{\sigma(n)-1} p(i)\Delta x(i) + \lambda \sum_{i=n}^{\sigma(n)-1} q(i)x(\sigma(i)) \le 0,$$

$$-a(n)\Delta x(n) - p(\sigma(n))x(\sigma(n)) + \lambda x(\sigma(n)) \sum_{i=n}^{\sigma(n)-1} q(i) \le 0,$$

$$\Delta x(n) + \tfrac{p(\sigma(n))}{a(n)} x(\sigma(n)) - \tfrac{\lambda}{a(n)} x(\sigma(n)) \sum_{i=n}^{\sigma(n)-1} q(i) \ge 0,$$

$$\Delta x(n) - \left[\frac{\lambda}{a(n)} \sum_{j=n}^{\sigma(n)-1} q(j) - \frac{p(\sigma(n))}{a(n)}\right] x(\sigma(n)) \ge 0. \tag{16}$$

By condition (III), the inequality (16) has no eventually positive solution. This is a contradiction. Thus $x(n)$ is an oscillatory solution of equation (1). Therefore the equation (1) is oscillatory when $\delta_1 = -1$ and $\delta_2 = 1$. The proof is complete. □

Theorem 4. *Assume that the following conditions hold*

$$\lim_{n\to\infty} \sup \left\{ \frac{\lambda}{a(n)} \sum_{j=\tau(n)}^{n-1} q(j)\left[\sigma(j) - \tau(n)\right] \right\} > 1 \tag{17}$$

and

$$\lim_{n\to\infty} \inf \left\{ \sum_{i=-\tau(n)}^{n-1} \left(\frac{\lambda}{a(\tau(i))} \left[\sum_{j=\tau(n)}^{n-1} q(j) - \frac{p(\tau(i))}{a(\tau(i))} \right] \right) \right\} > \frac{1}{e}, \tag{18}$$

where $\sigma(n) = n + \beta(n)$ hold. Then the equation (1) is oscillatory when $\delta_1 = \delta_2 = -1$.

Proof. Let $x(n)$ be a nonoscillatory solution of equation (1) when $\delta_1 = \delta_2 = -1$. Without loss of generality, assume that $x(n) > 0$ for $n \ge n_1$. Let $n_2 \ge n_1$ be chosen such that $x(\sigma(n)) > 0$ for $n \ge n_2$. First we claim that $\Delta x(n)$ is eventually of the same sign. To show this we assume that $\{\Delta x(n)\}$ is oscillatory. Then there exists an integer $n_1 \ge n_0$ such that $\Delta x(n_1) > 0$ or

$\Delta x(n_1) = 0$. When $\Delta x(n_1) > 0$, we obtain from (1),

$$\begin{aligned}\Delta\left[a(n_1)\Delta x(n_1)\right] =& p(n_1)\Delta x(n_1) + q(n_1)f(x(\sigma(n_1))),\\ \geq& p(n_1)\Delta x(n_1),\\ a(n_1+1)\Delta x(n_1+1) \geq& \left[a(n_1) + p(n_1)\right]\Delta x(n_1),\\ \Delta x(n_1+1) >& 0.\end{aligned}$$

By induction, we obtain $\Delta x(n) > 0$ for $n \geq n_1$. Now consider $\Delta x(n_1) = 0$. From equation (1), it follows that

$$\begin{aligned}\Delta\left[a(n_1)\Delta x(n_1)\right] =& q(n_1)f(x(\sigma(n_1)) > 0,\\ a(n_1+1)\Delta x(n_1+1) >& 0,\\ \Delta x(n) >& 0 \text{ for } n \geq n_1,\end{aligned}$$

which contradicts the assumption that $\{\Delta x(n)\}$ is oscillatory. Hence $\{\Delta x(n)\}$ is not oscillatory.

Now we consider the following two cases.

(i) $\Delta x(n) > 0$ eventually

(ii) $\Delta x(n) < 0$ eventually.

If (i) holds, then from equation (1) and the condition (H_3), we have

$$\Delta[a(n)\Delta x(n)] - \lambda q(n)x(\sigma(n)) \geq 0 \tag{19}$$

eventually.

By the discrete Taylor expansion for $x(\sigma(n))$, we deduce the inequality

$$x(\sigma(n)) \geq [\tau(n) - \sigma(s)][-\Delta x(\tau(n))] \tag{20}$$

for $\sigma(s) \leq \tau(n) \leq s \leq n$.

Summing the inequality (19) from $\tau(n)$ to $n-1$ and using (20) we obtain

$$\frac{\lambda}{a(n)}\left[\sum_{j=\tau(n)}^{n-1}[\sigma(j) - \tau(n)]q(j)\right] \leq 1. \tag{21}$$

Taking limit superior of both sides of the inequality (21) as $n \to \infty$ we obtain

$$\limsup_{n\to\infty}\frac{\lambda}{a(n)}\left[\sum_{j=\tau(n)}^{n-1}[\sigma(j) - \tau(n)]q(j)\right] \leq 1.$$

This contradicts the condition (17).

If (ii) holds, then by (H_3), the equation (1) becomes

$$\Delta[a(n)\Delta x(n)] - p(n)\Delta x(n) - \lambda q(n)x(\sigma(n)) \geq 0, \quad n \geq n_2. \tag{22}$$

Summing the inequality (22) from $\tau(n)$ to $n-1$ for $\tau(n) \leq n$, we have

$$\sum_{i=\tau(n)}^{n-1} \Delta\left[a(i)\Delta x(i)\right] - \sum_{i=\tau(n)}^{n-1} p(i)\Delta x(i) - \lambda \sum_{i=\tau(n)}^{n-1} q(i)x(\sigma(i)) \geq 0,$$

$$a(\tau(n))\Delta x(\tau(n)) \leq p(\tau(n))x(\tau(n)) - \lambda \sum_{i=\tau(n)}^{n-1} q(i)x(\sigma(i))$$

$$\Delta x(\tau(n)) + \left[\frac{\lambda}{a(\tau(n))} \sum_{i=\tau(n)}^{n-1} q(i) - \frac{p(\tau(n))}{a(\tau(n))}\right] x(\tau(n)) \leq 0$$

for $\sigma(s) \leq \tau(n) \leq s \leq n, x(\sigma(s)) \geq x(\tau(n))$.

We note that $x(n) \leq x(\tau(n))$ and $\Delta x(n) \leq \Delta x(\tau(n))$ for $\tau(n) \leq n$. Hence we obtain

$$\Delta x(n) + \left[\frac{\lambda}{a(\tau(n))} \sum_{i=\tau(n)}^{n-1} q(i) - \frac{p(\tau(n))}{a(\tau(n))}\right] x(\tau(n)) \leq 0.$$

By (18), the above inequality has no eventually positive solution.

From Lemma C and Theorem D, we get a contradiction. Thus $x(n)$ is an oscillatory solution for equation (1). Therefore equation (1) is oscillatory when $\delta_1 = \delta_2 = -1$. The proof is complete. □

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York (2000).

[2] R.P. Agarwal, Said R. Grace, Donald O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht (2000).

[3] R.P. Agarwal, S.R. Grace, Jelena V. Manojlovic, On the oscillatory properties of certain fourth order non linear difference equation, *J. Math. Anal. Appl.*, **322** (2006), 930-956.

[4] Leonid Berenzansky, Elena Braverman, On existence of positive solutions for linear difference equations with several delays, *Advances in Dynamical Systems and Applications*, **1**, No. 1 (2006), 29-47.

[5] Yasar Bolat, Omer Akin, Huseyin Yildirim, Oscillation criteria for a even order neutral difference equation with an oscillating coefficient, *Applied Mathematics Letters*, doi:10.1016/j.aml.2008.06.036.

[6] George E. Chatzarakis, Roman Koplatadze, Ioannis P. Stavroulakis, Optimal oscillation criteria for first order difference equations with delay argument, *Pacific Journal of Matheamtics*, **235**, No. 1 (2008), 15-33.

[7] Khadija Niri, Ioannis P. Stavroulakis, On the oscillation of the solutions to delay and difference equations, *Tatra Mt. Math. Publ.*, **43** (2009), 173-187.

[8] Samir H. Saker, Sui Sun Cheng, Oscillation criteria for difference equations with damping terms, *Applied Mathematics and Computation*, **148** (2004), 421-442.

[9] S.H. Saker, B.G. Zhang, Oscillation of second order non linear delay damped difference equations, *Acta Mathematica Sinica, English Series*, **23**, No. 4 (2007), 715-722.

[10] E. Thandapani, B.S. Lalli, Oscillation criteria for a second order damped difference equations, *Appli. Math. Lett.*, **8**, No. 1 (1995), 1-6.

[11] B.G. Zhang, Chuan Jun Tian, Oscillation criteria for difference equations with unbounded delay, *Computers Math. Appl.*, **35**, No. 4 (1998), 19-26.