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THE FOURIER TRANSFORM OF $\delta^{(k-1)}(M(x_1...x_n))$ AND $\delta^{(k-1)}(c^2+M(x_1...x_n))$

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Abstract: Let $M(x_1...x_n)$ and $V(x_1...x_n)$ be the quadratic forms defined by (3) and (4), respectively. In this paper we obtain the Fourier transform of $\delta^{(k-1)}(V(x_1...x_n)), \delta^{(k-1)}(M(x_1...x_n))$, and $\delta^{(k-1)}(c^2+M(x_1...x_n))$ and the expansion in series Taylor types of $\delta^{(k-1)}(c^2+M(x_1...x_n))$. Our formulae are generalization of the results that appear in [1], p. 126 and [4].

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1. Introduction

Let $x=(x_1,\dots,x_n)$ be a point of n-dimensional Euclidean space R^n , $(M(x))^\lambda_+$ and $(M(x))^\lambda_-$ defined by the following formulae

$$(M(x))^\lambda_+ = (M(x_1,\dots,x_n))^\lambda_+ = \begin{cases} (M(x_1,\dots,x_n))^\lambda_+, & \text{if } M(x_1,\dots,x_n) \geq 0, \\ 0, & \text{if } M(x_1,\dots,x_n) < 0, \end{cases} \quad (1)$$

and

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$$(M(x))^\lambda_- = (M(x_1, \dots, x_n))^\lambda_- = \begin{cases} 0, & \text{if } M(x_1,\dots,x_n) > 0, \\ (-M(x_1,\dots,x_n))^\lambda_-, & \text{if } M(x_1,\dots,x_n) \le 0, \end{cases} \quad (2)$$

where

$$M(x) = M(x_1,\dots,x_n) = (\sum_{i=1}^{n} a_i x_i^2)^m = (a_1x_1^2 + \cdots + a_\mu x_\mu^2 + a_{\mu+1}x_{\mu+1}^2 + \cdots + a_{\mu+v}x_{\mu+v}^2)^m, \quad (3)$$

$a_i, i = 1, 2, \dots, n$ are real numbers and $\mu + v = n$ dimension of the space. By putting $m = 1$ in (1), (2) and (3) we have

$$M(x) = V(x) = V(x_1,\dots,x_n) = a_1x_1^2 + \cdots + a_\mu x_\mu^2 + a_{\mu+1}x_{\mu+1}^2 + \cdots + a_{\mu+v}x_{\mu+v}^2 \quad (4)$$

$$(V(x))^\lambda_+ = (V(x_1,\dots,x_n))^\lambda_+ = \begin{cases} (V(x_1,\dots,x_n))^\lambda_+ \text{ if } V(x_1,\dots,x_n) \ge 0, \\ 0 \text{ if } V(x_1,\dots,x_n) < 0, \end{cases} \quad (5)$$

and

$$(V(x))^\lambda_- = (V(x_1,\dots,x_n))^\lambda_- = \begin{cases} 0 \text{ if } V(x_1,\dots,x_n) > 0, \\ (-V(x_1,\dots,x_n))^\lambda_- \text{ if } V(x_1,\dots,x_n) \le 0. \end{cases} \quad (6)$$

The generalized function $(V(x))^\lambda_+$ is defined by

$$\left\langle (V(x))^\lambda_+, \varphi \right\rangle = \int_{V(x)\ge 0} (V(x))^\lambda \varphi(x) dx \quad (7)$$

and appear in [4] and under conditions $a_1, \cdots a_\mu > 0$ and $a_{\mu+1}, \cdots a_{\mu+\nu} < 0$, we have the following properties,

$$\operatorname*{Res}_{\lambda=-k, k=1,2,\dots} (V(x))^\lambda_+ = \frac{(-1)^{(k-1)}}{(k-1)!} \delta^{(k-1)}(V(x)), \quad (8)$$

$$\operatorname*{Res}_{\lambda=-k, k=1,2,\dots} (V(x))^\lambda_+ = 0, \quad (9)$$

if μ is even and ν is odd,

$$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k, k=0,1,2,\dots} (V(x))^\lambda_+ = \frac{(-1)^{\frac{\nu}{2}} \pi^{\frac{n}{2}}}{2^{2k} k! \Gamma(\frac{n}{2}+k)} L_a^k \{\delta(x)\}, \quad (10)$$

if μ is odd and ν is even,

$$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k, k=0,1,2,\dots} (V(x))^\lambda_+ = \frac{(-1)(-1)^{\frac{\nu}{2}} \pi^{\frac{n}{2}}}{2^{2k} k! \Gamma(\frac{n}{2}+k)} L_a^k \{\delta(x)\}, \quad (11)$$

if μ and ν are both even and

$$\underset{\lambda=-\frac{n}{2}-k,k=0,1,2,\ldots}{\mathrm{Res}} (V(x))_+^\lambda = \frac{(-1)(-1)^{\frac{\nu+1}{2}}\pi^{\frac{n}{2}-1}}{2^{2k}k!\Gamma(\frac{n}{2}+k)} \left[\psi(\frac{\mu}{2}) - \psi(\frac{n}{2})\right] L_a^k \left\{\delta(x)\right\}, \tag{12}$$

if μ and ν are both odd, where L_a is the operator defined by

$$L_a = \frac{1}{a_1}\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{1}{a_\mu}\frac{\partial^2}{\partial x_\mu^2} + \frac{1}{a_{\mu+1}}\frac{\partial^2}{\partial x_\mu^2} + \cdots + \frac{1}{a_{\mu+\nu}}\frac{\partial^2}{\partial x_{\mu+\nu}^2}, \tag{13}$$

if $a_1, \cdots a_\mu > 0$ and $a_{\mu+1}, \cdots a_{\mu+\nu} < 0, \psi(x)$ is defined by

$$\psi(x) = \frac{\Gamma^{`}(x)}{\Gamma(x)} \tag{14}$$

for integral and half-integral values of the argument, $\psi(x)$ is given by

$$\psi(k) = -C + 1 + \frac{1}{2} + \cdots + \frac{1}{k-1}, \tag{15}$$

$k = 2, 3, \cdots$ and

$$\psi(k + \frac{1}{2}) = -C - 2\ln 2 + 2\left(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}\right), \tag{16}$$

$k = 1, 2, 3, \cdots$.

On the other hand, from [5] Section 1, pp. 209-247, $\delta^{(k)}(H(x_1 \cdots x_n))$ is defined for an infinitely differentiable function $H(x_1 \cdots x_n)$ such that the $H(x_1 \cdots x_n) = 0$ hypersurface has not singular points.

But if $H(x_1 \cdots x_n) = V(x_1, \ldots, x_n)$, where $V(x_1, \ldots, x_n)$ is defined by (4) then $H(x_1 \cdots x_n)$ has a singular point of the origen. For this case $\delta^{(k)}(H)$ can be defined (see [5], Section 1, pp. 209-247) in the following form

$$< \delta^{(k-1)}(V), \varphi >= (-1)^{k-1} \int \left[\frac{\partial^{k-1}}{\partial u_1^{k-1}}\{\varphi D\binom{x}{u}\}\right]_{u_1=0} du_2 \cdots du_n, \tag{17}$$

$k = 1, 2, \cdots$, where

$$\begin{aligned} u_1 &= V(x_1, \ldots, x_n), \\ u_2 &= x_2, \\ &\vdots \\ u_n &= x_n, \end{aligned} \tag{18}$$

$$D\binom{x}{u} = (D\binom{x}{u})^{-1} = \frac{1}{\frac{\partial V}{\partial x_1}}. \tag{19}$$

The integral in (4) is take over

$$V(x_1, \ldots, x_n) = 0. \tag{20}$$

We observed that the definition is independ of the particular choice of u_i coordinate for $i = 1, 2, \ldots, n$.

Now using the change of variable defined by (18) we have

$$< (M(x))_+^\lambda, \varphi >= \int_{M\geq 0} (M(x))^\lambda \varphi(x_1 \cdots x_n) dx_1 \cdots dx_n$$
$$= \int_{V\geq 0} (V(x))^{m\lambda} \varphi(x_1 \cdots x_n) dx_1 \cdots dx_n = \int_0^\infty u_1^{m\lambda} \psi(u_1) du_1, \quad (21)$$

and

$$< (M(x))_-^\lambda, \varphi >= \int_{-M\geq 0} (-M(x))^\lambda \varphi(x_1 \cdots x_n) dx_1 \cdots dx_n$$
$$= \int_{-V\geq 0} (-V(x))^{m\lambda} \varphi(x_1 \cdots x_n) dx_1 \cdots dx_n = \int_{-\infty}^0 (-u_1)^{m\lambda} \psi(u_1) du_1, \quad (22)$$

where

$$\Psi(u_1) = \int [\varphi_1(u_1, \ldots, u_n) D \binom{x}{u}]_{u_1=0} du_1 \cdots du_n. \quad (23)$$

On the other hand, the formula $< x_+^\lambda, \varphi >$, studied in [5], Chapter I, Section 3, is regular for all λ except at $\lambda = -1, -2, \ldots$, where it has simple poles. At these poles we have,

$$\underset{\lambda=-k, k=1,2,\ldots}{\text{Res}} < x_+^\lambda, \varphi >= \frac{\varphi^{(k-1)}(0)}{(k-1)!} \qquad \text{(see [5], p. 41)}, \quad (24)$$

where the function x_+^λ is equal to x^λ for $x > 0$ and to zero for $x \leq 0$. Since

$$\varphi^{(k-1)}(0) = (-1)^{(k-1)} < \delta^{(k-1)}(x), \varphi(x) > \text{ (see [5], p. 49)}, \quad (25)$$

from (24) and (25) we have,

$$\underset{\lambda=-k, k=1,2,\ldots}{\text{Res}} < x_+^\lambda, \varphi >= \frac{(-1)^{(k-1)}}{(k-1)!} < \delta^{(k-1)}(x), \varphi(x) > . \quad (26)$$

From (26) and considering (25) we get

$$\underset{\lambda=-\frac{k}{m}, k=1,2,\ldots}{\text{Res}} < (M(x))_+^\lambda, \varphi >= \underset{\lambda=-\frac{k}{m}, k=1,2,\ldots}{\text{Res}} < u_1^{\lambda m}, \Psi_1(u) >$$
$$\underset{\gamma=-k, k=1,2,\ldots}{\text{Res}} < u_1^\gamma, \Psi_1(u) >= \frac{1}{(k-1)!} \Psi_1^{(k-1)}(0). \quad (27)$$

From (27) and taking into account (26) we obtain

$$\underset{\lambda=-\frac{k}{m}, k=1,2,\ldots}{\text{Res}} < (M(x))_+^\lambda, \varphi >$$

$$= \int \left[\frac{\partial^{k-1}}{\partial u_1^{k-1}} \left\{ \varphi_1(u_1, \ldots, u_n) D\binom{x}{u} \right\} \right]_{u_1=0} du_2 \cdots du_n$$

$$= \frac{(-1)^{(k-1)}}{(k-1)!} < \delta^{(k-1)}(V(x)), \varphi(x) > \quad (28)$$

In consequence from (28) we arrive at the following formula

$$\operatorname*{Res}_{\lambda=-\frac{k}{m}, k=1,2,\ldots} (M(x))_+^\lambda = \frac{(-1)^{(k-1)}}{(k-1)!} \delta^{(k-1)}(V(x)) \quad (29)$$

where $(M(x))_+^\lambda$ is defined by (1).

Similarly, using the formula

$$\operatorname*{Res}_{\lambda=-k, k=1,2,\ldots} < x_-^\lambda, \varphi >= \frac{1}{(k-1)!} < \delta^{(k-1)}(x), \varphi(x) > \quad (30)$$

[5]), p. 58, where the function x_-^λ is equal $(-x)^\lambda$ for $x < 0$ and to zero for $x \geq 0$, we obtain the following formula

$$\operatorname*{Res}_{\lambda=-\frac{k}{m}, k=1,2,\ldots} (M(x))_-^\lambda = \frac{1}{(k-1)!} \delta^{(k-1)}(V(x)) \quad (31)$$

where $(M(x))_-^\lambda$ is defined by (2).

On the other hand, from (21) and using (23), (24), (25) and (26), we have

$$\operatorname*{Res}_{\lambda=-k, k=1,2,\ldots} < (M(x))_+^\lambda, \varphi >= \operatorname*{Res}_{\gamma=-mk, k=1,2,\ldots} < u_1^\gamma, \Psi_1(u) >$$

$$= \frac{1}{m(mk-1)!} \Psi_1^{(mk-1)}(0)$$

$$= \frac{1}{m(mk-1)!} \int \left[\frac{\partial^{mk-1}}{\partial u_1^{mk-1}} \{ \varphi(u_1, \cdots u_n) D\binom{x}{u} \} \right]_{u_1=0} du_2 \cdots du_n$$

$$= \frac{(-1)^{mk-1}}{m(mk-1)!} < \delta^{(mk-1)}(V(x)), \varphi(x) > . \quad (32)$$

Now considering the change of variable

$$\begin{aligned} y_1 &= M(x_1, \ldots, x_n), \\ y_2 &= x_2, \\ &\vdots \\ y_n &= x_n, \end{aligned} \quad (33)$$

where

$$D\binom{x}{y} = (D\binom{y}{x})^{-1} = \frac{1}{\frac{\partial M}{\partial x_1}} \quad (34)$$

and the definition (17), we have

$$< \delta^{(k-1)}(M), \varphi >= (-1)^{k-1} \int \left[\frac{\partial^{k-1}}{\partial y_1^{k-1}} \{\varphi_1 D \begin{pmatrix} x \\ y \end{pmatrix} \} \right]_{y_1=0} dy_2 \cdots dy_n, \qquad (35)$$

where

$$\varphi_1(y_1, \cdots y_n) = \varphi(x_1, \cdots x_n), \qquad (36)$$

and using the formula

$$< (M(x))_+^\lambda, \varphi >= \int_{M \geq 0} (M(x))^\lambda \varphi(x_1 \cdots x_n) dx_1 \cdots dx_n$$

$$= \int_0^\infty y_1^\lambda \psi(y_1) dy_1, \quad (37)$$

where

$$\Psi(y_1) = \int [\varphi_1(y_1, \ldots, y_n) D \begin{pmatrix} x \\ y \end{pmatrix}]_{y_1=0} dy_1 \cdots dy_n. \qquad (38)$$

From (36), (37) and (38) and considering (25) and (26), we have

$$\underset{\lambda=-k, k=1,2,\ldots}{\mathrm{Res}} < (M(x))_+^\lambda, \varphi >= \frac{(-1)^{k-1}}{(k-1)!} < \delta^{(k-1)}(M(x)), \varphi(x) > . \qquad (39)$$

From (32) and (39) we obtain the following formula

$$\delta^{(k-1)}((V(x))^m) = \frac{(k-1)!}{(-1)^{k-1}} \frac{(-1)^{mk-1}}{m(mk-1)!} \delta^{(mk-1)}(V(x)). \qquad (40)$$

On the other hand, from [3] we have the following formula

$$(M(x))_+^\lambda = \frac{\Gamma(-\lambda)\Gamma(1+\lambda)}{2\pi i} [e^{-\pi i \lambda}(M(x)+i0)^\lambda - e^{\pi i \lambda}(M(x)-i0)^\lambda] \qquad (41)$$

(see [3], p. 7, formula (44)) and

$$(M(x))_-^\lambda = -\frac{\Gamma(-\lambda)\Gamma(1+\lambda)}{2\pi i} [(M(x)+i0)^\lambda - (M(x)-i0)^\lambda] \qquad (42)$$

(see [3], p. 7, formula (45))

$$\mathcal{F}\{((a_1 x_1^2 + \cdots + a_\mu x_\mu^2 + a_{\mu+1} x_{\mu+1}^2 + a_{\mu+v} x_{\mu+v}^2)^m)^\lambda\}$$

$$= \frac{\Gamma(-\lambda)\Gamma(1+\lambda) 2^{2m\lambda+\frac{n}{2}} \pi^{\frac{n}{2}} \Gamma(m\lambda + \frac{n}{2})}{2\pi i \Gamma(-m\lambda)\sqrt{|\Delta|}}$$

$$\times \left\{ e^{-\left(\lambda+\frac{v}{2}\right)\pi i} (N - i0)^{-\lambda m - \frac{n}{2}} - e^{\left(\lambda+\frac{v}{2}\right)\pi i} (N + i0)^{-\lambda m - \frac{n}{2}} \right\}, \quad (43)$$

where

$$(M(x)+i0)^\lambda = \lim_{M' \to 0} (M \pm i M')^\lambda \qquad (44)$$

(see [3], p. 4, formulae (20) and (21)), $M = M(x)$ is defined by equation (3),

$$N = N(y) = N(y_1, \ldots, y_n) = \frac{1}{\alpha_1}y_1^2 + \cdots + \frac{1}{\alpha_\mu}y_\mu^2 + \frac{1}{\alpha_{\mu+1}}y_{\mu+1}^2 + \cdots + \frac{1}{\alpha_{\mu+\nu}}y_{\mu+v}^2 \quad (45)$$

if $a_1, \cdots a_\mu > 0$ and $a_{\mu+1}, \cdots a_{\mu+\nu} < 0$ (see [3], formula (36)),

$\mathcal{F}\{f\}$ is the Fourier transform defined by

$$\mathcal{F}\{f(x)\}(y) = \int_{R^n} f(x)e^{-i<x,y>}dx \quad (46)$$

and Δ is the determinant of the coefficients of $M(x)$.

From (8) and using the formula

$$\operatorname*{Res}_{z=-h,h=0,1,2,\ldots} \Gamma(z) = \frac{(-1)^h}{h!} \quad (47)$$

we have,

$$\delta^{(k-1)}(V(x)) = \lim_{\lambda\to -k} \frac{(V(x))_+^\lambda}{\Gamma(\lambda+1)}. \quad (48)$$

Similarly from (33) and (47) we have

$$\delta^{(k-1)}(V(x)) = \lim_{\lambda\to -\frac{k}{m}} \frac{(M(x))_+^\lambda}{\Gamma(\lambda m+1)}. \quad (49)$$

By putting $m = 1$ in (41), (43) and using (48), we obtain the following formula

$$\mathcal{F}\{\delta^{(k-1)}(V(x))\} = C(-k,n)[e^{-(-k+\frac{v}{2})\pi i}(N-i0)^{k-\frac{n}{2}} - e^{(-k+\frac{v}{2})\pi i}(N+i0)^{k-\frac{n}{2}}] \quad (50)$$

if $a_1, \cdots a_\mu > 0$ and $a_{\mu+1}, \cdots a_{\mu+\nu} < 0$, where

$$C(-k,n,) = \frac{1}{i}\frac{1}{2}\frac{\pi^{\frac{n-2}{2}}\Gamma(k)2^{n-2k}\Gamma(\frac{n}{2}-k)}{\Gamma(k)\sqrt{|\Delta|}}. \quad (51)$$

The formula (50) is a generalization of the formula (99) due to Manuel A. Aguirre in [1], p. 126.

In fact, by putting $a_1 = a_2 = \cdots = a_\mu = 1$ and $a_{\mu+1} = a_{\mu+2} = \cdots = a_{\mu+v} = -1$ in (50) and considering that

$$V(x_1 \cdots x_n) = P(x_1 \cdots x_n) = x_1^2 + \cdots + x_\mu^2 - x_{\mu+1}^2 - \cdots - x_{\mu+v}^2 \quad (52)$$

if $a_1 = a_2 = \cdots = a_\mu = 1$ and $a_{\mu+1} = a_{\mu+2} = \cdots = a_{\mu+v} = -1$, where $\mu+v = n$ dimension of the space, we get the formula

$$\mathcal{F}\{\delta^{(k-1)}(P(x_1\cdots x_n))\} = b(-k,n,)(-1)^k[e^{-\frac{\mu\pi i}{2}}(N-i0)^{k-\frac{n}{2}} - e^{\frac{\mu\pi i}{2}}(N+i0)^{k-\frac{n}{2}}]. \quad (53)$$

where

$$b(-k,n,)=\frac{1}{i}\frac{1}{2}\frac{\pi^{\frac{n-2}{2}}\Gamma(k)2^{n-2k}\Gamma(\frac{n}{2}-k)}{\Gamma(k)}. \tag{54}$$

The formula (53) appear in [1], p. 126.

From (40) and (50) we obtain the Fourier transform of $\delta^{(k-1)}((V(x))^m)$,

$$\mathcal{F}\{\delta^{(k-1)}(M(x))\}=\mathcal{F}\{\delta^{(k-1)}((V(x))^m)\}=\frac{(-1)^{k-1}}{(k-1)!}\frac{(-1)^{mk-1}}{m(mk-1)!}C(-mk,n)$$
$$\times[e^{-(-mk+\frac{v}{2})\pi i}(N-i0)^{k-\frac{n}{2}}-e^{(-mk+\frac{v}{2})\pi i}(N+i0)^{k-\frac{n}{2}}], \tag{55}$$

if $a_1,\cdots a_\mu>0$ and $a_{\mu+1},\cdots a_{\mu+\nu}<0$, where $C(-mk,n)$ is defined by(51).

It is clear that by putting $m=1$ in(55) we obtain the formula(50).

2. The Expansion in Series Taylor Type of $\delta^{(k-1)}(c^2+M(x))$

Let $(c^2+M(x)\pm i0)^\alpha$ be the distribution defined by

$$(c^2+M(x)\pm i0)^\alpha=\lim_{M'\to 0}(c^2+M(x)\pm iM'(x))^\alpha, \tag{56}$$

then the following formula is valid

$$(c^2+M(x)\pm i0)^\alpha=\sum_{v\geq 0}\frac{(c^2)^v}{v!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-v)}(M(x)\pm i0)^{\alpha-v} \tag{57}$$

if $M(x)\geq c^2$ and $\alpha\neq v-\frac{n}{2m}-k, k=0,1,2,\cdots$.

Proof. Let $H_\alpha(c^2+M(x)\pm iM'(x))$ be

$$H_\alpha(c^2+M(x)\pm iM'(x))=(c^2+M(x)\pm i0)^\alpha \tag{58}$$

and

$$H_\alpha(c^2+M(x)\pm i0)=(c^2+M(x)\pm i0)^\alpha. \tag{59}$$

From the definition we have

$$\lim_{M'\to 0}H_\alpha(c^2+M(x)\pm iM'(x))^\alpha=H_\alpha(c^2+M(x)\pm i0). \tag{60}$$

Using the formula

$$(1+z)^\alpha=\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-l)}\frac{z^l}{l!} \tag{61}$$

if $|z| < 1$ (see [6], p. 101), from (34) we have,

$$\begin{aligned} H_\alpha(c^2 + M(x) \pm iM'(x)) &= (c^2 + M(x) \pm iM'(x))^\alpha \\ &= (M(x) \pm iM'(x))^\alpha \textstyle\sum_{v=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-v)} \frac{(c^2)^v}{v!} (M(x) \pm iM'(x))^v. \end{aligned} \tag{62}$$

Let

$$H_\alpha(c^2 + M(x) \pm iM'(x)) = \sum_{v=0}^{j} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-v)} \frac{(c^2)^v}{v!} (M(x) \pm iM'(x))^{\alpha-v}. \tag{63}$$

Therefore, for $Re(\alpha - v) > 0$, we have,

$$\lim_{M'(x)\to 0} H_\alpha^j(c^2 + M(x) \pm iM'(x)) = \sum_{v=0}^{j} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-v)} \frac{(c^2)^v}{v!} (M(x) \pm i0)^{\alpha-v} \tag{64}$$

and by analytic continuation the formula (40) is valid for every α (complex numbers) so that $\alpha \neq v - \frac{n}{2m} - k, k = 0, 1, 2, ...$ and $m = 1, 2, 3...$ Therefore

$$\begin{aligned} &\lim_{j\to\infty} (\lim_{M'(x)\to 0} H_\alpha^j(c^2 + M(x) \pm iM'(x))) \\ &\qquad = \sum_{v=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-v)} \frac{(c^2)^v}{v!} (M(x) \pm i0)^{\alpha-v} \end{aligned} \tag{65}$$

From (41), (34), (35) and (36), we conclude

$$\begin{aligned} (c^2 + M(x) \pm iM'(x))^\alpha &= H_\alpha(c^2 + M(x) \pm i0) = \lim_{M'(x)\to 0} H_\alpha(c^2 + M(x) \pm iM'(x)) \\ &= \lim_{j\to\infty} \lim_{M'(x)\to 0} H_\alpha^j(c^2 + M(x) \pm iM'(x)) \\ &= \sum_{v=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-v)} \frac{(c^2)^v}{v!} (M(x) \pm i0)^{\alpha-v}. \end{aligned} \tag{66}$$

□

Lemma 1. *Let k be a positive integer and $\delta^{(k-1)}(M(x))$ the distribution defined by (27), then the following formula is valid*

$$\begin{aligned} &\delta^{(k-1)}(c^2 + (V(x))^m) = \delta^{(k-1)}(c^2 + M(x)) \\ &\sum_{v\geq 0} \frac{(c^2)^v}{v!} \delta^{(k+v-1)}(M(x)) = \sum_{v\geq 0} \frac{(c^2)^v}{v!} \delta^{(k+v-1)}((V(x))^m). \end{aligned} \tag{67}$$

Proof. Using the formula (20) for the case $c^2 + M(x)$, we obtain

$$\frac{2\pi i}{\Gamma(1-\alpha)} \cdot \frac{(c^2 + M(x))^{\alpha-1}}{\Gamma(\alpha)}$$

$$= e^{(\alpha-1)\pi i}(c^2 + M(x) - i0)^{\alpha-1} - e^{-(\alpha-1)\pi i}(c^2 + M(x) + i0)^{\alpha-1}. \quad (68)$$

Now using (60) we obtain

$$\frac{2\pi i}{\Gamma(1-\alpha)}.\frac{(c^2+M(x))^{\alpha-1}}{\Gamma(\alpha)}$$
$$= \sum_{v=0}^{\infty} \frac{(c^2)^v}{v!}\frac{\Gamma(\alpha)}{\Gamma(\alpha-v)}[e^{(\alpha-1)\pi i}(M(x)-i0)^{\alpha-v-1} - e^{-(\alpha-1)\pi i}(M(x)+i0)^{\alpha-v-1}]. \quad (69)$$

Now using the properties

$$e^{(\alpha-1)\pi i} = e^{(\alpha-1-v)\pi i}(-1)^v \quad (70)$$

and

$$e^{-(\alpha-1)\pi i} = e^{-(\alpha-1-v)\pi i}(-1)^v \quad (71)$$

for $v = 0, 1, 2\cdots$ and considering the formulae (41) and (42), we have

$$e^{(\alpha-1)\pi i}(M(x)-i0)^{\alpha-1-v} - e^{-(\alpha-1)\pi i}(M(x)+i0)^{\alpha-1-v}$$
$$= (-1)^v[e^{(\alpha-1-v)\pi i}(M(x)-i0)^{\alpha-1-v} - e^{-(\alpha-1-v)\pi i}(M(x)+i0)^{\alpha-1-v}]$$
$$= \frac{(-1)^v 2\pi i}{\Gamma(-\alpha+1+v)}.\frac{(M(x))_+^{\alpha-1-v}}{\Gamma(\alpha-v)}. \quad (72)$$

Reemplacing the formula (48) into (45) we obtain

$$\frac{(c^2+M(x))_+^{\alpha-1}}{\Gamma(\alpha)} = \sum_{v=0}^{\infty}\frac{(c^2)^v}{v!}\frac{(-1)^v\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha-v)\Gamma(-\alpha+1+v)}.\frac{(M(x))_+^{\alpha-v-1}}{\Gamma(\alpha-v)}. \quad (73)$$

From (49) and considering the formula

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha-v)} = \frac{(-1)^v\Gamma(-\alpha+k+1)}{\Gamma(1-\alpha)} \qquad \text{(see [6], p. 3)}, \quad (74)$$

from (73) and (74) we have

$$\lim_{\alpha\to-k}\frac{(c^2+M(x))_+^{\alpha-1}}{\Gamma(\alpha)} = \sum_{v=0}^{\infty}\frac{(c^2)^v}{v!}\lim_{\alpha\to-k}\frac{(M(x))_+^{\alpha-\nu-1}}{\Gamma(\alpha-v)}. \quad (75)$$

Taking into account that

$$\lim_{\alpha\to-k}\frac{(c^2+M(x))_+^{\alpha-1}}{\Gamma(\alpha)} = \delta^{(k-1)}(c^2+M(x)) \quad (76)$$

and using (8) we have

$$\lim_{\alpha\to-k}\frac{(M(x))_+^{\alpha-\nu-1}}{\Gamma(\alpha-v)} = \lim_{\lambda\to-k}\frac{(M(x))_+^{\lambda-\nu}}{\Gamma(\lambda-v+1)} = \lim_{\lambda\to-k-\nu}\frac{((V(x))^m)_+^{\gamma}}{\Gamma(\gamma+1)}$$

$$= \lim_{\gamma\to -k-\nu} \frac{((V(x))^m)_+^\gamma}{\Gamma(\gamma+1)} = \lim_{\gamma\to -k-\nu} \frac{(\gamma+k+\nu)((V(x))^m)_+^\gamma}{(\gamma+k+\nu)\Gamma(\gamma+1)}$$

$$= \frac{\lim_{\gamma\to -k-\nu}(\gamma+k+\nu)((V(x))^m)_+^\gamma}{\lim_{\gamma\to -k-\nu}(\gamma+k+\nu)\Gamma(\gamma+1)} = \frac{\lim_{\gamma\to -k-\nu}(\gamma+k+\nu)V_+^{m\gamma}}{\operatorname{Res}_{\gamma=-k-\nu}\Gamma(\gamma+1)}$$

$$= \frac{(k+\nu-1)!}{(-1)^{k+\nu-1}} \lim_{\beta\to -m(k+\nu)} (\frac{\beta}{m}+k+\nu)V_+^\beta = \frac{(k+\nu-1)!}{(-1)^{k+\nu-1}} \frac{1}{m} \operatorname{Res}_{\beta=-m(k+\nu)} V_+^\beta$$

$$= \frac{(k+\nu-1)!}{(-1)^{k+\nu-1}} \frac{1}{m} \frac{(-1)^{m(k+\nu)-1}}{(m(k+\nu)-1)!} \delta^{(m(k+\nu)-1)}(V(x)). \quad (77)$$

From (77) and considering (75), we get the following formula

$$\delta^{(k-1)}(c^2+(V(x))^m) = \delta^{(k-1)}(c^2+M(x))$$
$$= \sum_{v=0}^{\infty} \frac{(c^2)^v}{v!} \frac{(k+\nu-1)!}{(-1)^{k+\nu-1}} \frac{1}{m} \frac{(-1)^{m(k+\nu)-1}}{(m(k+\nu)-1)!} \delta^{(m(k+\nu)-1)}(V(x)). \quad (78)$$

We observe that by putting $c^2 = 0$ in (78) we obtain the formula (40).

Now from (78) and using the formula (40) we obtain

$$\delta^{(k-1)}(c^2+(V(x))^m) = \delta^{(k-1)}(c^2+M(x))$$
$$= \sum_{v=0}^{\infty} \frac{(c^2)^v}{v!} \delta^{(k+v-1)}(M(x)) = \sum_{v=0}^{\infty} \frac{(c^2)^v}{v!} \delta^{(k+v-1)}((V(x))^m), \quad (79)$$

under conditions $a_1, \cdots a_\mu > 0$ and $a_{\mu+1}, \cdots a_{\mu+\nu} < 0$. □

The formula (67) is a generalization of formula that appear in [4] due to Manuel A. Aguirre. In fact, by putting $m = 1$ in (67) and considering(4) we have

$$\delta^{(k-1)}(c^2+V(x)) = \sum_{v=0}^{\infty} \frac{(c^2)^v}{v!} \delta^{(k+v-1)}(V(x)). \quad (80)$$

The formula(80)appear in [4].

Remark 2. We observed that using the formula

$$\delta^{(k-1)}(u(x_1\cdots x_n)-t) = \sum_{l\geq 0} \frac{(-1)^l}{l!} \delta^{(k+l-1)}(u(x_1\cdots x_n))t^l \quad (81)$$

due to Manuel A. Aguirre (see [2], formula (46), where $u(x_1\cdots x_n)$ is C^∞ functions without singular point, we arrive at the same formula (67)and(80)).

From (67) and using (55) we obtain the Fourier transform of

$$\delta^{(k-1)}(c^2+M(x)),$$

$$\mathcal{F}\{\delta^{(k-1)}(c^2+M(x))\} = \sum_{l=0}^{\infty} \frac{(c^2)^l}{l!} \mathcal{F}\{\delta^{(k+l-1)}(M(x)\}$$

$$= \sum_{l=0}^{\infty} \frac{(c^2)^l}{l!} \frac{(k+l-1)!}{(-1)^{k+l-1}} \frac{(-1)^{m(k+l)-1}}{m(m(k+l)-1)!} C(-m(k+l),n)$$

$$\times [e^{-(-m(k+l)+\frac{v}{2})\pi i}(N-i0)^{(k+l)m-\frac{n}{2}} - e^{(-m(k+l)+\frac{v}{2})\pi i}(N+i0)^{(k+l)m-\frac{n}{2}}], \quad (82)$$

where

$$C(-m(k+l),n) = \frac{1}{i}\frac{1}{2}\frac{\pi^{\frac{n-2}{2}}\Gamma(k+l)2^{n-2m(k+l)}\Gamma(\frac{n}{2}-m(k+l))}{\Gamma(m(k+l))\sqrt{|\Delta|}}. \quad (83)$$

It is clear that by putting $m = 1$ in (82) and (83)we obtain the formula (50).

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