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A CLOSED FORM FUNDAMENTAL SOLUTION OF THE PARABOLIC EQUATION OF THE BI-HARMONIC TYPE

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Abstract: In this short notes, we present a closed form fundamental solution of the parabolic equation $u_t + u_{xxxx} = 0$; $x \in R$, $t > 0$. In general that of the heat equation is known but only the asymptotic forms of the bi-harmonic types are usually given.

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1. Introduction

The objective here is to construct a closed form fundamental solution of the parabolic equation of bi-harmonic type

$$u_t + u_{xxxx} = 0; \qquad x \in R,\ t > 0, \tag{1}$$

where $u_t = \frac{\partial u}{\partial t}$ and $u_{xxxx} = \frac{\partial^4 u}{\partial x^4}$ are respectively the partial derivative of u with respect to the time variable t and the fourth order partial derivative of u with respect to the spatial variable x. The closed form fundamental solution of the heat equation $u_t - u_{xx} = 0$; $x \in R$, $t > 0$ is well known. However, to the best of my knowledge that of equation (1) is yet to be given. An asymptotic expansion solution using the saddle-point method was given by Pólya [5] in the form

$$u(x,t) = \pi^{-1} \left(\frac{x}{t}\right)^{\frac{1}{3}} \exp\left(-3\,(2)^{\frac{11}{3}}\,\rho\right) \sum_{0}^{\infty} \alpha_k \left(\frac{2}{27}\right)^{(1+4k)/6} \rho^{-k-\frac{1}{2}} \times \cos\left(\beta\rho - \pi\,(1+4k)\,/6\right),$$

where

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$$\beta = 3^{\frac{3}{2}}(2)^{\frac{11}{3}},$$

$$\rho = \left(\frac{\mathrm{x}^4}{\mathrm{t}}\right)^{\frac{1}{3}},$$

$$\alpha_{\mathrm{k}} = \frac{2}{(2\mathrm{k})!}\int_0^\infty s^{4\mathrm{k}} H_{\mathrm{k}}\left(\frac{2\mathrm{s}}{\sqrt{6}}\right)\exp(-s^2)ds,$$

and H_k is the k-th Hermite polynomial.

Also under the requirement that the solution of (1) be infinitely differentiable and compactly supported on R, Fourier transform technique provides the integral solution of the form

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mathrm{x}\omega-\omega^4\mathrm{t}}d\omega. \tag{2}$$

It is shown in [3, 4] that (2) is infinitely differentiable and for an integer $m \geq 0$,

$$K(x,t) = D_{\mathrm{x}}^{\mathrm{m}}u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\omega)^{\mathrm{m}}\, e^{-i\mathrm{x}\omega-\omega^4\mathrm{t}}d\omega.$$

Letting $\xi = \mathrm{t}^{\frac{1}{4}}\omega$, it can be shown that

$$K(x,t) = \frac{\mathrm{t}^{-\frac{\mathrm{m}+1}{4}}}{2\pi}\int_{-\infty}^{\infty}(-i\xi)^{\mathrm{m}}\, e^{-i\mathrm{t}^{-\frac{1}{4}}\mathrm{x}\xi-\xi^4}d\xi,$$

and for $\alpha > 0$ the relation $\mathrm{K}(\alpha^{\frac{1}{4}}\mathrm{x},\alpha\mathrm{t})=\alpha^{-\frac{\mathrm{m}+1}{4}}\mathrm{K}(\mathrm{x},\mathrm{t})$, and hence

$$D_{\mathrm{x}}^{\mathrm{m}}u(x,t) = t^{-\frac{\mathrm{m}+1}{4}}K(xt^{-\frac{1}{4}},1), \qquad t > 0,$$

holds. Also in [2], a solution to the boundary value problem using the method of separation of variables is suggested. However, that of an infinite dimensional domain is not given. We shall give, under the method employed here, a closed form series representation of (2) from which the infinite differentiability of the fundamental solution is guaranteed by the convergence of the series for all values of x and for $\mathrm{t} > 0$.

2. Method

The function $u(x,t)=\varphi\left(\frac{x^4}{t}\right)$, depending only on $\frac{x^4}{t}$, is a solution of equation (1) in the half space $t>0 = \{x \in R,\ t > o\}$, where $\varphi \in C^4(R)$ and satisfies the ordinary differential equation

$$256z^3\varphi^{\mathrm{iv}}(z) + 1152z^2\varphi'''(z) + 816z\varphi''(z) + (24+z)\varphi'(z) = 0, \tag{3}$$

where

$$z = \frac{x^4}{t}, \qquad t > 0,$$

and $\varphi^n(z) = \frac{d^n\varphi}{d\,z^n}$. Hence $z \geq 0$ for all x. If we further let

$$F(z) = \varphi'(z), \tag{4}$$

then equation (3) is reduced to the following third order ordinary differential equation of Euler coefficients [1]:

$$256z^3F'''(z) + 1152z^2F''(z) + 816zF'(z) + (24+z)F(z) = 0. \tag{5}$$

We shall solve (5) by the series solution method outlined in [1] and many other ordinary differential equations text books. On that note, by letting $F(z) = \sum_{n=0}^{\infty} a_n z^{n+r}$ and substituting into (5) we have

$$a_0[256r(r-1)(r-2) + 1152r(r-1) + 816r + 24]z^r + \sum_{n=1}^{\infty} B(n,r)z^{n+r} = 0,$$

where

$B(n,r) = a_n\,[256(n+r)(n+r-1)(n+r-2)+1152(n+r)(n+r-1)+816(n+r)+24]\,a_{n-1}$,

from which we have the following indicia equation and recurrence relation:

$$32r^3 + 48r^2 + 22r + 3 = 0 \tag{6}$$

and

$$a_n = \frac{-a_{n-1}}{256\,(n+r)\,(n+r-1)\,(n+r-2)\,+1152\,(n+r)\,(n+r-1)+816\,(n+r)\,+24},$$

$$n \geq 1, \qquad a_0 \neq 0. \tag{7}$$

The roots of the indicia equation are $r = -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}$.

For $r = -\frac{3}{4}$,

$$a_n = \frac{(-1)^n\, a_0}{4^n\ \alpha_n},$$

where

$$\alpha_n = \left[(4n-3)(16n^2+2)+6\right]\left[(4n-7)(16(n-1)^2+2)+6\right]B(n),$$

$$B(n) = \Big[(4\text{n-}11)(16(\text{n-}2)^2)+2)+6\Big] ...(1320)(336)(24).$$

Hence the first solution of (5) is $\mathrm{F_1(z) = a_0 z^{-\frac{3}{4}}} \left[1+ \sum\limits_{n=1}^{\infty} \frac{(-1)^n z^n}{4^n\ \alpha_n}\right]$.

Since $\lim\limits_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}}$=0, the series converges for all z ≥ 0 and hence is integrable term by term. Thus, on using (4), we have that

$$\varphi_1(z) = 4a_0 z^{\frac{1}{4}} \left[1+\sum_{n=1}^{\infty} \frac{(-1)^n\, z^n}{4^n\,(4n+1)\,\alpha_n}\right] + C_1$$

and on replacing z we have

$$\varphi_1(\frac{x^4}{t}) = 4a_0 x t^{-\frac{1}{4}} \left[1+\sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{x^4}{4t}\right)^n}{(4n+1)\,\alpha_n}\right] + C_1. \tag{8}$$

The series on the right hand side of (8) converges uniformly for all x and t >0. Corresponding to the root r = -$\frac{1}{2}$, we proceed similarly to get

$$F_2(z) = a_0 z^{-\frac{1}{2}} \left[1+\sum_{n=1}^{\infty} \frac{(-1)^n\, z^n}{8^n \beta_n}\right],$$

where

$$\beta_n = \Big[(2\text{n-}1)(16\text{n}^2+8\text{n}+3)+3\Big] \Big[(2\text{n-}3)(16(\text{n-}1)^2+8\text{n-}5)+3\Big] ...(870)(252)(30).$$

Again , we note that since $\lim\limits_{n\to\infty} \frac{\beta_n}{\beta_{n+1}} = 0$, the series converges for all z $\geq$ 0. Hence term by term integration gives after replacing

$$z\varphi_2(\frac{x^4}{t}) = 2a_0 \left(xt^{-\frac{1}{4}}\right)^2 \left[1+\sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{x^4}{4t}\right)^n}{2^n(2n+1)\beta_n}\right] + C_2, \tag{9}$$

which again by the ratio test converges for all x and t > 0.

Finally for the root r = -$\frac{1}{4}$, we have $\mathrm{F_3(z)=a_0 z^{-\frac{1}{4}}} \left[1+ \sum\limits_{n=1}^{\infty} \frac{(-1)^n z^n}{4^n \gamma_n}\right]$, where γ_n = $\Big[2(4\text{n-}1)(8\text{n}^2+8\text{n}+3)+3\Big] \Big[2(4\text{n-}5)(8(\text{n-}1)^2+8\text{n-}5)+3\Big]$...(2181)(717)(117). Thus, on using (4) again and replacing z, we have that

$$\varphi_3(\frac{x^4}{t}) = \frac{4a_0}{3} \left(xt^{-\frac{1}{4}}\right)^3 \left[1+3\sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{x^4}{4t}\right)^n}{(4n+3)\,\gamma_n}\right] + C_3, \tag{10}$$

which also converges for all x and t > 0. On combining (8), (9), and (10), we have as the closed form series fundamental solution of (1) in the form u(x,t) =

$\varphi_1 + \varphi_2 + \varphi_3$. Substituting in the φ_i's, we have u(x,t) $= \frac{4a_0}{t^{\frac{1}{4}}} C_n +$ C;

$$C_n = x\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{x^4}{4t}\right)^n}{(4n+1)\,\alpha_n}\right] + \frac{x^2}{2t^{\frac{1}{4}}}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{x^4}{4t}\right)^n}{2^n(2n+1)\beta_n}\right] + \frac{x^3}{3t^{\frac{1}{2}}}\left[1+3\sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{x^4}{4t}\right)^n}{(4n+3)\,\gamma_n}\right].$$

If we now let

$$\Omega\left(xt^{-\frac{1}{4}}\right) = \left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{x^4}{4t}\right)^n}{(4n+1)\,\alpha_n}\right] + \frac{x}{2t^{\frac{1}{4}}}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{x^4}{4t}\right)^n}{2^n(2n+1)\beta_n}\right] + \frac{x^2}{3t^{\frac{1}{2}}}\left[1+3\sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{x^4}{4t}\right)^n}{(4n+3)\,\gamma_n}\right],$$

we can rewrite u(x,t) as u(x,t)$=4a_0 x t^{-\frac{1}{4}}\Omega\left(xt^{-\frac{1}{4}}\right)+$C. If we choose C $= 0$ and $a_0 = \pi^{-\frac{1}{4}}$ (since C and a_0 are arbitrary constants), we have u(x,t)$=\frac{4x}{\sqrt[4]{\pi t}}\Omega\left(xt^{-\frac{1}{4}}\right)$ as the closed form fundamental solution of (1).

Theorem. *Let $\phi = \phi(\frac{x^4}{t}) \in C^4(R)$ be a function depending only on $\frac{x^4}{t}$ and satisfying the 4-th order ordinary differential equation*

$$256z^3\phi^{iv}(z) + (1152z^2 - 1024z^3)\phi'''(z) + (776z - 3456z^2 + 1536z^3)\phi''(z)$$
$$+(14-1551z+3456z^2-1024z^3)\phi'(z)+(-14+775z-1152z^2+256z^3)\phi(z) = 0,$$

where $z = \frac{x^4}{t}$. Then u(x,t) $= \phi\left(\frac{x^4}{t}\right)e^{-\frac{x^4}{t}}$ is a fundamental solution of $u_t+u_{xxxx}=0$; x$\in$ R, $t > 0$.

Proof. Substitute u(x,t) $= \phi\left(\frac{x^4}{t}\right)e^{-\frac{x^4}{t}}$ into equation (1) and obtain the given differential equation. □

References

[1] W.E. Boyce, R.C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, Seventh Edition, John Wiley and Sons, New York-London-Sydney-Toronto (1969).

[2] R. Haberman, *Elementary Applied Partial Differential Equations with*

Fourier Series and Boundary Value Problems, Prentice-Hall, Englewood Cliff, New Jersey (1983).

[3] F.B. Jones, Jr., A class of singular integrals, *Amr. Jour. Math.*, **86**, No. 2 (1964), 441-462.

[4] T.A. Kwembe, Existence and uniqueness of global solutions for the parabolic equation of the bi-harmonic type, *J. Nonlinear Analysis*, **47** (2001), 1321-1332.

[5] P.C. Rosenbloom, Linear partial differential equations, *Numerical Analysis and Partial Differential Equations*, **5** (1958), 45-195.