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ON THE DISTRIBUTION $\left(m^2 - P \pm i0\right)^\lambda$

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Abstract: In this paper the well-known one dimensional distribution $\left(1-t^2\right)_+^\lambda$ is generalized for the distribution $\left(m^2 - P \pm i0\right)^\lambda$, where P is a non degenerate quadratic form and m is a real positive number.

Also we introduce the generalized functions $\left(m^2 - P\right)_+^\lambda$ and $\left(P - m^2\right)_+^\lambda$, and by using a method by Gelfand and Shilov, the Fourier transform of this new generalized function is obtained.

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1. Preliminaries

The unidimensional generalized functions $(1-t^2)_+^\lambda$, $(1+t^2)^\lambda$ and $(t^2-1)_+^\lambda$, for λ a complex number has been introduced by Gelfand and Shilov (cf. [1], p. 183). They also has evaluated their Fourier transform given in terms of Bessel functions. In particular for $(1-t^2)_+^\lambda$ we have for $\lambda \neq -1, -2, ...$

$$\mathcal{F}[(1-t^2)_+^\lambda] = \sqrt{\pi}\,\Gamma(\lambda+1)\left(\frac{\xi}{2}\right)^{-\lambda-\frac{1}{2}} J_{\lambda+\frac{1}{2}}(\xi) \tag{1.1}$$

cf. [1], p. 363, formula (39).

In this paper we introduce a generalized function in n variables that may be considerer as a generalization of $(1-t^2)_+^\lambda$. To do this let $t = (t_1, t_2, \dots, t_n)$

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be a point of $\mathbb{R}^n$ and let $P = P(t)$ be a non degenerate quadratic form in the form

$$P = P(t) = t_1^2 + ... + t_p^2 - t_{p+1}^2 - ... - t_{p+q}^2, \tag{1.2}$$

where $p + q = n$.

Gelfand and Shilov (cf. [2], p. 289) has defined the generalized function $\left(m^2 + P \pm i0\right)^\lambda$ formally analogue to the one considered in this paper by the following limit:

$$\left(m^2 + P \pm i0\right)^\lambda = \lim_{\varepsilon \to 0} \left(m^2 + P \pm i\varepsilon \left|t\right|^2\right)^\lambda, \tag{1.3}$$

where $\varepsilon > 0$, λ is a complex number and $\left|t\right|^2 = t_1^2 + ... + t_n^2$.

Frequently one may use an equivalent expression of $(m^2 + P \pm i0)^\lambda$ in terms of the $(m^2 + P)_+$ and $(m^2 + P)_-$ distributions that are defined

$$\left(m^2 + P\right)_+^\lambda = \begin{cases} \left(m^2 + P\right)^\lambda & \text{if } m^2 + P > 0, \\ 0 & \text{if } m^2 + P \leq 0, \end{cases} \tag{1.4}$$

and

$$\left(m^2 + P\right)_-^\lambda = \begin{cases} 0 & \text{if } m^2 + P \geq 0, \\ |m^2 + P|^\lambda & \text{if } m^2 + P < 0. \end{cases} \tag{1.5}$$

Then, we have that

$$(m^2 + P \pm i0)^\lambda = (m^2 + P)_+^\lambda \ + \ e^{\pm i\pi\lambda} \ (m^2 + P)_-^\lambda. \tag{1.6}$$

The Fourier transform of $(m^2 + P \pm i0)^\lambda$ is expressed as a function or generalized function involving the modified Bessel function of third kind and is given by the formula

$$\mathcal{F}[(m^2 + P \pm i0)^\lambda] = \frac{2^{\lambda+1}(2\pi)^{\frac{n}{2}} m^{\frac{n}{2}+\lambda}}{\Gamma(-\lambda)} \ \frac{K_{\frac{n}{2}+\lambda}\left(m(Q \mp i0)^{1/2}\right)}{(Q \mp i0)^{\frac{1}{2}\left(\frac{n}{2}+\lambda\right)}}, \tag{1.7}$$

where

$$K_v(z) = \frac{\pi}{2 \ sen v\pi}[I_{-v}(z) - I_v(z)]$$

and $\Gamma(z)$ is the Euler Gamma function, cf. [2], p. 365.

We now introduce the distribution $\left(m^2 - P \pm i0\right)^\lambda$ by the following

Definition 1. Let λ be a complex number and $P = P(t)$ given by (1.2), $\varepsilon > 0$ and $\left|t\right|^2 = t_1^2 + ... + t_n^2$. By the definition we have

$$\left(m^2 - P \pm i0\right)^\lambda = \lim_{\varepsilon \to 0} \left(m^2 - P \pm i\varepsilon \left|t\right|^2\right)^\lambda. \tag{1.8}$$

To see the existence of this limit, consider the hypersurface $m^2 - P = 0$, that is equal to $m^2 + Q = 0$ where $Q = -P$, and P has no singular points.

We also consider the $\left(m^2 - P\right)_+^\lambda$ distribution where P is a positive definite quadratic form like (1.2) and m is a real positive number. This distribution is defined for λ a complex number

$$\left(m^2 - P\right)_+^\lambda = \begin{cases} \left(m^2 - P\right)^\lambda & \text{for } m^2 \geq P, \\ 0 & \text{for } m^2 < P. \end{cases} \tag{1.9}$$

Also it is possible to define the $\left(P - m^2\right)_+^\lambda$ distribution

$$\left(P - m^2\right)_+^\lambda = \begin{cases} \left(P - m^2\right)^\lambda & \text{for } P \geq m^2, \\ 0 & \text{for } P < m^2. \end{cases} \tag{1.10}$$

From (1.9) and (1.10) we have the following identity

$$\left(m^2 - P \pm i0\right)^\lambda = \left(m^2 - P\right)_+^\lambda + e^{\pm i\pi\lambda}\left(P - m^2\right)_+^\lambda, \tag{1.11}$$

that is enterely analogue to (1.6).

2. The Fourier Transform of $(m^2 - P \pm io)^\lambda$

We begin this section remembering the definition of the Fourier transform of a function φ belonging to $\mathfrak{D}$. We have

$$\mathcal{F}[\varphi](\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{ix\xi} dx,$$

where

$$x\xi = \sum_{i=1}^n x_i \xi_i.$$

Following Gelfand and Shilov procedure we start considering the Fourier transform of the generalized function $(m^2 - P)_+^\lambda$ for $Re\lambda < -\frac{n}{2}$.

Then we have

$$\mathcal{F}\left[\left(m^2 - P\right)_+^\lambda\right] = \int \left(m^2 - P\right)_+^\lambda e^{i\langle t,\xi\rangle} dt, \tag{2.1}$$

where $\langle t, \xi\rangle = t_1\xi_1 + t_2\xi_2 + \ldots + t_n\xi_n$.

To evaluate this transform first we consider the case in which $P = t_1^2 + ... + t_n^2$. In this case we get that the generalized function depends only on $|\xi|$ and the integral (2.1) is reduced at

$$\int \left(m^2 - |t|^2\right)_+^\lambda e^{it_1|\xi|} dt = I. \tag{2.2}$$

In order to obtain the Fourier transform indicated in (2.2) we introduce polar coordinates. Then

$$I = \frac{2\left(\sqrt{\pi}\right)^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^m \int_0^\pi \left(m^2 - r^2\right)_+^\lambda e^{ir|\xi|\cos\varphi_1} \sin^{n-2}\varphi_1 \cdot r^{n-1} d\varphi_1 \, dr, \quad (2.3)$$

where we used that $\frac{2\ (\sqrt{\pi})^{n-1}}{\Gamma(\frac{n-1}{2})} = \Omega_{n-1}$, the surface of the unit sphere in $\mathbb{R}^{n-1}$. The interior integral in (2.3) is

$$\int_0^\pi e^{ir|\xi|\cos\varphi_1} \sin^{n-2}\varphi_1 \cdot d\varphi_1 = \frac{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}}{\left(\frac{r|\xi|}{2}\right)^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}\left(r\left|\xi\right|\right). \quad (2.4)$$

Replacing (2.4) in (2.3) we have

$$\int_0^m \left(m^2 - r^2\right)_+^\lambda r^{\frac{n}{2}} J_{\frac{n}{2}-1}\left(r\left|\xi\right|\right) dr = II \quad (2.5)$$

and making the change $\frac{r}{m} = x$, we obtain

$$\begin{aligned} II &= \int_0^1 \left(1 - x^2\right)^\lambda (mx)^{\frac{n}{2}} J_{\frac{n}{2}-1}\left(mx\left|\xi\right|\right) m \; dx \\ &= m^{2\lambda+\frac{n}{2}-1} \int_0^1 \left(1 - x^2\right)^\lambda x^{\frac{n}{2}} J_{\frac{n}{2}-1}\left(mx\left|\xi\right|\right) dx. \end{aligned}$$

Applying formulae (6.567.1) from [2] and considering $\nu + 1 = \frac{n}{2}$, $\mu = \lambda$, it results

$$II = m^{2\lambda+\frac{n}{2}+1} 2^\lambda \Gamma\left(\lambda+1\right) \left(m\left|\xi\right|\right)^{-(\lambda+1)} J_{\frac{n}{2}+\lambda}\left(m\left|\xi\right|\right). \quad (2.6)$$

Putting (2.5) in (2.2) we get

$$I = 2^{\frac{n}{2}+\lambda} \left(\sqrt{\pi}\right)^n m^{2\lambda+\frac{n}{2}} \Gamma\left(\lambda+1\right) \left|\xi\right|^{-\left(\lambda+\frac{n}{2}\right)} J_{\frac{n}{2}+\lambda}\left(m\left|\xi\right|\right). \quad (2.7)$$

Then, for $Re\lambda < -\frac{n}{2}$ and $\lambda \neq -1, -2, ...$, we have

$$\mathcal{F}[(m^2 - |t|^2)_+^\lambda] = 2^{\frac{n}{2}+\lambda}\ (\sqrt{\pi})^n\ m^{2\lambda+n}\ \Gamma(\lambda+1)\ |\xi|^{-(\lambda+\frac{n}{2})}\ J_{\frac{n}{2}+\lambda}(m|\xi|), \quad (2.8)$$

where $J_\upsilon(z)$ is the Bessel function of first kind

$$J_\upsilon(z) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\ \Gamma(\upsilon+m+1)}\ \left(\frac{z}{2}\right)^{\upsilon+2m}.$$

Now, let P be a positive definite quadratic form, and let Q be its dual. Then the Fourier transform of the distribution $\left(m^2 - P\right)_+^\lambda$ is

$$\begin{aligned} \mathcal{F}\left[\left(m^2 - P\right)_+^\lambda\right] =& 2^{\lambda+\frac{n}{2}} \left(\sqrt{\pi}\right)^n\ m^{2\lambda+\frac{n}{2}}\ \Gamma(\lambda+1) \\ & \left(Q^{\frac{1}{2}}\right)^{-\left(\lambda+\frac{n}{2}\right)} J_{\frac{n}{2}+\lambda}\left(m\left(Q^{\frac{1}{2}}\right)\right), \end{aligned} \quad (2.9)$$

$\lambda \neq -1, -2, \ldots$. And if now P is any quadratic form, for $\lambda \neq -1, -2, \ldots$, we have

$$\mathcal{F}\left[\left(m^2 - P \pm i0\right)^\lambda\right] = 2^{\lambda+\frac{n}{2}} \left(\sqrt{\pi}\right)^n \; m^{2\lambda+\frac{n}{2}} \; \Gamma(\lambda+1) \left((Q \mp i0)^{\frac{1}{2}}\right)^{-\left(\lambda+\frac{n}{2}\right)} J_{\frac{n}{2}+\lambda}\left(m\left((Q \mp i0)^{\frac{1}{2}}\right)\right). \quad (2.10)$$

After simple calculations we get the equivalent expression

$$J_{\frac{n}{2}+\lambda}\left(m\left((Q \mp i0)^{\frac{1}{2}}\right)\right) = \frac{1}{2^{\lambda+\frac{n}{2}} \left(\sqrt{\pi}\right)^n \; \Gamma(\lambda+1)} m^{-\left(2\lambda+\frac{n}{2}\right)} \left((Q \mp i0)^{\frac{1}{2}}\right)^{\lambda+\frac{n}{2}} \mathcal{F}\left[\left(m^2 - P \pm i0\right)^\lambda\right]. \quad (2.11)$$

3. The Fourier Transform of Power of $\left(P - m^2\right)_+^k$

If we consider the function $\left(P - m^2\right)_+^k$, for k a non negative integer defined by (1.9), the distribution $\left(m^2 - P\right)_+^k$ may be defined as

$$\left(m^2 - P\right)_+^k = \begin{cases} \left(m^2 - P\right)^k & \text{for } m^2 > P, \\ 0 & \text{for } m^2 < P, \end{cases} \quad (3.1)$$

and then we have

$$\left(m^2 - P\right)^k = \left(m^2 - P\right)_+^k + (-1)^k \left(P - m^2\right)_+^k. \quad (3.2)$$

Taking into account that the Fourier transform of $\left(m^2 - P\right)_+^k$ is given by (2.9) and that the Fourier transform of the distribution $\left(m^2 - P\right)^k$ may be considered as the one of a polynomial, then the following result is valid

$$\mathcal{F}\left[\left(m^2 - P\right)^k\right] = \left(m^2 + \square\right)^k \delta, \quad (3.3)$$

where $\square$ is the ultrahyperbolic differential operator

$$\square = \sum_{i=1}^{p} \frac{\partial^2}{\partial t_i^2} - \sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial t_i^2} \quad (3.4)$$

with $p+q = n$, the dimension of the space; and where δ is the Dirac distribution.

Then from (3.3) and (2.9) it results

$$\mathcal{F}\left[\left(P - m^2\right)_+^k\right] = (-1)^k \left\{\left(m^2 + \square\right)^k \delta - 2^{k+\frac{n}{2}} (\sqrt{\pi})^n k! m^{2k+\frac{n}{2}} \left((Q)^{\frac{1}{2}}\right)^{-k+\frac{n}{2}n} J_{\frac{n}{2}+k}\left(m\left(Q^{\frac{1}{2}}\right)\right)\right\}. \quad (3.5)$$

Particular Cases. If in (2.8) and (3.4) we consider $m = 1$, and the dimension of the space $n = 1$, taking into account that the quadratic form P reduce to t^2, we obtain

$$\mathcal{F}[(1-t^2)_+^\lambda] = 2^{\lambda+\frac{1}{2}}\sqrt{\pi}\Gamma(\lambda+1)(\xi)^{-(\lambda+\frac{1}{2})}J_{\lambda+\frac{1}{2}}(\xi) \tag{3.6}$$

and

$$\mathcal{F}[(t^2-1)_+^k] = (-1)^k \left\{ \left(1+\frac{d^2}{dt^2}\right)^k \delta - 2^{k+\frac{1}{2}}k!\sqrt{\pi}(\sigma)^{-k-\frac{1}{2}}J_{\frac{1}{2}+k}(\sigma) \right\}, \tag{3.7}$$

that are formulas (39) and (42) from [2], p. 363.

References

[1] I. Gelfand, G. Shilov, *Generalized Funtions*, Volume I, Academic Press (1964).

[2] I. Gradshteyn, I. Ryzhik, *Table of Integrals, Series and Products*, Academic Press (1994).