International Journal of Pure and Applied Mathematics

Volume 64 No. 3 2010, 347-353

COMMON FIXED POINTS FOR BANACH OPERATOR PAIRS AND WEAKLY COMPATIBLE MAPS

O.O. Owojori[1 §], G. Akinbo[2], A.O. Bosede[3]

[1,2]Department of Mathematics
Obafemi Awolowo University
Ile-Ife, NIGERIA
[1]e-mail: walejori@oauife.edu.ng
[2]e-mail: agnebg@yahoo.co.uk

[3]Department of Mathematics
Lagos State University
Ojoo, NIGERIA
e-mail: aolubosede@yahoo.co.uk

Abstract: In this work, we combine the notions of Banach operator pairs with weak compatibility to obtain common fixed point results for four selfmaps of a metric space without requiring their continuity. Our results extend those of Pathak et al [14], Singh and Kumar [16] and others.

AMS Subject Classification: 47H10, 54H25
Key Words: common fixed point, coincidence point, weakly compatible maps, Banach operator pairs

1. Introduction

Since 1976, when the concept of commuting mappings was first used as a tool to generalize the Banach Contraction Theorem (see Jungck [3]), a variety of extensions and generalizations have followed. Sessa in [15] defined two selfmaps f and g of a metric space X to be weakly commutative if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$.

Pant [9] introduced the concept of R-weakly commuting pairs, i.e., the pair (f, g) satisfying $d(fgx, gfx) \leq Rd(fx, gx),\ x \in X$ for some $R > 0$.

Received: May 29, 2009

§Correspondence author

In 1986 Jungck [4] came up with the notion of compatibility, and the study of this class of maps and its weaker forms has since then become central to the study of common fixed points for contractive-type maps. Some of these weaker forms include compatibility of type (A) introduced by Jungck et al [6], compatibility of types (B), (C) and (P) by Pathak, Khan and others (see [10]-[12]). Also Lal et al [8] introduced weak compatibility of type (A), and in 1998 Jungck and Rhoades [7] defined two selfmaps f and g of a metric space X to be weakly compatible if they commute at their coincidence points.

However, Singh and Kumar [16], in 2006, noted that commutativity requirement in common fixed point considerations could not be weaker than weak compatibility.

Recently, a new class of non-commuting maps namely, Banach operator pairs was introduced by Chen and Li [1] for which some common fixed point results were presented. In this paper, we combine the concepts of weak compatibility and Banach operator pair to prove existence of common fixed points for four selfmaps of a metric space without imposing their continuity.

2. Preliminaries

Definition 2.1. (see Chen and Li [1]) The ordered pair (f, g) of two selfmaps f ang g of a metric space X is called a Banach operator pair if the set F_g of fixed points of g is F-invariant.

The following characterization was made in [1].

Proposition 2.2. *Let f and g be selfmaps of a metric space X. The following conditions are equivalent:*

(i) the pair (f, g) is a Banach operator pair; (ii) f and g commute on the set F_g.

Examples 1 and 2 of [1] established the following

Remark 2.3. Let f and g be selfmaps of a metric space X. Then the Banach operator pair (f, g) need not to be compatible (and therefore need not to be weakly compatible), and vice versa.

Definition 2.4. Two selfmaps f and g of X are said to satisfy the (EA)-property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

We shall make use of the following extension of the notion of parametrically $\varphi(\epsilon, \delta; a)$-contraction mapping introduced in [14].

Definition 2.5. Let A, B, S and T be selfmaps of a metric space X such that $AX \subseteq TX$ *and* $BX \subseteq SX$, satisfying

$$ad(Ax, By) + (1-a)d(By, Ty) \leq \varphi(ad(Sx, Ty) + (1-a)d(Ax, Sx)), \qquad (1)$$

for some $a \in (\frac{1}{2}, 1]$, where $\varphi : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ is such that:

$(a_\varphi):$ φ is continuous, and $(b_\varphi):$ $\varphi(t) < t$ for all $t > 0$.

Remark 2.6. In [14], Pathak et al required the following additional condition to Definition 2.5 to guarantee existence of unique common fixed point:

$(c_\varphi):$ $\epsilon \leq d(By, Ty) < \delta(\epsilon)$ implies $\varphi(d(Ax, Sx)) < \epsilon$, where $\delta : (0, \infty) \longrightarrow (0, \infty)$ is such that $\delta(\epsilon) > \epsilon$ for all $\epsilon > 0$.

In this work, we shall do without this condition.

The following theorem is proved by Singh and Kumar in [16].

Theorem 2.7. *Let Y be an arbitrary nonempty set and (X, d) a metric space and $A, B, S, T : Y \longrightarrow X$ such that*

$$d(By, Ty) < \max\left\{d(Sx, Ty), \alpha d(Ax, Sx), \alpha d(By, Ty), \frac{d(By, Sy) + d(Ax, Ty)}{2}\right\}, \qquad (2)$$

where $0 \leq \alpha < 1$.

If one of the pairs (A,S) and (B,T) satisfies the (EA)-property and

$$\overline{AY} \subseteq TY,\ \overline{BY} \subseteq SY, \qquad (3)$$

then:

(i) A and S have a coincidence;

(ii) B and T have a coincidence.

Further, if Y=X, then:

(iii) A and S have a common fixed point provided that A and S commute at their coincidence point;

(iv) B and T have a common fixed point provided that B and T commute at their coincidence point;

(v) A, B, S and T have a unique common fixed point if the pairs (A, S) and (B, T) commute at their coincidences.

3. Main Results

The following lemma shall be useful in the proof of our main theorem.

Lemma 3.1. *Let $\varphi : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ be a continuous function satisfying $\varphi(t) < t$ for all $t > 0$. Then, $\varphi(0) = 0$. Hence, $\varphi(t) \leq t$ for all $t \geq 0$.*

Proof. Let, if possible, $\varphi(0) = p > 0$. Then, since $\frac{p}{2} > 0$, we have $\varphi(\frac{p}{2}) = q < \frac{p}{2}$, and

$$\left|\varphi(0) - \varphi(\frac{p}{2})\right| = |p - q| > \left|p - \frac{p}{2}\right| = \left|0 - \frac{p}{2}\right|.$$

This inequality $\left|\varphi(0) - \varphi(\frac{p}{2})\right| > \left|0 - \frac{p}{2}\right|$ contradicts the continuity of the real function φ at 0. Hence, $\varphi(0) = 0$. This completes the proof. □

In our results below, we shall require only the weak compatibility of (A, S), not both (A, S) and (B, T) as it is in many papers in the literature.

Theorem 3.2. *Let (X, d) be a metric space, and let the maps $A, B, S, T : X \longrightarrow X$ satisfying (1) and*

$$\overline{AX} \subseteq TX, \ \overline{BX} \subseteq SX, \tag{4}$$

be such that one of the pairs (A, S) and (B, T) satisfies the (EA)-property. Then:

(i) A and S have a coincidence;

(ii) B and T have a coincidence;

(iii) $F_A \cap F_S$ is not empty, provided A and S are weakly compatible.

Further:

(iv) if (B, A) and (B, S) are Banach operator pairs, $F_A \cap F_B \cap F_S$ is not empty;

(v) if (T, A) and (T, S) are Banach operator pairs, $F_A \cap F_B \cap F_T$ is not empty;

(vi) the set $F_A \cap F_B \cap F_S \cap F_T$ is a singleton.

Proof. Suppose the pair (B, T) satisfies the (EA)-property, then there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Since $\overline{BX} \subseteq SX$, for each x_n, there exists $y_n \in X$ such that $Bx_n = Sy_n$ and clearly, $Sy_n \to t$.

If we put $x = y_n, \ y = x_n$ into (1), we obtain

$$\begin{aligned} &ad(Ay_n, Bx_n) + (1 - a)d(Bx_n, Tx_n) \\ &\qquad \leq \varphi(ad(Sy_n, Tx_n) + (1 - a)d(Ay_n, Sy_n)), \end{aligned} \tag{5}$$

As $n \to \infty$, considering the continuity of φ, we have

$$\overline{\lim}_{n\to\infty} d(Ay_n, t) \le \varphi((1-a)\overline{\lim}_{n\to\infty} d(Ay_n, t)) \le (1-a)\overline{\lim}_{n\to\infty} d(Ay_n, t)$$

which implies that $\overline{\lim}_{n\to\infty} d(Ay_n, t) = 0$ since $a \in (\frac{1}{2}, 1]$. Hence, $Ay_n \to t$ as $n \to \infty$.

Since $t \in \overline{AX} \subseteq TX$, there exists some $u \in X$, such that $Tu = t$. Substituting $x = y_n,\ y = u$ into (1) yields

$$ad(Ay_n, Bu) + (1-a)d(Bu, Tu) \le \varphi(ad(Sy_n, Tu) + (1-a)d(Ay_n, Sy_n)), \quad (6)$$

Letting $n \to \infty$, $ad(t, Bu) + (1-a)d(Bu, t) \le \varphi(ad(t, t) + (1-a)d(t, t)) = \varphi(0) = 0$. That is, $Bu = t$.

Similarly, by the fact that $t \in \overline{BX} \subseteq SX$, we have $Sv = Bu = t$ for some $v \in X$. Therefore, putting $x = v,\ y = u$ into (1), we have $Av = t$. Hence, $Av = Sv = t = Bu = Tu$. This proves parts (i) and (ii) of the theorem.

Now suppose A and S commute at their coincidence point v, then $AAv = ASv = SAv = SSv$, and putting $x = Av,\ y = u$ into (1), we obtain

$$ad(AAv, t) + (1-a)d(t, t) \le \varphi(ad(SAv, t) + (1-a)d(AAv, SAv)).$$

Thus, $ad(At, t) \le \varphi(ad(At, t))$. Hence, $t = At = St$, proving part (iii).

Further, suppose (B, A) (and (B, S)) are Banach pairs, then by Proposition 2.2, $ABt = BAt = Bt$ (and $SBt = BSt = Bt$).

We now show that $Bt = t$, for if $d(Bt, t) > 0$, substituting Bt for x and u for y in (1), we shall obtain

$$ad(ABt, Bu) + (1-a)d(Bu, Tu) \le \varphi(ad(SBt, Tu) + (1-a)d(ABt, SBt)).$$

That is,

$$ad(Bt, t) + (1-a)d(t, t) \le \varphi(ad(Bt, t) + (1-a)d(Bt, Bt)),$$

which implies

$$ad(Bt, t) \le \varphi(ad(Bt, t)) < ad(Bt, t).$$

This is a contradiction. Therefore, $Bt = t$. This proves part (iv).

Similarly, if (T, A) (and (T, S)) are Banach pairs, then $ATt = TAt = Tt$ (and $STt = TSt = Tt$).

Substituting Tt for x and u for y in (1), we have $Tt = t$. This proves part (v).

Hence $t \in F_A \cap F_B \cap F_S \cap F_T$.

Finally, suppose there is another point $t' \in F_A \cap F_B \cap F_S \cap F_T$ such that $t' \ne t$. Then $d(t', t) > 0$, and (1) yields

$$ad(At', Bt) + (1-a)d(Bt, Tt) \le \varphi(ad(St', Tt) + (1-a)d(At', St')).$$

That is, $ad(t',t) \leq \varphi(ad(t',t)) < ad(t',t)$. This is also a contradiction.

Therefore, t is a unique common fixed point of A, B, S and T, and the proof is complete. □

The result that follows is a variant of Theorem 2.7. The proof is analogous to that of Theorem 3.2.

Theorem 3.3. *Let (X,d) be a metric space and let A, B, S, T be selfmaps of X satisfying (2) and (4). If one of (A,S) and (B,T) has the (EA)-property and (B,S) is a weakly compatible pair, then A, B, S, T have a unique common fixed point provided that (B,A), (B,S), (T,A) and (T,S) are Banach pairs.*

Corollary 3.4 below is motivated by Theorem 2 of [16]. The proof is obtained by making $B = T$ in Theorem 3.3.

Corollary 3.4. *Let (X,d) be a metric space and $A, B, S : X \longrightarrow X$ such that (2) holds with $S = T$. Let one of (A,S) and (B,S) satisfy the (EA)-property and $\overline{AX} \cup \overline{BX} \subset SX$. Then A, B, S have a coincidence. Further, if A and S are weakly compatible, and (B,A), (B,S) are Banach pairs, then A, B, S have a unique common fixed point.*

References

[1] J. Chen, Z. Li, Common fixed-points for Banach operator pairs in best approximation, *J. Math. Anal. Appl.*, **336** (2007), 1466-1475.

[2] C.O. Imoru, G. Akinbo, A.O. Bosede, On the fixed points for weak compatible type and parametrically $(\varphi, \epsilon; a)$-contraction mappings, *Math. Sci. Res. J.*, **10**, No. 10 (2006), 259-267.

[3] G. Jungck, Commuting mappings and fixed points, *Amer. Math. Monthly*, **83** (1976), 261-263.

[4] G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.*, **9**, No. 4 (1986), 771-779.

[5] G. Jungck, Coincidence and fixed points for compatible and relatively nonexpansive maps, *Internat. J. Math. Sci.*, **16**, No. 1 (1993), 95-100.

[6] G. Jungck, P.P. Murthy, Y.J. Cho, Compatible mappings of type (A) and common fixed points, *Math. Japonica*, **38**, No. 2 (1993), 381-390.

[7] G. Jungck, B.E. Rhoades, Fixed points for set-valued functions without continuity, *Indian J. Pure. Appl. Math.*, **29**, No. 3 (1998), 227-238.

[8] S.N. Lal, P.P. Murthy, Y.J. Cho, An extension of Telci, Tas and Fisher's Theorem, *J. Korean. Math. Soc.*, **33**, No. 4 (1996), 891-908.

[9] R.P. Pant, Common fixed points of four mappings, *Bull. Cal. Math. Soc.*, **90** (1998), 281-286.

[10] H.K. Pathak, M.S. Khan, Compatible mappings of type (B) and common fixed points of Gregus type, *Czech. Math. J.*, **45**, No. 120 (1995), 685-698.

[11] H.K. Pathak, Y.J. Cho, S. Chang, S.M. Kang, Compatible mappings of type (P) and fixed point theorem in metric spaces and probabilistic metric spaces, *Novi Sad J. Math.*, **26**, No. 2 (1996), 87-109.

[12] H.K. Pathak, Y.J. Cho, M.S. Khan, B. Madharia, Compatible mappings of type (C) and common fixed points of Gregus type, *Demons. Math.*, **31**, No. 3 (1998), 499-518.

[13] H.K. Pathak, M.S. Khan, S.M. Kang, Fixed and coincidence points for contraction and Parametrically nonexpansive mappings, *Math. Sci. Res. J.*, **8**, No. 1 (2004), 27-35.

[14] H.K. Pathak, R.K. Verma, S.M. Kang, M.S. Khan, Fixed points for weak compatible type and parametrically $\varphi(\epsilon, \delta; a)$-contraction mappings, *Internat. J. Pure and Appl. Math.*, **26**, No. 2 (2006), 247-263.

[15] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math. (N.S., Beograd)*, **32**, No. 46 (1982), 149-153.

[16] S.L. Singh, A. Kumar, Common fixed point theorems for contractive maps, *Matematnykn Bechnk*, **58** (2006), 85-90.