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SOME CONVERGENCE RESULTS FOR TWO NEW ITERATION PROCESSES IN UNIFORMLY CONVEX BANACH SPACE

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Abstract: In this paper, following the concepts in [6, 8], we shall establish some convergence results for nonexpansive operators in a uniformly convex Banach space. Two new iteration processes will be considered for this purpose.

Our results improve, generalize and extend those of [4, 7, 8, 9, 12, 13].

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1. Introduction

Suppose that $A = (a_{nk})$ is an infinite, lower triangular, regular row-stochastic matrix, E a closed convex subset of a Banach space and T is a continuous mapping of E into itself and $x_1 \in E$. Then, the general Mann iteration process $M(x_1, A, T)$ which was introduced in Mann [9] is defined by

$$v_n = \sum_{k=1}^{n} a_{nk} x_k, \quad x_{n+1} = T v_n, \quad n = 1, 2, \cdots . \tag{1}$$

If A is the identity matrix, then each sequence of $M(x_1, A, T)$ becomes the sequence of Picard iterates of T at x_1. It was established in [9] that if either of the sequences $\{x_n\}$ and $\{v_n\}$ converges, then the other also converges to the

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same point, and their common limit is a fixed point of T.

In [6, 8], it is said that the matrix A is *segmenting* for the Mann process if $a_{n+1,k} = (1 - a_{n+1,n+1})a_{nk}$ for $k \leq n$. In this case, v_{n+1} lies on the segment joining v_n and Tv_n:

$$v_{n+1} = (1 - d_n)v_n + d_n T v_n, \qquad n = 1, 2, \cdots, \tag{2}$$

where $d_n = a_{n+1,n+1}$.

A segmenting matrix is determined by its sequence of diagonal elements. Some authors including [4, 12, 13] have investigated the case $d_n = \lambda,\ 0 < \lambda < 1$, while Mann [9] approximated the fixed points of continuous functions on a closed interval of the real line using the segmenting matrix determined by $d_n = \frac{1}{n},\ \forall\ n$. Dotson [7] considered the case when d_n is bounded away from 0 and 1. Groetsch [8] generalized the results of [4, 7, 9, 12, 13] in a uniformly convex Banach space by employing (2) and assuming that A is a segmenting matrix for which $\sum_{n=1}^{\infty} d_n(1 - d_n) = \infty$.

We shall give another definition of a segmenting matrix in the next section with a view to generalizing and extending Groetsch [8] and others mentioned earlier in this paper.

2. Preliminaries

We shall introduce and employ the following iteration processes: Let E be a Banach space, $T_i : E \to E$ $(i = 0, 1, \cdots, k)$ selfmaps of E and $x_0 \in E$. Then, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$\left.\begin{aligned} x_{n+1} &= \alpha_{n,\ 0}\ x_n + \sum_{i=1}^{k} \alpha_{n,\ i}\ T_i y_n,\ \ \sum_{i=0}^{k} \alpha_{n,\ i} = 1,\ \ n = 0, 1, 2, \ldots, \\ y_n &= \sum_{j=0}^{s} \beta_{n,\ j}\ T_j x_n,\ \ \sum_{j=0}^{s} \beta_{n,\ j} = 1, \end{aligned}\right\} \tag{3}$$

$k \geq s,\ \alpha_{n,\ i} \geq 0,\ \alpha_{n,\ 0} \neq 0,\ \beta_{n,\ j} \geq 0,\ \beta_{n,\ 0} \neq 0,\ \alpha_{n,\ i},\ \beta_{n,\ j} \in [0, 1]$, where k and s are fixed integers and T_0 is an identity operator.

If $s = 0$ in (3), we also obtain the following interesting iteration process in a Banach space:

$$x_{n+1} = \sum_{i=0}^{k} \alpha_{n,\ i}\ T_i x_n,\ \ \sum_{i=0}^{k} \alpha_{n,\ i} = 1,\ \ n = 0, 1, 2, ..., \tag{4}$$

$\alpha_{n,\ i} \geq 0,\ \alpha_{n,\ 0} \neq 0,\ \alpha_{n,\ i} \in [0, 1]$, where k is a fixed integer and T_0 is an identity operator.

(i) If $s = 0,\ k = 1$ in (3), then we have $y_n = \beta_{n,\ 0}\ x_n = x_n,\ \beta_{n,\ 0} = 1$ and $x_{n+1} = (1 - \alpha_{n,\ 1})x_n + \alpha_{n,\ 1}T_1x_n$, which is the usual Mann iteration process with $\sum_{i=0}^{1} \alpha_{n,\ i} = 1,\ \alpha_{n,\ 1} = \alpha_n$.

(ii) Also, if $s = k = 1$, in (3), we get

$$\left.\begin{aligned} x_{n+1} &= (1 - \alpha_{n,\ 1})x_n + \alpha_{n,\ 1}T_1y_n \\ y_n &= (1 - \beta_{n,\ 1})x_n + \beta_{n,\ 1}T_1x_n, \end{aligned}\right\}$$

which is the usual Ishikawa iteration process with $\sum_{i=0}^{1} \alpha_{n,\ i} = \sum_{i=0}^{1} \beta_{n,\ i} = 1$, $\alpha_{n,\ 1} = \alpha_n,\ \beta_{n,\ 1} = \beta_n$.

(iii) If $s = 0,\ \alpha_{n,\ i} = \alpha_i$ and $T_i = T^i$ in (3), then we obtain the usual Kirk's iteration process

$$x_{n+1} = \sum_{i=0}^{k} \alpha_i\ T^i x_n,\ \ \sum_{i=0}^{k} \alpha_i = 1,\ n = 0, 1, 2, \cdots, \tag{5}$$

with $y_n = \beta_{n,\ 0}\ x_n = x_n,$ since $\beta_{n,\ 0} = 1$.

Equation (4) is also a generalization of Picard, Schaefer, Mann and the Kirk's iteration processes. See Berinde [1] for detail on the various already existing fixed point iteration processes.

In this paper, we shall establish some convergence results for nonexpansive operators in a uniformly convex Banach space using the newly introduced iteration processes defined in (3) and (4). We shall assume that A is a segmenting matrix such that $\sum_{n=0}^{\infty} \alpha_{n,\ 0}(1 - \alpha_{n,\ 0}) = \infty$.

Our results improve, generalize and extend those of [4, 7, 8, 9, 12, 13].

Lemma 2.1. (see Groetsch [5, 8]) *Let X be a uniformly convex Banach space and let $x,\ y \in X$. If $||x|| \le 1,\ ||y|| \le 1$ and $||x - y|| \ge \epsilon > 0$, then*

$$||\lambda x + (1 - \lambda)y|| \le 1 - 2\lambda(1 - \lambda)\delta(\epsilon) \quad \text{for} \quad 0 \le \lambda < 1.$$

3. The Main Results

Theorem 3.1. *Let E be a convex subset of a uniformly convex Banach space X and $T_i : E \to E$ $(i = 0, 1, 2, \cdots, k)$ nonexpansive mappings with at least a common fixed point. Let $\{v_n\}_{n=0}^{\infty}$ be the sequence defined by (3). Then, the sequence $\left\{(I - T_i^j)v_n\right\}_{n=0}^{\infty}$, for each $j \in \mathbb{N},\ 1 \le j \le k$, converges to $0 \in E$ for each i such that $\sum_{n=0}^{\infty} \alpha_{n,\ 0}(1 - \alpha_{n,\ 0}) = \infty$.*

Proof. If p is a common fixed point of T_i for each i, then

$$\begin{aligned}
||v_{n+1} - p|| &= ||\alpha_{n,\,0} v_n + \sum_{i=1}^{k} \alpha_{n,\,i} T_i y_n - \sum_{i=0}^{k} \alpha_{n,\,i} T_i p|| \\
&= ||\alpha_{n,\,0}(v_n - p) + \sum_{i=1}^{k} \alpha_{n,\,i}(T_i y_n - T_i p)|| \\
&\leq ||\alpha_{n,\,0}(v_n - p)|| + ||\sum_{i=1}^{k} \alpha_{n,\,i}(T_i y_n - T_i p)|| \\
&\leq \alpha_{n,\,0}||v_n - p|| + \sum_{i=1}^{k} \alpha_{n,\,i}||T_i y_n - T_i p|| \\
&\leq \alpha_{n,\,0}||v_n - p|| + \sum_{i=1}^{k} \alpha_{n,\,i}||y_n - p|| \\
&= \alpha_{n,\,0}||v_n - p|| + \sum_{i=1}^{k} \alpha_{n,\,i}||\sum_{j=0}^{s} \beta_{n,\,j} T_j v_n - \sum_{j=0}^{s} \beta_{n,\,j} T_j p|| \\
&= \alpha_{n,\,0}||v_n - p|| + \sum_{i=1}^{k} \alpha_{n,\,i}||\sum_{j=0}^{s} \beta_{n,\,j}(T_j v_n - T_j p)|| \\
&\leq \alpha_{n,\,0}||v_n - p|| + \sum_{i=1}^{k} \alpha_{n,\,i}\sum_{j=0}^{s} \beta_{n,\,j}||T_j v_n - T_j p|| \\
&\leq \alpha_{n,\,0}||v_n - p|| + \sum_{i=1}^{k} \alpha_{n,\,i}\sum_{j=0}^{s} \beta_{n,\,j}||v_n - p|| \\
&= \alpha_{n,\,0}||v_n - p|| + \sum_{i=1}^{k} \alpha_{n,\,i}||v_n - p|| = \sum_{i=0}^{k} \alpha_{n,\,i}||v_n - p|| = ||v_n - p||.
\end{aligned}$$

Now

$$\begin{aligned}
||(I - T_i^j)v_n|| &= ||v_n - T_i^j v_n|| \leq ||v_n - p|| + ||p - T_i^j v_n|| \\
&= ||v_n - p|| + ||T_i^j p - T_i^j v_n|| \leq ||v_n - p|| + ||p - v_n|| = 2||v_n - p||.
\end{aligned}$$

Since $||v_n - T_i^j v_n|| \leq 2||v_n - p||$, we may assume that there is an $a > 0$ such that $||v_n - p|| \geq a$, $\forall\ n$. If $\left\{(I - T^j)v_n\right\}_{n=0}^{\infty}$ does not converge to 0, then there is an $\epsilon > 0$ such that $||v_n - T_i^j v_n|| \geq \epsilon$, $\forall\ n$.

Let

$$b = 2\delta\left(\frac{\epsilon}{||v_0 - p||}\right),\ x_n = \frac{v_n - p}{||v_n - p||} \text{ and } z_n = \frac{\sum_{i=1}^{k} \alpha_{n,\ i}(T_i y_n - T_i p)}{(1 - \alpha_{n,\ 0})||v_n - p||}.$$

Then, we have

$$||x_n|| = ||\left(\frac{v_n - p}{||v_n - p||}\right)|| \leq \frac{||v_n - p||}{||v_n - p||} = 1$$

and

$$\begin{aligned}
||z_n|| = ||\left(\frac{\sum_{i=1}^{k} \alpha_{n,\ i}(T_i y_n - T_i p)}{(1 - \alpha_{n,\ 0})||v_n - p||}\right)|| &\leq \frac{||\sum_{i=1}^{k} \alpha_{n,\ i}(T_i y_n - T_i p)||}{(1 - \alpha_{n,\ 0})||v_n - p||} \\
&\leq \frac{\sum_{i=1}^{k} \alpha_{n,\ i}||T_i y_n - T_i p||}{(1 - \alpha_{n,\ 0})||v_n - p||} \leq \frac{\sum_{i=1}^{k} \alpha_{n,\ i}||y_n - p||}{(1 - \alpha_{n,\ 0})||v_n - p||} \\
&= \frac{\sum_{i=1}^{k} \alpha_{n,\ i}||\sum_{j=0}^{s} \beta_{n,\ j} T_j v_n - \sum_{j=0}^{s} \beta_{n,\ j} T_j p||}{(1 - \alpha_{n,\ 0})||v_n - p||} \\
&= \frac{\sum_{i=1}^{k} \alpha_{n,\ i}||\sum_{j=0}^{s} \beta_{n,\ j}(T_j v_n - T_j p)||}{(1 - \alpha_{n,\ 0})||v_n - p||} \\
&\leq \frac{\sum_{i=1}^{k} \alpha_{n,\ i}\sum_{j=0}^{s} \beta_{n,\ j}||T_j v_n - T_j p||}{(1 - \alpha_{n,\ 0})||v_n - p||} \\
&\leq \frac{\sum_{i=1}^{k} \alpha_{n,\ i}\sum_{j=0}^{s} \beta_{n,\ j}||v_n - p||}{(1 - \alpha_{n,\ 0})||v_n - p||} = \frac{\sum_{i=1}^{k} \alpha_{n,\ i}||v_n - p||}{(1 - \alpha_{n,\ 0})||v_n - p||} = 1,
\end{aligned}$$

since $\sum_{i=1}^{k} \alpha_{n,\ i} = 1 - \alpha_{n,\ 0}$.

Hence, we have from (4) that

$$\begin{aligned}
||v_{n+1} - p|| &= ||\alpha_{n,\ 0} v_n + \sum_{i=1}^{k} \alpha_{n,\ i} T_i y_n - \sum_{i=0}^{k} \alpha_{n,\ i} T_i p|| \\
&= ||\alpha_{n,\ 0}(v_n - p) + \sum_{i=1}^{k} \alpha_{n,\ i}(T_i y_n - T_i p)|| \\
&= ||(||v_n - p||)[\alpha_{n,\ 0}\frac{(v_n - p)}{||v_n - p||} + (1 - \alpha_{n,\ 0})\frac{\sum_{i=1}^{k} \alpha_{n,\ i}(T_i y_n - T_i p)}{(1 - \alpha_{n,\ 0})||v_n - p||}]|| \\
&= ||(||v_n - p||)[\alpha_{n,\ 0} x_n + (1 - \alpha_{n,\ 0}) z_n]|| \\
&\leq ||v_n - p||\ ||\alpha_{n,\ 0} x_n + (1 - \alpha_{n,\ 0}) z_n||. \quad (6)
\end{aligned}$$

Using Lemma 2.1 in (6) yields

$$||v_{n+1} - p|| \leq [1 - \alpha_{n,\ 0}(1 - \alpha_{n,\ 0})b]||v_n - p||$$

$$= ||v_n - p|| - b\alpha_{n,\ 0}(1-\alpha_{n,\ 0})||v_n - p||$$
$$\le ||v_{n-1} - p|| - b\alpha_{n-1,\ 0}(1-\alpha_{n-1,\ 0})||v_{n-1} - p|| - b\alpha_{n,\ 0}(1-\alpha_{n,\ 0})||v_n - p||.$$

Repeating this process inductively leads to

$$a \le ||v_{n+1}-p|| \le ||v_0-p||-b[\alpha_{0,\ 0}(1-\alpha_{0,\ 0})||v_0-p||+\alpha_{1,\ 0}(1-\alpha_{1,\ 0})||v_1-p||+\cdots + \alpha_{n,\ 0}(1-\alpha_{n,\ 0})||v_n - p||\ ]$$
$$= ||v_0 - p|| - b\sum_{r=0}^{n} \alpha_{r,\ 0}(1-\alpha_{r,\ 0})||v_r - p||$$
$$\le ||v_0 - p|| - ab\sum_{r=0}^{n} \alpha_{r,\ 0}(1-\alpha_{r,\ 0}).$$

Therefore, we obtain $a[1 + b\sum_{r=0}^{n} \alpha_{r,\ 0}(1-\alpha_{r,\ 0})] \le ||v_0 - p||$, from which it follows that

$$a \le \frac{||v_0 - p||}{1 + b\sum_{r=0}^{n} \alpha_{r,\ 0}(1-\alpha_{r,\ 0})} \to 0 \text{ as } n \to \infty,$$

leading to a contradiction. Therefore, we have $a = 0$. Hence,

$$\lim_{n\to\infty} ||v_n - T_i^j v_n|| = 0,$$

for each i. □

Remark 3.2. Theorem 3.1 is a generalization of the results of [4, 7, 8, 9, 12, 13].

Theorem 3.3. Let E be a convex subset of a uniformly convex Banach space X and $T_i : E \to E$ $(i = 0, 1, 2, \cdots, k)$ nonexpansive mappings with at least a common fixed point. Let $\{v_n\}_{n=0}^{\infty}$ be the sequence defined by (4). Then, the sequence $\left\{(I - T_i^j)v_n\right\}_{n=0}^{\infty}$, for each $j \in \mathbb{N}$, $1 \le j \le k$, converges to $0 \in E$ for each i such that $\sum_{n=0}^{\infty} \alpha_{n,\ 0}(1-\alpha_{n,\ 0}) = \infty$.

Proof. The proof of this theorem is similar to that of Theorem 3.1. □

Remark 3.4. Theorem 3.3 is also a generalization of the results of [4, 7, 8, 9, 12, 13].

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