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UPPER MINUS DOMINATION NUMBER IN K-REGULAR GRAPHS

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Abstract: Let $G = (V, E)$ be a graph. A function $f : V(G) \to \{-1, 0, 1\}$ defined on the the vertices of G is a minus domination function, if the sum of its function values over any closed neighborhood is at least one. The weight of a minus domination function is $\omega(f) = f(V) = \sum_{v \in V} f(v)$. The upper minus domination number of a graph G, denoted $\Gamma^-(G)$, equals the maximum weight of all the minimal minus domination functions of G.

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1. Introduction

In this paper all graphs under consideration are undirected, finite and simple. A graph, denoted by $G = (V, E)$, consists of a non-empty set V of vertices, and a set E of edges. The order of G is $n = |V|$. For a vertex $v \in V$, the open neighborhood of v is $\{N(v) = \{u \in V | uv \in E\}$, and the closed neighborhood of v is $N[v] = N(v) \bigcup v$. The degree of v in G is denoted by $d(v)$, a graph G is called k-regular if $\forall v \in V, d(v) = k$.

For a subset $S \subseteq V$, we use $d_S(v)$ to denote the number of vertices in S that are adjacent to v. For disjoint subsets U and W of vertices, we let $e(U, W)$ to denote the number of edges between U and W.

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Let $f : V \to \{-1, 0, 1\}$ be a function which assigns to each vertex of G an element in the set $\{-1, 0, 1\}$, then f is called a minus domination function of G, if $f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1$ for every $v \in V$. A minus domination function f is minimal if every minus domination function g satisfying $g(v) \leq f(v)$ for every $v \in V$, is equal to f. The minus domination number of a graph G, denoted by $\gamma^-(G)$ is the minimum weight of a minus dominating function of G. The upper minus domination number of a graph G, denoted by $\Gamma^-(G)$ equals the maximum weight of all the minimal minus domination functions of G. That is

$$\gamma^-(G) = \min\{\omega(f) \mid f \text{ is a minus domination function on } G\},$$

$$\Gamma^-(G) = \max\{\omega(f) \mid f \text{ is a minimal minus domination function on } G\}.$$

The weight of f on a graph $G = (V,\ E)$ is $\omega(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V)$. For a vertex $v \in V$, we denote: $f(N[v])$ by $f[v]$ for notational convenience. We call a minimal minus domination function of weight $\Gamma^-(G)$ a $\Gamma^-(G)-$function.

Let P, Q and M denote the sets of those vertices in G, which are assigned under f the value $1, 0, -1$, that is;

$$\begin{aligned} P &= \{v \in V|\ f(v) = 1\}, \\ Q &= \{v \in V|\ f(v) = 0\}, \\ M &= \{v \in V|\ f(v) = -1\}. \end{aligned}$$

Let $|P| = p,\ |Q| = q,\ |M| = m$. Thus $n = p + q + m$, $\omega(f) = |P| - |M| = n - q - 2m$.

Further we define:

$$\begin{aligned} P_{ij} &= \{v \in P|\ d_Q(v) = i, d_M(v) = j\}, \\ Q_{ij} &= \{v \in Q|\ d_P(v) = i, d_M(v) = j\}, \\ M_{ij} &= \{v \in M|\ d_P(v) = i, d_Q(v) = j\}. \end{aligned}$$

Liying Kang et al [3] proved that if G is a cubic graph of order n, then $\Gamma^-(G) \leq \frac{5}{8}n$. Hong Yan et al [2] proved the upper bounds on the upper minus total domination number of 3 and 4 regular graph. Many results on minus domination or signed domination or total domination in graphs have been presented by various authors [4], [6], [2], [7]. Based on these results, in this paper, we discuss the domination number of k-regular graph.

Lemma 1. (see [1]) *A minus domination function on a graph $G = (V, E)$ is minimal if and only if for every vertex $v \in V$ with $f(v) \geq 0$, there exists $u \in N[v]$, such that $f[u] = 1$.*

2. Main Result

Theorem 2. *If G is a k-regular graph of order n, then*

$$\Gamma^-(G) \leq \frac{2k-1}{2(k+1)}n.$$

Proof. Let f be a $\Gamma^-(G)-$function. Then $\Gamma^-(G) = |P| - |M| = p - m$. Hence we can partition P, Q and M into the following sets, respectively.

$$P_{ij} = \{v \in P|\ d_Q(v) = i, d_M(v) = j, \text{where } 0 \leq j \leq \lfloor\frac{k}{2}\rfloor, 0 \leq i \leq k-2j\},$$

$$Q_{ij} = \{v \in Q|\ d_P(v) = i, d_M(v) = j, \text{where } 0 \leq j \leq \lfloor\frac{k}{2}\rfloor, j+1 \leq i \leq k-j\},$$

$$M_{ij}$$
$$= \{v \in M|\ d_P(v) = i, d_Q(v) = j, \text{where } 0 \leq j \leq \lceil\frac{k}{2}\rceil, \lfloor\frac{k-j}{2}\rfloor \leq i \leq k-j\}\}.$$

Let $|P_{ij}| = p_{ij}, |Q_{ij}| = q_{ij}, |M_{ij}| = m_{ij}$. Further, we write

$$P^{'} = \bigcup_{k-2j-i=0} P_{ij},\ Q^{'} = \bigcup_{i-j=1} Q_{ij},\ M^{'} = \bigcup_{2i+j-k=2} M_{ij}.$$

Clearly, $\forall\ v\ \in P^{'} \bigcup Q^{'} \bigcup M^{'}, f[v] = 1$, while $\forall\ v \in V - (P^{'} \bigcup Q^{'} \bigcup M^{'}), f[v] \geq 2$. By counting the edge number, we can immediately get the following equalities:

$$\sum jq_{ij} = e(Q, M) = \sum jm_{ij},$$
$$\sum jp_{ij} = e(P, M) = \sum im_{ij} = km - \sum (k-i)m_{ij},$$
$$\sum ip_{ij} = e(P, Q) = \sum iq_{ij} = kq - \sum (k-i)q_{ij}.$$

By Lemma 1, for every vertex $v \in P - P'$, there exists a vertex $u \in N[v]$ such that $f[u] = 1$. It follows that for every vertex $v \in P - P'$, there must exist a neighbor v' of v that belongs to $P' \bigcup Q' \bigcup M'$. So we can get

$$\sum_{k-2j-i\neq 0} p_{ij} \leq e(P - P',\ P' \bigcup Q' \bigcup M')$$

$$= e(P - P',\ P') + e(P - P',\ Q' \bigcup M'),$$

$\forall\ v\ \in P_{ij} \subset P'$. There exists a neighbor v' of v satisfying $f[v'] = 1$, that is $v' \in P' \bigcup Q' \bigcup M'$. If $v' \in P'$ then v is adjacent to at most $k - i - j$ vertices of $P - P'$. If $v' \in Q' \bigcup M'$ then v is adjacent to at most $k - i - j$ vertices of $P - P'$. Hence, we can write P_{ij} as the disjoint union of two sets

$$P^{'}_{ij} = \{v \in P_{ij}|\ d_{P-P'}(v) = k - i - j\} \text{ and } P''_{ij} = P_{ij} - P'_{ij}.$$

Let $|P'_{ij}| = p_{ij}$, and so $|P''_{ij}| = p_{ij} - p'_{ij}$. Since each vertex $v \in P'_{ij}$ is adjacent to at least one vertex of $Q' \bigcup M'$, it follows that

$$p'_{ij} \leq e(P'_{ij},\ Q' \bigcup M'),$$

then

$$\begin{aligned} e(P - P',\ P') &= e(P - P',\ \bigcup_{k-2j-i=0} P_{ij}) \\ &= e(P - P',\ \bigcup_{k-2j-i=0} P'_{ij}) + e(P - P',\ \bigcup_{k-2j-i=0} P''_{ij}) \\ &= \sum_{k-2j-i=0} (k-j-i)p'_{ij} + \sum_{k-2j-i=0} (k-j-i-1)p''_{ij} \\ &\leq \sum_{k-2j-i=0} (k-j-i-1)p_{ij} + \sum_{k-2j-i=0} e(P'_{ij}, Q' \bigcup M') \\ &\leq \sum_{k-2j-i=0} (j-1)p_{ij} + \sum_{k-2j-i=0} e(P'_{ij}, Q' \bigcup M'). \end{aligned}$$

So we have

$$\begin{aligned} &\sum_{k-2j-i\neq 0} p_{ij} \\ &\leq \sum_{k-2j-i=0} (j-1)p_{ij} + \sum_{k-2j-i=0} e(P'_{ij}, Q' \bigcup M') + e(P - P', Q' \bigcup M') \\ &= \sum_{k-2j-i=0} (j-1)p_{ij} + e(\bigcup_{k-2j-i=0} P'_{ij} \bigcup (P - P'), Q' \bigcup M') \\ &= \sum_{k-2j-i=0} (j-1)p_{ij} + \sum_{i-j=1} iq_{ij} + \sum_{2i+j-k=2} im_{ij}. \end{aligned}$$

Using the above equalities and inequalities, we immediately get

$$\begin{aligned} n &= (q+m) + p = (q+m) + \sum p_{ij} \\ &= (q+m) + \sum_{k-2j-i\neq 0} p_{ij} + \sum_{k-2j-i=0} p_{ij} \\ &= (q+m) + \sum_{k-2j-i=0} jp_{ij} + \sum_{i-j=1} iq_{ij} + \sum_{2i+j-k=2} im_{ij} - (k-1)p_{k0} \\ &= (q+m) + km - \sum (k-i)m_{ij} + kq - \sum (k-i)q_{ij} - (k-1)p_{k0} \\ &\leq (k+1)(q+m) - [\ \sum (k-i)m_{ij} + \sum (k-i)q_{ij}\ ] - (k-1)p_{k0}. \end{aligned}$$

We obtain

$$q+m \geq \frac{1}{k+1}n+\frac{1}{k+1}[\sum(k-i)m_{ij}+\sum(k-i)q_{ij} \ +(k-1)p_{k0}].$$

So

$$\begin{aligned} p &= n-(q+m) \\ &\leq \frac{k}{k+1}n-\frac{1}{k+1}[\sum(k-i)m_{ij}+\sum(k-i)q_{ij} \ +(k-1)p_{k0}] \\ &=\leq \frac{k}{k+1}n-\frac{1}{k+1}a, \end{aligned}$$

where $a=\sum(k-i)m_{ij}+\sum(k-i)q_{ij} \ +(k-1)p_{k0}$.

On the other hand, we can get

$$\begin{aligned} p &= \sum_{k-2j-i\neq 0} p_{ij}+\sum_{k-2j-i=0} p_{ij} \\ &\leq \sum_{k-2j-i=0}(j-1)p_{ij}+\sum_{k-2j-i=0} p_{ij}+\sum_{i-j=1} iq_{ij}+\sum_{2i+j-k=2} im_{ij} \\ &= \sum_{k-2j-i=0} jp_{ij}+\sum_{i-j=1} iq_{ij}+\sum_{2i+j-k=2} im_{ij}+p_{k0} \\ &\leq km-\sum(k-i)m_{ij}+\sum_{2i+j-k=2} im_{ij}+\sum_{i-j=1} iq_{ij}+p_{k0}. \end{aligned}$$

As

$$\begin{aligned} \sum jq_{ij} &= \sum jm_{ij} \\ &= \sum_{i-j=1} jq_{ij}+\sum_{i-j\neq 1} jq_{ij} \\ &= \sum_{i-j=1} q_{ij}+\sum jq_{ij}-\sum_{i-j\neq 1} jq_{ij}, \end{aligned}$$

$$\begin{aligned} \sum(k-i)m_{ij}-\sum_{2i+j-k=2} im_{ij} &= \sum_{2i+j-k\neq 2} m_{ij}+\sum_{2i+j-k=2}(k-i)m_{ij}-\sum_{2i+j-k=2} im_{ij} \\ &= \sum_{2i+j-k\neq 2}(k-i)m_{ij}+\sum_{2i+j-k=2} m_{ij}. \end{aligned}$$

Thus

$$\begin{aligned} p &\leq km-\sum_{2i+j-k=2}(k-2i)m_{ij}-\sum_{2i+j-k\neq o}(k-i)m_{ij}+p_{k0}+\sum_{i-j=1} q_{ij}+\sum jq_{ij} \\ &-\sum_{i-j\neq 1} jq_{ij} = km-\sum_{2i+j-k=2}(k-2i)m_{ij}-\sum_{2i+j-k\neq 2}(k-i)m_{ij}+p_{k0}+\sum jm_{ij} \end{aligned}$$

$$-\sum_{i-j\neq 1} jq_{ij}+\sum_{i-j=1} q_{ij}=km-\sum_{2i+j-k=2}(k-i)m_{ij}-\sum_{2i+j-k\neq 2} km_{ij}+p_{k0}$$
$$+\sum(j+i)m_{ij}-\sum_{i-j\neq 1} jq_{ij}+\sum_{i-j=1} q_{ij}$$
$$\leq 2km-[\sum_{2i+j-k=2}(k-i)m_{ij}+\sum_{2i+j-k\neq 2} km_{ij}+\sum_{i-j\neq 1} jq_{ij}-\sum_{i-j=1} q_{ij}-p_{k0}],$$

which implies

$$m\geq \frac{1}{2k}p+\frac{1}{2k}[\sum_{2i+j-k=2}(k-i)m_{ij}+\sum_{2i+j-k\neq 2} km_{ij}+\sum_{i-j\neq 1} jq_{ij}-\sum_{i-j=1} q_{ij}-p_{k0}]$$
$$\geq \frac{1}{2k}p+\frac{1}{2k}b,$$

where

$$b=\sum_{2i+j-k=2}(k-i)m_{ij}+\sum_{2i+j-k\neq 2} km_{ij}+\sum_{i-j\neq 1} jq_{ij}-\sum_{i-j=1} q_{ij}-p_{k0}.$$

Therefore

$$\Gamma^{-}(G)=p-m$$
$$\leq \frac{2k-1}{2k}p-\frac{1}{2k}[\sum_{2i+j-k=2}(k-i)m_{ij}+\sum_{2i+j-k\neq 2} km_{ij}+\sum_{i-j\neq 1} jq_{ij}-\sum_{i-j=1} q_{ij}-p_{k0}]$$
$$\leq \frac{2k-1}{2k}\{\frac{k}{k+1}n-\frac{1}{k+1}a-\frac{1}{2k}b\}$$
$$=\frac{2k-1}{2(k+1)}n-\frac{2k-1}{2k(k+1)}a-\frac{1}{2k}b=\frac{2k-1}{2(k+1)}n-\frac{1}{2k(k+1)}[(2k-1)a+(k+1)b].$$

As

$$(2k-1)a+(k+1)b=(2k-1)\sum(k-i)m_{ij}$$
$$+(2k-1)\sum(k-i)q_{ij}+(2k-1)(k-1)p_{k0}+(k+1)\sum_{2i+j-k=2}(k-i)m_{ij}$$
$$+(k+1)\sum_{2i+j-k\neq 2} km_{ij}+(k+1)\sum_{i-j\neq 1} jq_{ij}-\sum_{i-j=1} q_{ij}-(k+1)p_{k0}$$
$$=(2k-1)\sum(k-i)m_{ij}+(2k-1)\sum(k-i)q_{ij}+2k(k-1)p_{k0}+(k+1)\sum(k-i)m_{ij}$$
$$+(k+1)\sum_{2i+j-k\neq 2} im_{ij}+(k+1)\sum_{i-j\neq 1} jq_{ij}-\sum_{i-j=1} q_{ij}$$
$$=3k\sum(k-i)m_{ij}+(k+1)\sum_{2i+j-k\neq 2} im_{ij}$$

$$+ (2k-1)\sum(k-i-1)q_{ij} + (2k-1)\sum_{i-j\neq 1} q_{ij} + (k+1)\sum_{i-j\neq 1} jq_{ij} + 2k(k-1)p_{k0}$$

$$\geq 0.$$

Therefore

$$\Gamma^-(G) \leq \frac{2k-1}{2(k-1)}n. \qquad \square$$

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