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A NOTE ON THE STRONG LAW OF LARGE NUMBERS FOR RANDOM FIELDS

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Abstract: In this note, we present a strong law of large numbers for blockwise-independent (resp. m-dependent) random fields under generalized Chung's type conditions.

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1. Introduction

The aim of this paper is to study certain strong law of large numbers of random fields (in abbreviation: r.f.) under generalized Chung's type conditions (see Cvetan et al [2] and Yang [4]).

Assume $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a r.f.'s on probability space $(\Omega, \mathcal{F}, P)$ with partial sums $S_{\mathbf{n}} = \sum_{\mathbf{i} \prec \mathbf{n}} X_{\mathbf{i}}$, where the random fields or index set $\mathbb{Z}_+^d$ be a positive integer with coordinatewise partial ordering $\prec$, that is, $\mathbf{m} \prec \mathbf{n}$ means that $m_i \leq n_i, 1 \leq i \leq d$, where $\mathbf{m} = (m_1, m_2, \cdots, m_d)$ and $\mathbf{n} = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$. Let $\{\alpha_i, 1 \leq i \leq d\}$ be positive constants. We denote

$$|\mathbf{n}| = \prod_{i=1}^{d} n_i, \quad |\mathbf{n}(\alpha)| = \prod_{i=1}^{d} n_i^{\alpha_i},$$

$$I(\mathbf{n}) = \{(a_1, \cdots, a_d) \in \mathbb{Z}_+^d : a^{n_i - 1} \leq a_i < a^{n_i},\ 1 \leq i \leq d\}, \quad a > 1.$$

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Definition 1. (see Thanh [3]) A r.f.'s $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ is said to be blockwise independent, if for each $\mathbf{k} \in \mathbb{Z}_+^d$, the random variables $\{X_{\mathbf{i}}, \mathbf{i} \in I(\mathbf{k})\}$ are independent.

Definition 2. (see Móricz et al [1]) A r.f.'s $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ is said to be blockwise m-dependent, if for each $\mathbf{k} \in \mathbb{Z}_+^d$, the random variables $\{X_{\mathbf{k}}, X_{\mathbf{l}} \in I(\mathbf{k})\}$ is independent for all $\mathbf{k}, \mathbf{l}$ with $\min_{1 \le i \le d}\{|k_i - l_i|\} > m$.

An article which is important for this paper is that of Thanh [3]. He established a lemma that is an important tool for deriving results on strong limit theorems for r.f.'s. In order to prove our main results, we first give the lemma (a minima modification), and it will be shown that it plays a key role in the proof.

Lemma 1. (see Móricz et al [1] and Thanh [3]) *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a r.f.'s with mean 0 blockwise independent (resp. m-dependent) random variables and let $\{\alpha_i, 1 \le i \le d\}$ be positive constants. If*

$$\sum_{\mathbf{n} \in \mathbb{Z}_+^d} \frac{E|X_{\mathbf{n}}|^p}{|\mathbf{n}(\alpha)|^p} < \infty \text{ for some } 0 < p \le 2, \tag{1.1}$$

then SLLN

$$\lim_{|\mathbf{n}| \to \infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}(\alpha)|} = 0 \quad a.s. \tag{1.2}$$

In the case $0 < p \le 1$, the independent (resp. m-dependent) hypothesis and the hypothesis that $EX_{\mathbf{n}} = 0, \mathbf{n} \in \mathbb{Z}_+^d$ are superfluous.

2. The Main Results

Now we come to our main result.

Theorem 1. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a r.f.'s and let $\{\alpha_i, 1 \le i \le d\}$ be positive constants. Assume that $\varphi_{\mathbf{n}} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be Borel functions.*

If either (1)

$$\varphi_{\mathbf{n}}(t) \nearrow, \text{ and } t_1 \le t_2 \quad \Rightarrow \quad \frac{t_1^{a_{\mathbf{n}}}}{\varphi_{\mathbf{n}}(t_1)} \le K_{\mathbf{n}} \frac{t_2^{a_{\mathbf{n}}}}{\varphi_{\mathbf{n}}(t_2)}, \tag{2.1}$$

where $0 < a_{\mathbf{n}} \le 1, K_{\mathbf{n}} > 0, \mathbf{n} \in \mathbb{Z}_+^d$, or (2) $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a r.f.'s of blockwise independent (resp. m-dependent) random variables and $EX_{\mathbf{n}} = 0$, and

$$t_1 \le t_2 \quad \Rightarrow \quad \frac{\varphi_{\mathbf{n}}(t_1)}{t_1^{a_{\mathbf{n}}}} \le K_{\mathbf{n}} \frac{\varphi_{\mathbf{n}}(t_2)}{t_2^{a_{\mathbf{n}}}} \tag{2.2}$$

and

$$\frac{t_1^{b_{\mathbf{n}}}}{\varphi_{\mathbf{n}}(t_1)} \leq M_{\mathbf{n}} \frac{t_2^{b_{\mathbf{n}}}}{\varphi_{\mathbf{n}}(t_2)} \tag{2.3}$$

where $a_{\mathbf{n}} \geq 1, b_{\mathbf{n}} \leq 2, K_{\mathbf{n}} > 0, M_{\mathbf{n}} \geq 1, \mathbf{n} \in \mathbb{Z}_+^d$.

Set

$$\mathcal{D} = \left\{ \omega : \sum_{\mathbf{n} \in \mathbb{Z}_+^d} D_{\mathbf{n}} \frac{E(\varphi_{\mathbf{n}}(|X_{\mathbf{n}}|))}{\varphi_{\mathbf{n}}(|\mathbf{n}(\alpha)|)} < \infty \right\}, \tag{2.4}$$

where $D_{\mathbf{n}} = \max\{K_{\mathbf{n}}, M_{\mathbf{n}}\}, \mathbf{n} \in \mathbb{Z}_+^d$, *then SLLN*

$$\lim_{|\mathbf{n}| \to \infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}(\alpha)|} = 0 \quad a.s. \quad \omega \in \mathcal{D}. \tag{2.5}$$

Proof. We will follow the same way as Thanh [3] using the truncation method. For $\mathbf{n} \in \mathbb{Z}_+^d$, set $Y_{\mathbf{n}} = X_{\mathbf{n}} 1_{(|X_{\mathbf{n}}| \leq |\mathbf{n}(\alpha)|)}, Z_{\mathbf{n}} = X_{\mathbf{n}} 1_{(|X_{\mathbf{n}}| > |\mathbf{n}(\alpha)|)}$. It is easy to see that $Y_{\mathbf{n}}$ and $Z_{\mathbf{n}}$ are also blockwise independent (resp. m-dependent). Consider the case (1). It follows from (2.2) and (2.4) that

$$\sum_{\mathbf{n} \in \mathbb{Z}_+^d} \frac{E|Y_{\mathbf{n}}|}{|\mathbf{n}(\alpha)|} \leq \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \frac{E|Y_{\mathbf{n}}|^{a_{\mathbf{n}}}}{|\mathbf{n}(\alpha)|^{a_{\mathbf{n}}}} \leq \sum_{\mathbf{n} \in \mathbb{Z}_+^d} K_{\mathbf{n}} \frac{E(\varphi_{\mathbf{n}}(|Y_{\mathbf{n}}|))}{\varphi_{\mathbf{n}}(|\mathbf{n}(\alpha)|)} < \infty, \quad \omega \in \mathcal{D}. \tag{2.6}$$

By Lemma 1,

$$\lim_{|\mathbf{n}| \to \infty} \frac{\sum_{\mathbf{i} \prec \mathbf{n}} Y_{\mathbf{i}}}{|\mathbf{n}(\alpha)|} = 0 \quad a.s. \ \omega \in \mathcal{D}. \tag{2.7}$$

On the other hand

$$\begin{aligned}
\sum_{\mathbf{n} \in \mathbb{Z}_+^d} P\{X_{\mathbf{n}} \neq Y_{\mathbf{n}}\} &= \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \int_{(|X_{\mathbf{n}}| > |\mathbf{n}(\alpha)|)} dP \\
&\leq \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \int_{(|X_{\mathbf{n}}| \geq |\mathbf{n}(\alpha)|)} \frac{\varphi_{\mathbf{n}}(|X_{\mathbf{n}}|)}{\varphi_{\mathbf{n}}(|\mathbf{n}(\alpha)|)} dP \\
&\leq \sum_{\mathbf{n} \in \mathbb{Z}_+^d} D_{\mathbf{n}} \frac{E(\varphi_{\mathbf{n}}(|X_{\mathbf{n}}|))}{\varphi_{\mathbf{n}}(|\mathbf{n}(\alpha)|)} < \infty, \quad \text{a.s.} \quad \omega \in \mathcal{D}.
\end{aligned}$$

By the Borel-Cantelli Lemma,

$$\lim_{|\mathbf{n}| \to \infty} \frac{\sum_{\mathbf{i} \prec \mathbf{n}} (X_{\mathbf{i}} - Y_{\mathbf{i}})}{|\mathbf{n}(\alpha)|} = 0, \quad \text{a.s.} \quad \omega \in \mathcal{D}. \tag{2.8}$$

The conclusion (2.5) follows from (2.7) and (2.8).

Next, consider the case (2). Noticing that

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(Y_\mathbf{n}-EY_\mathbf{n})^2}{|\mathbf{n}(\alpha)|^2} &\le \sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(Y_\mathbf{n})^2}{|\mathbf{n}(\alpha)|^2}\\ &\le \sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(Y_\mathbf{n})^{b_\mathbf{n}}}{|\mathbf{n}(\alpha)|^{b_\mathbf{n}}}\\ &\le \sum_{\mathbf{n}\in\mathbb{Z}_+^d}M_\mathbf{n}\frac{E(\varphi_\mathbf{n}(|X_\mathbf{n}|))}{\varphi_\mathbf{n}(|\mathbf{n}(\alpha)|)}<\infty,\quad \omega\in\mathcal{D}.\end{aligned}$$

By Lemma 1,

$$\lim_{|\mathbf{n}|\to\infty}\frac{\sum_{\mathbf{i}\prec\mathbf{n}}(Y_\mathbf{i}-EY_\mathbf{i})}{|\mathbf{n}(\alpha)|}=0\ \text{ a.s. }\ \omega\in\mathcal{D} \tag{2.9}$$

and

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E|Z_\mathbf{n}-EZ_\mathbf{n}|}{|\mathbf{n}(\alpha)|} &\le 2\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E|Z_\mathbf{n}|}{|\mathbf{n}(\alpha)|}=2\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E|Y_\mathbf{n}|}{|\mathbf{n}(\alpha)|}\\ &\le 2\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E|Y_\mathbf{n}|^{a_\mathbf{n}}}{|\mathbf{n}(\alpha)|^{a_\mathbf{n}}}\le 2\sum_{\mathbf{n}\in\mathbb{Z}_+^d}M_\mathbf{n}\frac{E(\varphi_\mathbf{n}(|Y_\mathbf{n}|))}{\varphi_\mathbf{n}(|\mathbf{n}(\alpha)|)}\\ &\le 2\sum_{\mathbf{n}\in\mathbb{Z}_+^d}D_\mathbf{n}\frac{E(\varphi_\mathbf{n}(|X_\mathbf{n}|))}{\varphi_\mathbf{n}(|\mathbf{n}(\alpha)|)}<\infty\quad \omega\in\mathcal{D},\end{aligned}$$

which implies

$$\lim_{|\mathbf{n}|\to\infty}\frac{\sum_{\mathbf{i}\prec\mathbf{n}}(Z_\mathbf{i}-EZ_\mathbf{i})}{|\mathbf{n}(\alpha)|}=0\ \text{ a.s. }\ \omega\in\mathcal{D}. \tag{2.10}$$

(2.5) follows immediately from (2.9) and (2.10). □

Corollary 1. *Let $\{X_\mathbf{n},\mathbf{n}\in\mathbb{Z}_+^d\}$ be a r.f.'s of blockwise independent (resp. m-dependent) mean 0 random variables and let $\{\alpha_i, 1\le i\le d\}$ be positive constants. If*

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(|X_\mathbf{n}|\log_a(|X_\mathbf{n}|))}{|\mathbf{n}(\alpha)|\log_a(|\mathbf{n}(\alpha)|)}<\infty, \tag{2.11}$$

then SLLN

$$\lim_{|\mathbf{n}|\to\infty}\frac{S_\mathbf{n}}{|\mathbf{n}(\alpha)|}=0\ \text{ a.s.} \tag{2.12}$$

Proof. Let $\varphi_\mathbf{n}(t)=t\log_a t$ in Theorem 1. □

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