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THE FOURIER TRANSFORM OF $((\sum_{i=1}^{n} a_i x_i^2)^m \pm i0))^\lambda$

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Abstract: Let H be a quadratic form defined by (1). In this paper we give a sense to Fourier transform of H^λ and $((\sum_{i=1}^{n} a_i x_i^2)^m \pm i0)^\lambda$, where λ is a complex number and $m = 1, 2, ...$

As consequence we obtain the Fourier transform of the distributions family defined by (50).

Our results (cf. formulae (35) and (37)) are a generalization of the Fourier transform of distributions $(x_1^2 + .. + x_p^2 - x_{p+1}^2 - .. - x_{p+q}^2 \pm i0)^\lambda$, which appears in [2], p. 284.

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1. Introduction

Let H be a quadratic form defined by

$$H = \sum_{j=1}^{n} \alpha_j x_j^2. \tag{1}$$

Associated to H considering H_m defined by

$$H_m = (\sum_{j=1}^{n} \alpha_j x_j^2)^m \quad \text{for } m = 1, 2, ... \tag{2}$$

with complex coefficients, and let $Im\ (H)$ be a positive definite quadratic form, i.e. $Im\ \alpha_j > 0,\ j = 1, 2, ..., n$. The generalized function H_m^λ and therefore also it is Fourier transform $\mathcal{F}\{H_m^\lambda\}$, are analytic functions of the α_j in the region

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$Im\ \alpha_j > 0$. Therefore in order to find $\mathcal{F}\{H_m^\lambda\}$ we need only to treat the case in which all the α_j are imaginary, and we shall write

$$\alpha_j = ib_j \quad \text{with } b_j > 0.$$

For the case, we have,

$$\mathcal{F}\{H_m^\lambda\} = \int H_m^\lambda e^{i<x,s>} dx$$
$$= \int (\sum_{j=1}^n \alpha_j x_j^2)^{m\lambda} e^{j<x,s>} dx = \int (\sum_{j=1}^n \alpha_j x_j^2)^{m\lambda} e^{j(x_1 s_1 + ... + x_n s_n)} dx. \quad (3)$$

A suitable change of variables in the integrand transform this to th form

$$\mathcal{F}\{H_m^\lambda\} = \frac{e^{\frac{\lambda m\pi i}{2}}}{\sqrt{b_1}...\sqrt{b_n}} \int ((r^2)^m)^\lambda e^{i(y_1 \frac{s_1}{\sqrt{b_1}} + ... + y_n \frac{s_n}{\sqrt{b_n}})} dx, \quad (4)$$

where

$$r^2 = y_1^2 + ... + y_n^2\ . \quad (5)$$

We know from [2], p. 194, that the Fourier transform of r^λ is given by the following formula

$$\mathcal{F}\{r^\lambda\} = \frac{2^{\lambda+n} \pi^{\frac{n}{2}} \Gamma(\frac{\lambda+n}{2})}{\Gamma(-\frac{\lambda}{2})} \rho^{-\lambda-n} \quad \text{(see [2], p. 194, formula (2))}, \quad (6)$$

where

$$\rho^2 = \sigma_1^2 + ... + \sigma_n^2\ . \quad (7)$$

From (4) and (6) we have,

$$\mathcal{F}\{H_m^\lambda\} = \frac{e^{\frac{\lambda m\pi i}{2}}}{\sqrt{b_1}...\sqrt{b_n}} \frac{2^{2m\lambda+n} \pi^{\frac{n}{2}} \Gamma(\frac{2m\lambda+n}{2})}{\Gamma(\frac{-2m\lambda}{2})} \left(\frac{s_1^2}{b_1} + ... + \frac{s_n^2}{b_n}\right)^{-\frac{2m\lambda-n}{2}}$$
$$= \frac{e^{\frac{\lambda m\pi i}{2}}}{\sqrt{b_1}...\sqrt{b_n}} \frac{2^{2m\lambda+n} \pi^{\frac{n}{2}} \Gamma(m\lambda + \frac{n}{2})}{\Gamma(-m\lambda)} \left(\frac{s_1^2}{b_1} + ... + \frac{s_n^2}{b_n}\right)^{-\frac{2m\lambda-n}{2}}. \quad (8)$$

Now using

$$\sqrt{b_1}...\sqrt{b_n} = \sqrt{-i\alpha_1}.\sqrt{-i\alpha_2}...\sqrt{-i\alpha_n}, \quad (9)$$

we have

$$\left(\frac{s_1^2}{b_1} + ... + \frac{s_n^2}{b_n}\right)^{-m\lambda-\frac{n}{2}} = e^{-\frac{\lambda m\pi i}{2}} e^{-\frac{n\pi i}{4}} \left(\frac{s_1^2}{\alpha_1} + ... + \frac{s_n^2}{\alpha_n}\right)^{-m\lambda-\frac{n}{2}}. \quad (10)$$

From (8) and (10) we arrive at the following formula

$$\mathcal{F}\{H_m^\lambda\} = \frac{e^{-\frac{n\pi i}{4}}}{\sqrt{-i\alpha_1}.\sqrt{-i\alpha_2}...\sqrt{-i\alpha_n}}$$

$$\times \frac{2^{m\lambda+n}\pi^{\frac{n}{2}}\Gamma(m\lambda+\frac{n}{2})}{\Gamma(-m\lambda)}\left(\frac{s_1^2}{\alpha_1}+...+\frac{s_n^2}{\alpha_n}\right)^{-m\lambda-\frac{n}{2}}. \quad (11)$$

2. The Fourier Transform of $((a_1x_1^2+..+a_px_p^2+a_{p+1}x_{p+1}^2+..+a_{p+q}x_{p+q}^2)^m \pm i0)^\lambda$

For our propuse consider the quadratic form

$$G = P + iP', \quad (12)$$

where

$$P = \left(\sum_{j=1}^{n} \lambda_j x_j^2\right)^m, \quad (13)$$

$$P' = \varepsilon\left(\sum_{k=1}^{n} x_k^2\right)^m, \quad (14)$$

and ε is a real number such that $\varepsilon > 0$.

Using the formula

$$(a_1+...+a_r)^n = \sum_{n_1+...+n_r=n} \frac{n!}{n_1!...n_r!} a_1^{n_1}.a_2^{n_2}...a_r^{n_r}, \quad (15)$$

we have

$$P = (\lambda_1 x_1^2+...+\lambda_n x_n^2)^m = \sum_{m_1+...+m_n=m} \frac{m!}{m_1!...m_n!}(\lambda_1 x_1^2)^{m_1}...(\lambda_n x_n^2)^{m_n} \quad (16)$$

and

$$P' = \varepsilon(x_1^2+...+x_n^2)^m = \varepsilon \sum_{m_1+...+m_n=m} \frac{m!}{m_1!...m_n!}(x_1^2)^{m_1}...(x_n^2)^{m_n}. \quad (17)$$

Now $P + iP'$ can be rewritten in the following form

$$P + iP' = \sum_{m_1+...+m_n=m} \frac{m!}{m_1!...m_n!}$$

$$\times \left\{(\lambda_1{}^{m_1}+i\varepsilon)(x_1^2)^{m_1}+(\lambda_2{}^{m_2}+i\varepsilon)(x_2^2)^{m_2}+...+(\lambda_n{}^{m_n}+i\varepsilon)(x_n^2)^{m_n}\right\}. \quad (18)$$

By the definition we have

$$(P+i0)^\lambda = \lim_{P'\to 0}(P+iP')^\lambda \quad \text{(see [2], p. 275)}. \quad (19)$$

Similarly, we shall define the generalized function $(P - i0)^\lambda$ as the limit of the generalized function $(P - iP')^\lambda$ as the coefficient of P' converges to a zero,

$$(P - i0)^\lambda = \lim_{P' \to 0} (P - iP')^\lambda. \tag{20}$$

On the other hand, using that. We have

$$\sqrt{z} = |z|^{\frac{1}{2}}\, e^{\frac{i}{2}\arg z}, \tag{21}$$

where z is a complex number, we have,

$$\sqrt{-i\alpha_1}.\sqrt{-i\alpha_2}...\sqrt{-i\alpha_n} = e^{-\frac{n\pi i}{4}}\sqrt{|\Delta|}e^{\frac{i}{2}(\arg\alpha_1+...+\arg\alpha_n)} \tag{22}$$

where Δ is the determinant of the cofficients of H.

We observe that the equation (2) using (15) can be rewritten in the following form

$$H_m = \left(\sum_{i=1}^{n}\alpha_i x_i^2\right)^m = \sum_{m_1+...+m_r=m}\frac{m!}{m_1!...m_r!}(\alpha_1 x_1^2)^{m_1}...(\alpha_n x_n^2)^{m_r} \tag{23}$$

From (18) and (23) we have the following relations

$$\begin{aligned} \lambda_1^{m_1} + i\varepsilon &= \alpha_1^{m_1}, \\ &\vdots \\ \lambda_p^{m_p} + i\varepsilon &= \alpha_p^{m_p}, \\ \lambda_{p+1}^{m_{p+1}} + i\varepsilon &= \alpha_{p+1}^{m_{p+1}}, \\ &\vdots \\ \lambda_{p+q}^{m_{p+q}} + i\varepsilon &= \alpha_{p+q}^{m_{p+q}}. \end{aligned} \tag{24}$$

Take limit from (24) when $\varepsilon \to 0$, we have,

$$\begin{aligned} \lambda_1^{m_1} + i0 &= \alpha_1^{m_1}, \\ &\vdots \\ \lambda_p^{m_p} + i0 &= \alpha_p^{m_p}, \\ \lambda_{p+1}^{m_{p+1}} + i0 &= \alpha_{p+1}^{m_{p+1}}, \\ &\vdots \\ \lambda_{p+q}^{m_{p+q}} + i0 &= \alpha_{p+q}^{m_{p+q}}. \end{aligned} \tag{25}$$

Now using the property

$$(x - i0)^\lambda = \begin{cases} x^\lambda & \text{for } x > 0, \\ e^{-i\pi\lambda}\left|x^\lambda\right|, & \text{for } x < 0 \quad \text{(see [2], p. 59)}, \end{cases} \tag{26}$$

from (25) we have

$$\begin{aligned} \alpha_1^{m_1} &= \lambda_1^{m_1} + i0 = (\lambda_1 + i0)^{m_1} = \lambda_1^{m_1}, \\ &\vdots \\ \alpha_p^{m_p} &= \lambda_p^{m_p} + i0 = (\lambda_p + i0)^{m_p} = \lambda_p^{m_p}, \end{aligned} \tag{27}$$

if $\lambda_1, \lambda_p, ..., \lambda_p > 0$ and

$$\begin{aligned} \alpha_{p+1}^{m_{p+1}} &= \lambda_{p+1}^{m_{p+1}} + i0 = (\lambda_{p+1} + i0)^{m_{p+1}} = e^{i\pi(m_{p+1})} \left|\lambda_{p+1}^{m_{p+q}}\right|, \\ &\vdots \\ \alpha_{p+q}^{m_{p+q}} &= \lambda_{p+q}^{m_{p+q}} + i0 = (\lambda_{p+q} + i0)^{m_{p+q}} = e^{i\pi(m_{p+q})} \left|\lambda_{p+q}^{m_{p+q}}\right|, \end{aligned} \tag{28}$$

if $\lambda_{p+1}, .., \lambda_{p+q} < 0$.

Put

$$M = M(x_1...x_n) = \sum_{i=1}^{n} a_i x_i^2 \tag{29}$$

and

$$M_m = M_m(x_1...x_n) = \left(\sum_{i=1}^{n} a_i x_i^2\right)^m = (a_1 x_1^2 + ... + a_n x_n^2)^m \tag{30}$$

where $a_i = \mathrm{Re}(\alpha_i)$, $a_1, ..., a_p > 0$ and $a_{p+1}, ..., a_{p+q} < 0$.

By putting the imaginary part of quadratic form now approaching zero in equation (11) and using (19) we have

$$\mathcal{F}\left\{(M_m(x_1...x_n) + i0)^{\lambda}\right\}$$
$$\frac{e^{\frac{-n\pi i}{4}} 2^{2m\lambda+n} \pi^{\frac{n}{2}} \Gamma(m\lambda + \frac{n}{2})}{\Gamma(-m\lambda) e^{\frac{-n\pi i}{4}} \sqrt{|\Delta|} e^{\frac{i}{2}(\arg\alpha_1 + ... + \arg\alpha_n)}} \left(\frac{s_1^2}{\alpha_1} + ... + \frac{s_n^2}{\alpha_n}\right)^{-\lambda m - n} . \tag{31}$$

Take into account that

$$e^{\frac{i}{2}(\arg\alpha_1 + ... + \arg\alpha_p + \arg\alpha_{p+1} + ... + \arg\alpha_{p+q})}$$
$$= e^{\frac{i}{2}(\arg\lambda_1 + ... + \arg\lambda_p + \arg\lambda_{p+1} + ... + \arg\lambda_{p+q})} = e^{\frac{q\pi i}{2}}, \tag{32}$$

if $\lambda_1, ..., \lambda_p > 0$ and $\lambda_{p+1}, ..., \lambda_{p+q} < 0$. Now denoting

$$c_j = \frac{1}{\lambda_j} \quad , \; j = 1, 2...n \tag{33}$$

and considering the formulae (27) and (28), we have

$$\begin{aligned} \left(\frac{s_1^2}{\alpha_1} + ... + \frac{s_n^2}{\alpha_n}\right)^{-\lambda m - n} &= [(\frac{1}{\lambda_1} - i0)s_1^2 + ... + (\frac{1}{\lambda_p} - i0)s_p^2 \\ &+ (\frac{1}{\lambda_{p+1}} - i0)s_{p+1}^2 + ... + (\frac{1}{\lambda_{p+q}} - i0)s_{p+q}^2]^{-\lambda m - \frac{n}{2}} \\ &= \left[\frac{1}{\lambda_1}s_1^2 + ... + \frac{1}{\lambda_p}s_p^2 + \frac{1}{\lambda_{p+1}}s_{p+1}^2 + ... + \frac{1}{\lambda_{p+q}}s_{p+q}^2 - i0\right]^{-\lambda m - \frac{n}{2}} \end{aligned}$$

$$= \left[\frac{1}{\alpha_1}s_1^2 + ... + \frac{1}{\alpha_p}s_p^2 + \frac{1}{\alpha_{p+1}}s_{p+1}^2 + ... + \frac{1}{\alpha_{p+q}}s_{p+q}^2 - i0\right]^{-\lambda m - \frac{n}{2}}. \quad (34)$$

From (30) and (33) we obtain the Fourier transform of $(M_m(x_1...x_n) + i0)^\lambda$,

$$\mathcal{F}\left\{(M_m(x_1...x_n) + i0)^\lambda\right\} = \frac{2^{2m\lambda+n}\pi^{\frac{n}{2}}\Gamma(m\lambda + \frac{n}{2})}{\Gamma(-m\lambda)\sqrt{|\Delta|}e^{+\frac{q\pi i}{2}}}(N(s_1, ..., s_n) - i0)^{-\lambda m - \frac{n}{2}}, \quad (35)$$

if $a_1, ..a_p > 0$ and $a_{p+1}, ..a_{p+q} < 0$,where $M(x_1...x_n)$ and $M_m(x_1...x_n)$ are defined by (29) and (30) respectively, $N(s_1, ..., s_n)$ is defined by

$$N(s_1, ..., s_n) = \frac{1}{\alpha_1}s_1^2 + ... + \frac{1}{\alpha_n}s_n^2 = \sum_{j=1}^{n}\frac{1}{\alpha_j}s_j^2 \quad (36)$$

and Δ is the determinant of coefficients of $M_m(x_1...x_n)$. Similarly using (20) and (32) we obtain the Fourier transform of $(M_m(x_1...x_n) - i0)^\lambda$,

$$\mathcal{F}\left\{(M_m(x_1...x_n) - i0)^\lambda\right\} = \frac{2^{2m\lambda+n}\pi^{\frac{n}{2}}\Gamma(m\lambda + \frac{n}{2})}{\Gamma(-m\lambda)\sqrt{|\Delta|}e^{-\frac{q\pi i}{2}}}(N(s_1, ..., s_n) + i0)^{-\lambda m - \frac{n}{2}}, \quad (37)$$

if $a_1, ..a_p > 0$ and $a_{p+1}, ..a_{p+q} < 0$, where $M_m(x_1...x_n)$ is defined by (30), $N(s_1, ..., s_n)$ is defined by (36). The results are a generaliztion of the formulae (3) and (3') of [2]), p. 284.

In fact by letting $a_1 = a_2 = ...a_p = 1; a_{p+1} = ... = a_{p+q} = -1$ and $m = 1$ in (35) and (36), we obtain the Fourier transform of $(x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2 \pm i0)$:

$$\mathcal{F}\left\{(x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2 \pm i0)^\lambda\right\} = \frac{2^{2\lambda+n}\pi^{\frac{n}{2}}\Gamma(\lambda + \frac{n}{2})}{\Gamma(-\lambda)e^{\pm\frac{q\pi i}{2}}}(s_1^2 + ... + s_p^2 - s_{p+1}^2 - ... - s_{p+q}^2 \mp i0)^{-\lambda - \frac{n}{2}}. \quad (38)$$

The formula (38) appear in [2]), p. 284, formula (3) and (3').

On the other hand, the generalized function $(M_m(x_1...x_n) + i0)^\lambda$ can be expresed in terms of $(M_m(x_1...x_n))_+^\lambda$ and $(M_m(x_1...x_n))_-^\lambda$:

$$(M_m(x_1...x_n) + i0)^\lambda = (M_m(x_1...x_n))_+^\lambda + e^{\pi i\lambda}(M_m(x_1...x_n))_-^\lambda \quad (39)$$

and

$$(M_m(x_1...x_n) - i0)^\lambda = (M_m(x_1...x_n))_+^\lambda + e^{-\pi i\lambda}(M_m(x_1...x_n))_-^\lambda. \quad (40)$$

Indeed, for $Re(\lambda) > 0$ the functionals $((M_m(x_1...x_n))_+^\lambda, \varphi)$ and $((M_m(x_1...x_n))_-^\lambda, \varphi)$

correspond to the function,

$$(M_m(x_1...x_n))_+^\lambda = \begin{cases} (M_m(x_1...x_n))^\lambda \text{ if } M_m(x_1...x_n) \geq 0, \\ 0 \text{ if } M_m(x_1...x_n) \leq 0, \end{cases} \tag{41}$$

and

$$(M_m(x_1...x_n))_-^\lambda = \begin{cases} 0 \text{ if } M_m(x_1...x_n) \geq 0, \\ (-M_m(x_1...x_n))^\lambda \text{ if } M_m(x_1...x_n) \leq 0. \end{cases} \tag{42}$$

The properties (39) and (40) follow directly from definitions (19) and (20) respectively.

From (39) and (40), using the formula

$$\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin \pi\lambda} \tag{43}$$

(see [1], p. 3, formula (6)), we have the following formulae

$$(M_m(x_1...x_n))_+^\lambda = \frac{\Gamma(-\lambda)\Gamma(1+\lambda)}{2\pi i} \times \left\{ e^{-\pi i\lambda}(M_m(x_1...x_n)+i0)^\lambda - e^{\pi i\lambda}(M_m(x_1...x_n)-i0)^\lambda \right\} \tag{44}$$

and

$$(M_m(x_1...x_n))_-^\lambda = \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{2\pi i} \left\{ (M_m(x_1...x_n)+i0)^\lambda - (M_m(x_1...x_n)-i0)^\lambda \right\}. \tag{45}$$

From (44) and (45), using the formulae (35) and (37) we obtain the Fourier transform of $(M_m(x_1...x_n))_\pm^\lambda$,

$$\mathcal{F}\{((a_1x_1^2+...+a_nx_n^2)^m)_+^\lambda\} = \frac{\Gamma(-\lambda)\Gamma(1+\lambda)}{2\pi i} \frac{2^{2m\lambda+n}\pi^{\frac{n}{2}}\Gamma(m\lambda+\frac{n}{2})}{\Gamma(-m\lambda)\sqrt{|\Delta|}} \times \left\{ e^{-(\lambda+\frac{q}{2})\pi i}(N-i0)^{-\lambda m-\frac{n}{2}} - e^{(\lambda+\frac{q}{2})\pi i}(N+i0)^{-\lambda m-\frac{n}{2}} \right\} \tag{46}$$

and

$$\mathcal{F}\{((a_1x_1^2+...+a_nx_n^2)^m)_+^\lambda\} = \frac{\Gamma(\lambda)\Gamma(1-\lambda)\Gamma(m\lambda+\frac{n}{2})}{2\pi i} \times \frac{2^{2m\lambda+n}\pi^{\frac{n}{2}}}{\Gamma(-m\lambda)\sqrt{|\Delta|}} \left\{ e^{-\frac{q\pi i}{2}}(N-i0)^{-\lambda m-\frac{n}{2}} - e^{\frac{q\pi i}{2}}(N+i0)^{-\lambda m-\frac{n}{2}} \right\}. \tag{47}$$

By letting $a_1 = a_2 = ... = a_p = 1; a_{p+1} = a_{p+2} = ... = a_{p+q} = -1$ in (44) and (45) we have

$$\mathcal{F}\{(x_1^2+...+x_p^2-x_{p+1}^2-...-x_{p+q}^2)_+^\lambda\} = \frac{\Gamma(1+\lambda)\Gamma(\lambda+\frac{n}{2})2^{2\lambda+n}\pi^{\frac{n}{2}-1}}{2i}\{e^{-(\lambda+\frac{q}{2})\pi i} \times (s_1^2+...+s_p^2-s_{p+1}^2-...-s_{p+q}^2-i0)^{-\lambda-\frac{n}{2}})$$

$$- e^{-(\lambda+\frac{q}{2})\pi i}(s_1^2 + ... + s_p^2 - s_{p+1}^2 - ... - s_{p+q}^2 + i0)^{-\lambda-\frac{n}{2}}\} \quad (48)$$

and

$$\mathcal{F}\{(x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2)_-^\lambda\} = \frac{\Gamma(1+\lambda)\Gamma(\lambda+\frac{n}{2})2^{2\lambda+n}\pi^{\frac{n}{2}-1}}{2i}\{e^{-\frac{q\pi i}{2}}$$
$$\times (s_1^2 + ... + s_p^2 - s_{p+1}^2 - ... - s_{p+q}^2 - i0)^{-\lambda-\frac{n}{2}})$$
$$- e^{-\frac{q\pi i}{2}}(s_1^2 + ... + s_p^2 - s_{p+1}^2 - ... - s_{p+q}^2 + i0)^{-\lambda-\frac{n}{2}}\}. \quad (49)$$

The formulae (48) and (49) appear in ([2]), page 284, formulae (4) and (4') respectively.

3. The Distributional Family $F_\alpha^\pm(M \pm i0, m, n)$

Let $F_\alpha^\pm(M \pm i0, m, n)$ be the distributions family defined by

$$F_\alpha^\pm(M \pm i0, m, n) = F_{\alpha,m}^\pm = A_{\alpha,m}^\pm(M \pm i0)^{\frac{\alpha-n}{2m}}, \quad (50)$$

where

$$A_{\alpha,m}^\pm = \frac{e^{\frac{\alpha\pi i}{2}}\Gamma(-m(\frac{\alpha-n}{2m}))\sqrt{|\Delta|}e^{\pm\frac{q\pi i}{2}}}{2^{2m\frac{\alpha-n}{2m}+n}\pi^{\frac{n}{2}}\Gamma(m(\frac{\alpha-n}{2m})+\frac{n}{2})}, \quad (51)$$

$(M_m + i0)^\lambda$ is defined by (19), $(M_m - i0)^\lambda$ by (20), $M_m(x_1...x_n)$ by (29) and Δ is the determinant of coefficient of $M_m(x_1...x_n)$.

Using the formulae (35) and (37) we obtain the Fourier transform of $E_{\alpha,m}^\pm$,

$$\mathcal{F}\{F_{\alpha,m}^\pm\} = e^{\frac{\alpha\pi i}{2}}(N(s_1, ..., s_n) \mp i0)^{-\frac{\alpha}{2}}, \quad (52)$$

where $N(s_1, ..., s_n)$ is defined by (36).

By letting $m = 1, a_1 = a_2 = ... = a_p = 1; a_{p+1} = a_{p+2} = ... = a_{p+q} = -1$ in (50) and (51) we have

$$F_{\alpha,1}^\pm = \frac{e^{\frac{\alpha\pi i}{2}}\Gamma(\frac{n-\alpha}{2})e^{\pm\frac{q\pi i}{2}}}{2^\alpha\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}(x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2 \pm i0)^{\frac{\alpha-n}{2}}. \quad (53)$$

The family defined by (53) was introduced S.E. Trione in [4].

We observe that for the case

$$P = P(x_1, ..., x_n) = \sum_{i,j} g_{i,j}x_i\alpha_j, \quad (54)$$

where $g_{i,j} \in \mathbb{R}$ (real numbers) the distributions family $H_\alpha(P \pm i0, n)$ defined by

$$H_\alpha(P \pm i0, n) = \frac{e^{\frac{\alpha\pi i}{2}}e^{\pm\frac{q\pi i}{2}}\sqrt{|\Delta|}\Gamma(\frac{n-\alpha}{2})}{2^\alpha\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}(P \pm i0)^{\frac{\alpha-n}{2}} \quad (55)$$

was introduced by Cristian Oliver in [3].

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