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EXACT TRAVELLING WAVE SOLUTIONS OF FITZHUGH–NAGUMO EQUATION

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Abstract: In this work we use a modifed extended tanh method (METF) and modified tanh–coth method to solve the Fitzhugh–Nagumo equation. The main idea is to take full advantage of the Riccati equation that the tanh-function satisfies. So multiple travelling wave solutions of the Fitzhugh–Nagumo equation is presented and implemented in a computer algebraic system. So the efficiency of the methods can be demonstrated.

AMS Subject Classification: 35Q53
Key Words: modified extended tanh method, modified extended tanh–coth, Fitzhugh–Nagumo equation, Newell–Whitehead equation

1. Introduction

The investigation of the exact solutions to nonlinear equations plays an important role in the study of nonlinear physical phenomena. Among these are Backlund transformation, Darboux transformation, the inverse scattering method, Hirota's bilinear method, the tanh method, the sine–cosine method, the homogeneous balance method, and the Riccati expansion method with constant coefficients. To date, various nonlinear equations were presented, which described, for example, the motion of the isolated waves, and in many fields such as hydrodynamic, plasma physics, nonlinear optic, etc.

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The Fitzhugh–Nagumo equation is

$$u_t - u_{xx} = u(u - \alpha)(1 - u), \tag{1.1}$$

where α is an arbitrary constant and $0 < \alpha < 1$. When $\alpha = -1$, equation (1.1) reduces to the real Newell–Whitehead equation. equation (1.1) is an important nonlinear reaction–diffusion equation and applied to model the transmission of nerve impulses [5], [11], also is used in biology and the area of population genetics, in circuit theory [14]. Equation (1.1) was solved by using the Hirota method [8], the inverse variational method [7], the tanh-coth method [16], the (constrained) canonical reduction [13], Lie group method [2], the Jacobi elliptic function [12]; the first integral method [10].

Exact travelling wave solutions of nonlinear PDEs is one of the fundamental objects of study in mathematical physics. These exact solutions when they exist can help one to well understand the mechanism of the complicated physical phenomena and dynamically processes modelled by the nonlinear evolution equations. The extended tanh function method, the modified extended tanh function method and F-expansion method belong to a class of methods called the sub-equation method for which there appears some basic relationships among the complicated nonlinear evolution equations in study and some simple and solvable nonlinear ordinary equations. Modified extended tanh method has some merits in contrast with the tanh function method. It only uses a simpler algorithm to yield an algebraic system and also yields singular soliton solutions with no extra effort. El-Wakil and Abdou [3] and Soliman [15] proposed the METF method and obtained some new exact solutions. Recently, an extended tanh-function method and symbolic computation have been suggested for solving the some nonlinear PDEs to obtain four kinds of soliton solutions [1], [9], [4], [6], [17], [3], [15]. Modified tanh-coth have been studied by Wazzan [19], [18].

The paper is organized as follows: In Section 2 and Section 3, simple description of the modified extended tanh method and modified extended tanh-coth method will be given. The travelling wave solutions are obtained for the Fitzhugh–Nagumo equation. The conclusions are then given in the final Section 4.

2. Modified Extended tanh Method

First, we will give a simple description of the modified extended tanh method. For doing this, we consider a partial differential equation (for short, PDE) in two variables given by

$$Q(u, u_t, u_x, u_{xx}, ...) = 0. \tag{2.2}$$

We first consider its travelling solutions $u(x,t) = u(z), z = x + ct$ or $z = x - ct$. Then equation (2.2) becomes an ordinary differential equation. The next important step is to expand the solution u in the form

$$u(z) = a_0 + \sum_{i=1}^{M} \left(a_i w^i + b_i w^{-i} \right),$$

where

$$w' = b + w^2, \tag{2.3}$$

b is a parameter to be determined later, $w = w(z)$, $\frac{dw}{dz} = w'$. The parameter M can be found by balancing the highest order linear term with the nonlinear terms. Inserting (2.2) and (2.3) into the ordinary differential equation will yield a system of algebraic equations with respect to a_i, b_i, b and c (where $i = 1...M$) since all the coefficients of w^i have to disappear. With the aid of *Maple*, one can determine a_i, b_i, b and c. The Riccati equation (2.3) has the following general solutions:

I. If $b < 0$,

$$w = -\sqrt{-b}\tanh(\sqrt{-b}z), \qquad w = -\sqrt{-b}\coth(\sqrt{-b}z)$$

it depends on the initial conditions.

II. If $b > 0$,

$$w = \sqrt{b}\tan(\sqrt{b}z), \qquad w = -\sqrt{b}\cot(\sqrt{b}z).$$

III. If $b = 0$,

$$w = -\frac{1}{z}.$$

2.1. Implementation of METF to Fitzhugh-Nagumo Equation

We consider the equation (1.1). To investigate the travelling wave solution of equation (1.1), we use the transformation $u(x,t) = u(w)$, $z = x - ct$. Then equation (1.1) becomes

$$-cu' - u'' - u^2(\alpha + 1) + \alpha u + u^3 = 0. \tag{2.4}$$

Balancing the linear term of the highest order with the nonlinear term yields $M = 1$. So take the ansatz

$$u(z) = a_0 + a_1 w + b_1 w^{-1}. \tag{2.5}$$

Substituting equation (2.5) into equation (2.4) and using equation (2.3), we obtain following set of nonlinear algebraic equations for a_0, a_1, b_1, b and c:

$-2a_1 + a_1^3 = 0,$
$-ca_1 - \alpha a_1^2 + 3a_0\alpha^2 - a_1^2 = 0,$
$-2a_0a_1\alpha - 2a_1b - 2a_0a_1 + \alpha a_1 + 3a_0^2a_1 + 3a_1^2b_1 = 0,$
$cb_1 + 6a_0a_1b_1 - 2\alpha a_1b_1 + \alpha a_0 - a_0^2 - \alpha a_0^2 - ca_1b + a_0^3 - 2a_1b_1 = 0,$
$3a_1b_1^2 + 3a_0^2b_1 - 2a_0b_1 - 2b_1b - 2\alpha a_0b_1 + \alpha b_1 = 0,$
$b_1^3 - 2b_1b^2 = 0.$

Solving this set of algebraic equation by the use of *Maple*, we obtain:

a. $a_0 = \frac{1}{2}, a_1 = 0, b_1 = \pm\frac{\sqrt{2}}{8}, b = -\frac{1}{8}, c = \pm\frac{\sqrt{2}}{2}(2\alpha - 1).$
b. $a_0 = \frac{\alpha}{2}, a_1 = 0, b_1 = \pm\frac{\sqrt{2}}{8}\alpha^2, b = -\frac{\alpha^2}{8}, c = \pm\frac{\sqrt{2}}{2}(\alpha - 2).$
c. $a_0 = \frac{\alpha+1}{2}, a_1 = 0, b_1 = \pm\frac{\sqrt{2}}{8}(\alpha - 1)^2, b = -\frac{(\alpha-1)^2}{8}, c = \pm\frac{\sqrt{2}}{2}(\alpha + 1).$
d. $a_0 = \frac{1}{2}, a_1 = \pm\sqrt{2}, b_1 = 0, b = -\frac{1}{8}, c = \pm\frac{\sqrt{2}}{2}(2\alpha - 1).$
e. $a_0 = \frac{\alpha}{2}, a_1 = \pm\sqrt{2}, b_1 = 0, b = -\frac{\alpha^2}{8}, c = \pm\frac{\sqrt{2}}{2}(\alpha - 2).$
f. $a_0 = \frac{\alpha+1}{2}, a_1 = \pm\sqrt{2}, b_1 = 0, b = -\frac{(\alpha-1)^2}{8}, c = \pm\frac{\sqrt{2}}{2}(\alpha + 1).$
g. $a_0 = \frac{1}{2}, a_1 = \pm\sqrt{2}, b_1 = \pm\frac{\sqrt{2}}{32}, b = -\frac{1}{32}, c = \pm\frac{\sqrt{2}}{2}(2\alpha - 1).$
h. $a_0 = \frac{\alpha}{2}, a_1 = \pm\sqrt{2}, b_1 = \pm\frac{\sqrt{2}}{32}\alpha^2, b = -\frac{\alpha^2}{32}, c = \pm\frac{\sqrt{2}}{2}(\alpha - 2).$
i. $a_0 = \frac{\alpha+1}{2}, a_1 = \pm\sqrt{2}, b_1 = \pm\frac{\sqrt{2}}{32}(\alpha - 1)^2, b = -\frac{(\alpha-1)^2}{32}, c = \pm\frac{\sqrt{2}}{2}(\alpha + 1).$

Case 1. From (a), (d) and (g), if $b = -\frac{1}{8}$ and $c = \pm\frac{\sqrt{2}}{2}(2\alpha - 1)$

$$u(x,t) = \frac{1}{2}(1 \pm \coth(\frac{z}{2\sqrt{2}})), \tag{2.6}$$

$$u(x,t) = \frac{1}{2}(1 \pm \tanh(\frac{z}{2\sqrt{2}})), \tag{2.7}$$

and

$$u(x,t) = \frac{1}{2}(1 \pm \operatorname{csch}(\frac{z}{2\sqrt{2}})), \tag{2.8}$$

where $z = x \pm \frac{\sqrt{2}}{2}(2\alpha - 1)t$.

Case 2. From (b), (e) and (h), if $b = -\frac{\alpha^2}{8}$ and $c = \pm\frac{\sqrt{2}}{2}(\alpha - 2)$

$$u(x,t) = \frac{\alpha}{2}(1 \pm \coth(\frac{z}{2\sqrt{2}})), \tag{2.9}$$

$$u(x,t) = \frac{\alpha}{2}(1 \pm \tanh(\frac{\alpha z}{2\sqrt{2}})), \tag{2.10}$$

and

$$u(x,t) = \frac{\alpha}{2}(1 \pm \operatorname{csch}(\frac{\alpha z}{2\sqrt{2}})), \tag{2.11}$$

where $z = x \pm \frac{\sqrt{2}}{2}(\alpha - 2)t$.

Case 3. From (c), (f) and (i), if $b = -\frac{(\alpha-1)^2}{8}$ and $c = \pm\frac{\sqrt{2}}{2}(\alpha + 1)$

$$u(x,t) = \frac{\alpha+1}{2} \pm \frac{\alpha-1}{2}\coth(\frac{(\alpha-1)z}{2\sqrt{2}})), \tag{2.12}$$

$$u(x,t) = \frac{\alpha+1}{2} \pm \frac{\alpha-1}{2}\tanh(\frac{(\alpha-1)z}{2\sqrt{2}})), \tag{2.13}$$

and

$$u(x,t) = \frac{\alpha+1}{2} \pm \frac{\alpha-1}{2}\operatorname{csch}(\frac{(\alpha-1)z}{2\sqrt{2}})), \tag{2.14}$$

where $z = x \pm \frac{\sqrt{2}}{2}(\alpha + 1)t$.

3. Modified Extended tanh-coth Method

We consider a given PDE in two independent variables given by

$$F(u, u_t, u_x, u_{xx}, ...) = 0. \tag{3.15}$$

In order to apply the tanh–coth method, the independent variables, x and t, are combined into a new variable, $\zeta = k(x - wt)$ where k and w represent the wave number and velocity of the travelling wave, respectively. Both are undetermined parameters with the assumption that $k > 0$. Therefore, $u(x,t)$ is replaced by $u(\zeta)$, which defines the travelling wave solutions of equation (3.15). Equations such as equation (3.15) are then transformed into

$$F(u, -kwu', ku', k^2u'', ...) = 0. \tag{3.16}$$

Hence, under the transformation $\zeta = k(x-wt)$, the PDE in equation (3.15) has been reduced to an ordinary differential equation (ODE) given by equation (3.16). The resulting ODE is then solved by the tanh–coth method [10], which admits the use of a finite series of functions of the form:

$$u(x,t) = u(\zeta) = a_0 + \sum_{i=1}^{M}\left(a_j Y^j + b_i Y^{-j}\right) \tag{3.17}$$

and the Riccati equation

$$Y' = A + BY + CY^2, \tag{3.18}$$

where $' := \frac{d}{d\zeta}$; and A, B, and C are constants to be prescribed later. The parameter M is a positive constant that can be determined by balancing the linear term of highest order with the nonlinear term in equation (3.16). Inserting equation (3.17) into the ODE in equation (3.16) and using equation (3.18) results in an algebraic equation in powers of Y. Since all coefficients of Y^j must vanish. This will give a system of algebraic equations with respect to parameters a_j, b_j, k and w. With the aid of *Maple*, we can determine a_j, b_j, k and w. We will consider the following special solutions of the Riccati equation (3.18):

I. $A = B = 1$ and $C = 0$, equation (3.18) has the solution $Y = e^{\zeta} - 1$.

II. $A = 1/2, B = 0$ and $C = -1/2$, equation (3.18) has the solutions $Y = \cot h\zeta \pm \mathrm{csch}\zeta$ or $Y = \tan h\zeta \pm i \sec h\zeta$, where $i^2 = -1$.

3.1. Implementation of Modified tanh-coth to Fitzhugh-Nagumo Equation

We consider the Eq (1.1). In order to obtain travelling wave solutions for equation (1.1), we use

$$u(x,t) = u(\zeta), \qquad \zeta = k(x - wt), \qquad k > 0. \tag{3.19}$$

Substituting equation (3.19) into equation (1.1), we obtain

$$-wku' - k^2u'' - u^2(\alpha + 1) + u\alpha + u^3 = 0. \tag{3.20}$$

Balancing the order of the nonlinear term u^3 with the linear term u'' in equation (3.20), we obtain $M = 1$. Thus, the solution in equation (3.17) has the form

$$u(z) = a_0 + a_1Y + b_1Y^{-1}. \tag{3.21}$$

Substituting equation (3.21) into equation (3.20) and using equation (3.18), collecting the coefficients of Y, all coefficients of Y^j have to vanish, yields to a system of algebraic equations in $a_0, a_1, b_1, A, B, C, k$ and w of the form

$-2k^2C^2a_1 + a_1^3 = 0,$

$-wkCa_1 - \alpha a_1^2 - 3k^2BCa_1 - a_1^2 + 3a_0a_1^2 = 0,$

$-k^2B^2a_1 - 2\alpha a_0a_1 - 2a_0a_1 + \alpha a_1 + 3a_0^2a_1 + 3a_1^2b_1 - wka_1B - 2k^2CAa_1 = 0,$

$wkb_1C + 6a_0a_1b_1 + \alpha a_0 - wka_1A - \alpha a_0^2 - 2a_1b_1 - k^2ABa_1 - a_0^2 - 2\alpha a_1b_1 + a_0^3 - k^2b_1BC = 0,$

$3a_0^2b_1 - 2a_0b_1\alpha - 2k^2CAb_1 + \alpha b_1 + b_1wkB - 2a_0b_1 + 3a_1b_1^2 - k^2B^2b_1 = 0,$

$3a_0b_1^2 - 3k^2b_1AB - b_1^2 + wkb_1A - \alpha b_1^2 = 0,$

$-2k^2b_1A^2 + b_1^3 = 0.$

I. If we set $A = B = 1$ and $C = 0$ in equation (3.20) and solve the resulting system, we get the six sets of solutions for $k > 0$,

a. $a_0 = 0, a_1 = 0, b_1 = -1, k = \frac{\sqrt{2}}{2}, w = \pm\frac{\sqrt{2}}{2}(2\alpha - 1)$.

b. $a_0 = 0, a_1 = 0, b_1 = -\alpha, k = \frac{\sqrt{2}}{2}\alpha, w = \pm\frac{\sqrt{2}}{2}(\alpha - 2)$.

c. $a_0 = 1, a_1 = 0, b_1 = 1, k = \frac{\sqrt{2}}{2}, w = \pm\frac{\sqrt{2}}{2}(2\alpha - 1)$.

d. $a_0 = 1, a_1 = 0, b_1 = -\alpha + 1, k = \frac{\sqrt{2}}{2}(\alpha - 1), w = \pm\frac{\sqrt{2}}{2}(\alpha + 1)$.

e. $a_0 = \alpha, a_1 = 0, b_1 = \alpha, k = \frac{\sqrt{2}}{2}\alpha, w = \pm\frac{\sqrt{2}}{2}(\alpha - 2)$.

f. $a_0 = \alpha, a_1 = 0, b_1 = \alpha - 1, k = \frac{\sqrt{2}}{2}(\alpha - 1), w = \pm\frac{\sqrt{2}}{2}(\alpha + 1)$.

Substituting these values in equation (3.21) and $Y = e^{\zeta} - 1$, after some simplifications, we obtain

Case 1. From (a) and (c), if $k = \frac{\sqrt{2}}{2}, w = \pm\frac{\sqrt{2}}{2}(2\alpha - 1)$

$$\begin{aligned} u(x,t) &= -(e^{\zeta} - 1)^{-1} \\ &= \frac{1}{2}(1 + \tanh(\frac{\zeta}{2})) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} u(x,t) &= 1 + (e^{\zeta} - 1)^{-1} \\ &= \frac{1}{2}(1 - \tanh(\frac{\zeta}{2})), \end{aligned} \tag{3.23}$$

where $\zeta = \frac{\sqrt{2}}{2}(x \pm \frac{\sqrt{2}}{2}(2\alpha - 1)t)$.

Case 2. From (b) and (e), if $k = \frac{\sqrt{2}}{2}\alpha, w = \pm\frac{\sqrt{2}}{2}(\alpha - 2)$

$$\begin{aligned} u(x,t) &= \alpha(1 - e^{\zeta})^{-1} \\ &= \frac{\alpha}{2}(1 + \tanh(\frac{\zeta}{2})), \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} u(x,t) &= \alpha(1 + (e^{\zeta} - 1)^{-1}) \\ &= \frac{\alpha}{2}(1 - \tanh(\frac{\zeta}{2})), \end{aligned} \tag{3.25}$$

where $\zeta = \frac{\sqrt{2}\alpha}{2}(x \pm \frac{\sqrt{2}}{2}(\alpha - 2)t)$.

Case 3. From (d) and (f), if $k = \frac{\sqrt{2}}{2}(\alpha - 1), w = \pm\frac{\sqrt{2}}{2}(\alpha + 1)$

$$\begin{aligned} u(x,t) &= 1 + (1 - \alpha)(e^{\zeta} - 1)^{-1} \\ &= \frac{\alpha + 1}{2} + \frac{\alpha - 1}{2}\tanh(\frac{\zeta}{2}), \end{aligned} \tag{3.26}$$

and

$$u(x,t) = \alpha + (\alpha-1)(e^{\zeta}-1)^{-1} = \frac{\alpha+1}{2} + \frac{\alpha-1}{2}\tanh(\frac{\zeta}{2}), \quad (3.27)$$

where $\zeta = \frac{\sqrt{2}(\alpha-1)}{2}(x \pm \frac{\sqrt{2}}{2}(\alpha+1)t)$.

II. If we set $A = \frac{1}{2}, B = 0$ and $C = -\frac{1}{2}$ in equation (3.20) and solve the resulting system, we get the nine sets of solutions for $k > 0$.

a. $a_0 = \frac{1}{2}, a_1 = \pm\frac{1}{2}, b_1 = 0, k = \frac{\sqrt{2}}{2}, w = \pm\frac{\sqrt{2}}{2}(2\alpha-1)$.

b. $a_0 = \frac{\alpha}{2}, a_1 = \pm\frac{\alpha}{2}, b_1 = 0, k = \frac{\sqrt{2}}{2}\alpha, w = \pm\frac{\sqrt{2}}{2}(\alpha-2)$.

c. $a_0 = \frac{\alpha+1}{2}, a_1 = \pm\frac{\alpha-1}{2}, b_1 = 0, k = \frac{\sqrt{2}}{2}(\alpha-1), w = \pm\frac{\sqrt{2}}{2}(\alpha+1)$.

d. $a_0 = \frac{1}{2}, a_1 = 0, b_1 = \pm\frac{1}{2}, k = \frac{\sqrt{2}}{2}, w = \pm\frac{\sqrt{2}}{2}(2\alpha-1)$.

e. $a_0 = \frac{\alpha}{2}, a_1 = 0, b_1 = \pm\frac{\alpha}{2}, k = \frac{\sqrt{2}}{2}\alpha, w = \pm\frac{\sqrt{2}}{2}(\alpha-2)$.

f. $a_0 = \frac{\alpha+1}{2}, a_1 = 0, b_1 = \pm\frac{\alpha-1}{2}, k = \frac{\sqrt{2}}{2}(\alpha-1), w = \pm\frac{\sqrt{2}}{2}(\alpha+1)$.

g. $a_0 = \frac{1}{2}, a_1 = \pm\frac{1}{4}, b_1 = \pm\frac{1}{4}, k = \frac{\sqrt{2}}{4}, w = \pm\frac{\sqrt{2}}{2}(2\alpha-1)$.

h. $a_0 = \frac{\alpha}{2}, a_1 = \pm\frac{\alpha}{4}, b_1 = \pm\frac{\alpha}{4}, k = \frac{\sqrt{2}}{4}\alpha, w = \pm\frac{\sqrt{2}}{2}(\alpha-2)$.

i. $a_0 = \frac{\alpha+1}{2}, a_1 = \pm\frac{\alpha-1}{4}, b_1 = \pm\frac{\alpha-1}{4}, k = \frac{\sqrt{2}}{4}(\alpha-1), w = \pm\frac{\sqrt{2}}{2}(\alpha+1)$.

Substituting these values in equation (3.21) and $Y = \coth\zeta \pm \mathrm{csch}\zeta$ or $Y = \tanh\zeta \pm i\sec h\zeta$ after some simplifications, we get

Case 1. From (a), (d) and (g), for $w = \pm\frac{\sqrt{2}}{2}(2\alpha-1)$

$$u(x,t) = \frac{1}{2}[1 \pm \coth\zeta \pm \mathrm{csch}\zeta] \quad (3.28)$$

$$u(x,t) = \frac{1}{2}[1 \pm \frac{1}{\coth\zeta \pm \mathrm{csch}\zeta}] \quad (3.29)$$

$$u(x,t) = \frac{1}{2} \pm \frac{1}{4}(\coth\zeta \pm \mathrm{csch}\zeta)] \pm \frac{1}{4(\coth\zeta \pm \mathrm{csch}\zeta)}. \quad (3.30)$$

After some simplification, we get

$$u(x,t) = \frac{1}{2}(1 \pm \tanh(\frac{\zeta}{2})),\ k = \frac{\sqrt{2}}{2}$$

$$u(x,t) = \frac{1}{2}(1 \pm \coth(\frac{\zeta}{2})),\ k = \frac{\sqrt{2}}{2}$$

$$u(x,t) = \frac{1}{2}(1 \pm \mathrm{csch}\zeta),\ k = \frac{\sqrt{2}}{4}$$

where $\zeta = k(x \pm \frac{\sqrt{2}}{2}(2\alpha - 1)t)$

Case 2. From (b), (e) and (h), for $w = \pm\frac{\sqrt{2}}{2}(\alpha - 2)$

$$u(x,t) = \frac{\alpha}{2}[1 \pm (\coth\zeta \pm \mathrm{csch}\zeta)], \tag{3.31}$$

$$u(x,t) = \frac{\alpha}{2}[1 \pm \frac{1}{(\coth\zeta \pm \mathrm{csch}\zeta)}], \tag{3.32}$$

$$u(x,t) = \frac{\alpha}{2}[1 \pm \frac{1}{2}(\coth\zeta \pm \csc h\zeta) \pm \frac{1}{2(\coth\zeta \pm \mathrm{csch}\zeta)}]. \tag{3.33}$$

After some simplification, we get

$$\begin{aligned} u(x,t) &= \frac{\alpha}{2}(1 \pm \tanh(\frac{\zeta}{2})),\ k = \frac{\sqrt{2}}{2}\alpha \\ u(x,t) &= \frac{\alpha}{2}(1 \pm \coth(\frac{\zeta}{2})),\ k = \frac{\sqrt{2}}{2}\alpha \\ u(x,t) &= \frac{\alpha}{2}(1 \pm \mathrm{csch}\zeta),\ k = \frac{\sqrt{2}}{4}\alpha \end{aligned}$$

where $\zeta = k(x \pm \frac{\sqrt{2}}{2}(\alpha - 2)t)$.

Case 3. From (c), (f) and (i), if $w = \pm\frac{\sqrt{2}}{2}(\alpha + 1)$

$$u(x,t) = \frac{\alpha+1}{2} \pm \frac{\alpha-1}{2}(\coth\zeta \pm \mathrm{csch}\zeta) \tag{3.34}$$

$$u(x,t) = \frac{\alpha+1}{2} \pm \frac{\alpha-1}{2(\coth\zeta \pm \mathrm{csch}\zeta)} \tag{3.35}$$

$$u(x,t) = \frac{\alpha+1}{2} \pm \frac{\alpha-1}{4}[(\coth\zeta \pm \mathrm{csch}\zeta) \pm \frac{1}{(\coth\zeta \pm \mathrm{csch}\zeta)}]. \tag{3.36}$$

After some simplification, we get

$$u(x,t) = \frac{\alpha+1}{2} \pm \frac{\alpha-1}{2}\tanh(\frac{\zeta}{2}),\ k = \frac{\sqrt{2}}{2}(\alpha - 1)$$

$$u(x,t) = \frac{\alpha+1}{2} \pm \frac{\alpha-1}{2}\coth(\frac{\zeta}{2}),\ k = \frac{\sqrt{2}}{2}(\alpha - 1)$$

$$u(x,t) = \frac{\alpha+1}{2} \pm \frac{\alpha-1}{2}\mathrm{csch}\zeta,\ k = \frac{\sqrt{2}}{4}(\alpha - 1).$$

where $\zeta = k(x \pm \frac{\sqrt{2}}{2}(\alpha + 1)t)$.

4. Conclusion

In this article, the METF and modified tanh–coth method have been successfully implemented to find travelling wave solutions for the Fitzhugh–Nagumo equation. The results show that these methods are a powerful mathematical tool for obtaining exact solutions for the Fitzhugh–Nagumo equation. The travelling wave solutions obtained by the METF are obtained using by the modified extended tanh–coth method.

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