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EVALUATION OF BETA-FUNCTION B-SPLINES, I: LOCAL MONOMIAL BASES

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Abstract: This is the first one of a sequence of three papers addressing the evaluation of *Beta function B-splines* (BFBS) earlier introduced and studied in [2, 4, 3]. BFBS are a special case of *generalized expo-rational B-splines* (GERBS) [2, 4], and are a practically important instance of smooth GERBS which are not infinitely smooth true *expo-rational B-splines* (ERBS) [1, 5]. Compared to ERBS, BFBS exhibit similar properties, cf. [5, 3], but with more limited range, due to the polynomial nature of BFBS compared to the exponential nature of ERBS. On the other hand, the integrals in the definition of BFBS can be solved exactly in elementary functions, and the resulting representation is computationally efficient, while with ERBS the respective integrals are special functions which can be computed efficiently, yet approximately. This makes BFBS more applicable than ERBS, e.g., in topics related to refinement, subdivision, multiresolution and other multilevel techniques.

The present sequence of three papers is dedicated to the derivation of explicit representations of BFBS yielding computationally efficient explicit formulae for evaluation of BFBS in terms of polynomial bases used in data interpolation, data fitting and geometric modelling, as well as in the design of multilevel constructions such as, e.g., multiwavelets. This is the first paper of the sequence, and here we derive a representation of BFBS in terms of local monomial bases. This is an essentially interpolatory representation; in the second article of the sequence, a Bezier type representation of BFBS will be

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considered, which is suitable for geometric modelling and data fitting; the third and last paper of the sequence will be dedicated to a representation in global monomial bases, suitable for use, e.g., in relevance to certain operational calculi.

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1. Introduction

Expo-rational B-splines (ERBS) were introduced in [1] and their properties were studied in considerable detail in [5]. Applications of ERBS in geometric modelling with parametric curves and tensor-product surfaces were considered in [1, 5, 9, 6, 7, 8].

Definition 1. (see [5]) Let $t_k \in \mathbb{R}$ and $t_k < t_{k+1}$ for $k = 0, 1, 2, ..., n+1$. Consider the strictly increasing knot-vector $\{t_k\}_{k=0}^{n+1}$. The ERBS $B_k(t)$, $k = 0, \ldots, n+1$, associated with this knot-vector are defined, as follows.

$$B_k(t) = \begin{cases} \int\limits_{t_{k-1}}^{t} \varphi_{k-1}(s)ds, & t_{k-1} < t \leq t_k \\ 1 - \int\limits_{t_{k-1}}^{t} \varphi_k(s)ds, & t_k < t < t_{k+1} \\ 0, & \text{otherwise} \end{cases}, \tag{1}$$

with

$$\varphi_k(t) = \frac{e^{-\beta_k \frac{[t-((1-\lambda_k)t_k+\lambda_k t_{k+1})]^{2\sigma_k}}{((t-t_k)(t_{k+1}-t)^{\gamma_k})^{\alpha_k}}}}{\int_{t_k}^{t_{k+1}} e^{-\beta_k \frac{[s-((1-\lambda_k)t_k+\lambda_k t_{k+1})]^{2\sigma_k}}{((s-t_k)(t_{k+1}-s)^{\gamma_k})^{\alpha_k}}} ds}, \tag{2}$$

where

$$\alpha_k > 0, \quad \beta_k > 0, \quad \gamma_k > 0, \quad 0 \leq \lambda_k \leq 1, \quad \sigma_k \geq 0,$$

are the intrinsic parameters, defaulting to: $\alpha_k = \beta_k = \gamma_k = \sigma_k = 1, \lambda_k = \frac{1}{2}$.

Generalized expo-rational B-splines (GERBS) [2], [4] are a generalization of expo-rational B-splines (ERBS) [1], [5] which includes the polynomial simplified modifications of ERBS, termed Euler Beta-function B-splines (BFBS) in [2], [4], and which also includes other basis/blending functions with relevant properties such as minimal support (as a 1st-degree piecewise affine B-spline), value 1 at the central knot, value zero at all other knots, and (at least one) derivative equal to zero at all knots.

Definition 2. (see [3, Definition 3]) GERBS is any piecewise monotone reparametrization of a piecewise affine B-spline which preserves the intervals of monotonicity of the latter, as well as the range of the latter in each of these intervals.

In the case of ERBS, one important topic is the reliability of the evaluation of the B-spline because of the possibility for overflow, underflow and division by zero in computing the exponent of a rational function near its poles. In the case of BFBS, reliability of the evaluation is not such an issue because BFBS are polynomial between adjacent knots. This was the reason why the author of the present work proposed the introduction of BFBS as early as in 2005, which chronologically preceded the introduction of the more general GERBS class in [2] where BFBS was formally introduced as a particular case of GERBS, essentially complementary to ERBS. A more detailed justification of the definition of BFBS and exposition of the basic properties of BFBS was given in [3].

Definition 3. (see [3]) A Beta-function B-spline (BFBS), associated with three strictly increasing knots t_{k-1}, t_k and t_{k+1}, $B_k(t) = B_k(i_{k-1}, i_k, i_{k+1}; t)$ is defined by

$$B_k(t) = \begin{cases} S_{k-1} \int\limits_{t_{k-1}}^{t} \psi_{k-1}(s)ds, & \text{if} \quad t \in (t_{k-1}, t_k), \\ S_k \int\limits_{t}^{t_{k+1}} \psi_k(s)ds, & \text{if} \quad t \in (t_k, t_{k+1}), \\ 1, & \text{if} \quad t = t_k, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

with

$$S_k = \left[\int\limits_{t_k}^{t_{k+1}} \psi_k(t)dt \right]^{-1}, \tag{4}$$

and

$$\psi_k(t) = C_k \frac{(t - t_k)^{i_k} (t_{k+1} - t)^{i_{k+1}}}{(t_{k+1} - t_k)^{i_k + i_{k+1}}}, \quad t \in [t_k, t_{k+1}], \tag{5}$$

where

$$C_k = \binom{i_k + i_{k+1}}{i_k}, \tag{6}$$

and

$$i_l > 0, \quad l = k-1, k, k+1. \tag{7}$$

It is possible to approach the computation of ERBS and BFBS by using numerical quadratures. The Romberg quadrature process which was proposed in [1], [5] for approximate computation of ERBS is highly efficient, and very rapidly convergent, in view of the infinite smoothness of the integrands involved. If applied to the numerical evaluation of BFBS, adaptively on every interval between the knots (with local choice of the order of the Romberg quadrature which matches the degree of the polynomial representing the BFBS on this interval), the numerical quadrature is *exact* (up to round-off errors). Thus, we have a uniform numerical procedure for approximate computation which works at least for all absolutely continuous GERBS [4], including, in particular, both ERBS and BFBS.

Apart of the numerical computation, however, it is of considerable interest, both theoretical and computational, to find explicit representations of BFBS between the adjacent knots in terms of polynomial bases that are typically used in interpolation (e.g., Lagrange, Hermite, Abel-Goncharov, Birkhoff interpolation), geometric modelling and in computing images of BFBS in operator calculus (the importance of the latter being, e.g., in the study of prospective BFBS-based constructions of multiwavelets). This will be the purpose of a sequence of three papers by this author, of which this is the first paper.

Here we shall provide some preliminary computations (Section 2) which will be used in the remaining part of this paper and in both of the other two subsequent papers of the sequence. Then, in Section 3, we shall develop a representation of BFBS in terms of *local monomial* polynomial bases, i.e., the polynomial bases appearing in the Taylor polynomial expansion around the central knot of the BFBS. This representation is the main result of the present paper. The last Section 4 contains some orientation about the other two papers in the sequence and some additional concluding remarks.

2. Preliminaries

In this section, the next section, and in the two following papers of this sequence we shall evaluate a BFBS $B_k(t) = B_k(i_{k-1}, i_k, i_{k+1}; t)$ (Definition 3) for the intervals (t_k, t_{k+1}) and (t_{k-1}, t_k), considering the case of a non-uniform knot vector $\{t_k\}_{k=0}^{n+1}$.

First, here we shall provide explicit expressions for the multiplicative factors appearing in the formulae for $B_k(t)$ in the two intervals (t_k, t_{k+1}) and (t_{k-1}, t_k), which will be used for all the calculations in the remaining part of the paper and in the following two papers of the sequence.

Lemma 1. *Formula (3) is equivalent to*

$$B_k(t) = \begin{cases} S_{k-1} d_{k-1} \int\limits_{t_{k-1}}^{t} \varphi_{k-1}(\tau) d\tau, , & \textit{if} \quad t \in (t_{k-1}, t_k), \\ S_k d_k \int\limits_{t}^{t_{k+1}} \varphi_k(\tau) d\tau, , & \textit{if} \quad t \in (t_k, t_{k+1}), \\ 1, & \textit{if} \quad t = t_k, \\ 0, & \textit{otherwise}, \end{cases} \tag{8}$$

where

$$d_{k-1} = \frac{(i_{k-1} + i_k)!}{i_{k-1}! i_k!} \frac{1}{(t_k - t_{k-1})^{i_{k-1}+i_k}}, \tag{9}$$

$$S_{k-1} = \frac{i_{k-1} + i_k + 1}{t_k - t_{k-1}}, \tag{10}$$

$$d_k = \frac{(i_k + i_{k+1})!}{i_k! i_{k+1}!} \frac{1}{(t_{k+1} - t_k)^{i_k+i_{k+1}}}, \tag{11}$$

$$S_k = \frac{i_k + i_{k+1} + 1}{t_{k+1} - t_k}, \tag{12}$$

$k = 0, \ldots, n.$

Proof. From the definition of BFBS (Definition 3) for $B_k(t)$ in (t_k, t_{k+1}) it follows

$$B_k(t) = S_k \int\limits_{t}^{t_{k+1}} \psi_k(\tau) d\tau, \tag{13}$$

where

$$\psi_k(t) = C_k \frac{(t-t_k)^{i_k}(t_{k+1}-t)^{i_{k+1}}}{(t_{k+1}-t_k)^{i_k+i_{k+1}}}, \tag{14}$$

$$S_k = \left[\int_{t_k}^{t_{k+1}} \psi_k(t)dt \right]^{-1}, \tag{15}$$

and

$$C_k = \binom{i_k + i_{k+1}}{i_k}. \tag{16}$$

$B_k(t)$ can be rewritten, as follows.

$$\begin{aligned} B_k(t) &= S_k \int_t^{t_{k+1}} \psi_k(\tau)d\tau \\ &= S_k \int_t^{t_{k+1}} C_k \frac{(\tau-t_k)^{i_k}(t_{k+1}-\tau)^{i_{k+1}}}{(t_{k+1}-t_k)^{i_k+i_{k+1}}} d\tau \\ &= S_k \int_t^{t_{k+1}} d_k(\tau-t_k)^{i_k}(t_{k+1}-\tau)^{i_{k+1}} d\tau \\ &= S_k d_k \int_t^{t_{k+1}} (\tau-t_k)^{i_k}(t_{k+1}-\tau)^{i_{k+1}} d\tau \\ &= S_k d_k \int_t^{t_{k+1}} \varphi_k(\tau)d\tau. \end{aligned}$$

Hence,

$$B_k(t) = S_k d_k \int_t^{t_{k+1}} \varphi_k(\tau)d\tau, \tag{17}$$

where

$$d_k = C_k \frac{1}{(t_{k+1}-t_k)^{i_k+i_{k+1}}} = \frac{(i_k+i_{k+1})!}{i_k! i_{k+1}!} \frac{1}{(t_{k+1}-t_k)^{i_k+i_{k+1}}}, \tag{18}$$

and

$$\varphi_k(\tau) = (\tau-t_k)^{i_k}(t_{k+1}-\tau)^{i_{k+1}}. \tag{19}$$

The calculation of S_k is in terms of the Euler Beta function:

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^1 u^{m-1}(1-u)^{n-1}du,$$

where Γ is the Gamma function

$$\Gamma(m) = (m-1)!.$$

Therefore,

$$\begin{aligned}
[S_k]^{-1} &= \int_{t_k}^{t_{k+1}} \psi_k(t)dt \\
&= \int_{t_k}^{t_{k+1}} C_k \frac{(t-t_k)^{i_k}(t_{k+1}-t)^{i_{k+1}}}{(t_{k+1}-t_k)^{i_k+i_{k+1}}}dt
\end{aligned}$$

Let us set

$$u = \frac{(t-t_k)}{(t_{k+1}-t_k)}, \quad t = t_k + (t_{k+1}-t_k)u,$$

mapping $[t_k, t_{k+1}]$ onto $[0,1]$. With this, we obtain

$$\begin{aligned}
[S_k]^{-1} &= C_k \int_{t_k}^{t_{k+1}} \frac{(t-t_k)^{i_k}(t_{k+1}-t)^{i_{k+1}}}{(t_{k+1}-t_k)^{i_k+i_{k+1}}}dt \\
&= C_k(t_{k+1}-t_k)\int_0^1 u^{i_k}(1-u)^{i_{k+1}}du \\
&= C_k(t_{k+1}-t_k)\frac{\Gamma(i_k+1)\Gamma(i_{k+1}+1)}{\Gamma(i_k+i_{k+1}+2)} \\
&= C_k(t_{k+1}-t_k)\frac{i_k!i_{k+1}!}{(i_k+i_{k+1}+1)!} \\
&= \binom{i_k+i_{k+1}}{i_k}(t_{k+1}-t_k)\frac{i_k!i_{k+1}!}{(i_k+i_{k+1}+1)!} \\
&= (t_{k+1}-t_k)\frac{(i_k+i_{k+1})!}{i_k!i_{k+1}!}\frac{i_k!i_{k+1}!}{(i_k+i_{k+1}+1)!} \\
&= \frac{t_{k+1}-t_k}{i_k+i_{k+1}+1}
\end{aligned}$$

So, for S_k we have:

$$S_k = \frac{i_k + i_{k+1} + 1}{t_{k+1} - t_k}. \tag{20}$$

In the same way, with corresponding modifications, we derive for $B_k(t)$ in (t_{k-1}, t_k):

$$B_k(t) = S_{k-1} \int_{t_{k-1}}^{t} \psi_{k-1}(\tau) d\tau, \tag{21}$$

where

$$\psi_{k-1}(t) = C_{k-1} \frac{(t - t_{k-1})^{i_{k-1}} (t_k - t)^{i_k}}{(t_k - t_{k-1})^{i_{k-1}+i_k}}, \tag{22}$$

$$S_{k-1} = \left[\int_{t_{k-1}}^{t} \psi_{k-1}(t) dt \right]^{-1}, \tag{23}$$

and

$$C_{k-1} = \binom{i_{k-1} + i_k}{i_{k-1}}. \tag{24}$$

Or,

$$B_k(t) = S_{k-1} d_{k-1} \int_{t_{k-1}}^{t} \varphi_{k-1}(\tau) d\tau, \tag{25}$$

where

$$\varphi_{k-1}(\tau) = (\tau - t_{k-1})^{i_{k-1}} (t_k - \tau)^{i_k}, \tag{26}$$

$$d_{k-1} = C_{k-1} \frac{1}{(t_k - t_{k-1})^{i_{k-1}+i_k}} = \frac{(i_{k-1} + i_k)!}{i_{k-1}! i_k!} \frac{1}{(t_k - t_{k-1})^{i_{k-1}+i_k}}, \tag{27}$$

and

$$S_{k-1} = \frac{i_{k-1} + i_k + 1}{t_k - t_{k-1}}. \tag{28}$$

□

3. BFBS Evaluation in Local Monomial Bases

Here we obtain the main result: a representation of $B_k(t)$ on the intervals of its support in the local monomial bases around its central knot:

$$1,\ t-t_k,\ (t-t_k)^2,\ \ldots,\ (t-t_k)^{i_{k-1}+i_k}, \quad t \in (t_{k-1}, t_k), \tag{29}$$

and

$$1,\ t-t_k,\ (t-t_k)^2,\ \ldots,\ (t-t_k)^{i_k+i_{k+1}}, \quad t \in (t_k, t_{k+1}). \tag{30}$$

Remark 1. Clearly, the local monomial bases in (29, 30) coincide, modulo normalization, with the bases in the Taylor interpolation polynomials at the the central knot t_k of degree $i_{k-1}+i_k$ and i_k+i_{k+1}, respectively.

Theorem 1. *Under the conditions of Definition 3, let $k = 1, \ldots, n$.*

(i) If $t \in (t_{k-1}, t_k)$, then,

$$B_k(t) = S_{k-1}d_{k-1} \sum_{l=i_k}^{i_{k-1}+i_k} \frac{1}{l+1}\left[(t_k - t_{k-1})^{l+1} - (t-t_k)^{l+1}\right] \times \left[\binom{i_{k-1}}{l-i_k}(t_k - t_{k-1})^{i_{k-1}+i_k-l}(-1)^{i_k}\right], \tag{31}$$

where

$$S_{k-1}d_{k-1} = \binom{i_{k-1}+i_k}{i_{k-1}} \frac{i_{k-1}+i_k+1}{(t_k-t_{k-1})^{i_{k-1}+i_k+1}}. \tag{32}$$

(ii) If $t \in (t_{k-1}, t_k)$, then,

$$B_k(t) = S_k d_k \sum_{l=i_k}^{i_k+i_{k+1}} \frac{1}{l+1}\left[(t_{k+1} - t_k)^{l+1} - (t-t_k)^{l+1}\right] \times \left[\binom{i_{k+1}}{l-i_k}(t_{k+1} - t_k)^{i_k+i_{k+1}-l}(-1)^{l-i_k}\right], \tag{33}$$

where

$$S_k d_k = \binom{i_k+i_{k+1}}{i_k} \frac{i_k+i_{k+1}+1}{(t_{k+1}-t_k)^{i_k+i_{k+1}+1}}. \tag{34}$$

Proof. Case (ii): $t \in (t_k, t_{k+1})$.
By Lemma 1,

$$B_k(t) = S_k d_k \int_t^{t_{k+1}} \varphi_k(\tau)d\tau$$

Evaluation of $\varphi_k(\tau)$ yields:

$$\begin{aligned}\varphi_k(\tau) &= (\tau - t_k)^{i_k}(t_{k+1} - \tau)^{i_{k+1}} \\ &= (\tau - t_k)^{i_k}[(t_{k+1} - t_k) - (\tau - t_k)]^{i_{k+1}} \\ &= (\tau - t_k)^{i_k} \sum_{j=0}^{i_{k+1}} \binom{i_{k+1}}{j} (t_{k+1} - t_k)^{i_{k+1}-j}(-1)^j(\tau - t_k)^j \\ &= \sum_{j=0}^{i_{k+1}} \binom{i_{k+1}}{j} (t_{k+1} - t_k)^{i_{k+1}-j}(\tau - t_k)^{i_k+j}(-1)^j\end{aligned}$$

After the following change of index j

$$\begin{aligned}&i_k + j = l, \qquad l = i_k + 0, \dots, i_k + j, \quad j = 0, \dots, i_{k+1}, \\ &j = l - i_k,\end{aligned}$$

it follows that

$$\varphi_k(\tau) = \sum_{l=i_k}^{i_k+i_{k+1}} (\tau - t_k)^l \left[\binom{i_{k+1}}{l - i_k} (t_{k+1} - t_k)^{i_k+i_{k+1}-l}(-1)^{l-i_k} \right].$$

Now integrate, to get

$$\begin{aligned}B_k(t) &= S_k d_k \int_t^{t_{k+1}} \varphi_k(\tau) d\tau \\ &= S_k d_k \int_t^{t_{k+1}} \sum_{l=i_k}^{i_k+i_{k+1}} (\tau - t_k)^l \left[\binom{i_{k+1}}{l - i_k} (t_{k+1} - t_k)^{i_k+i_{k+1}-l}(-1)^{l-i_k} \right] d\tau \\ &= S_k d_k \sum_{l=i_k}^{i_k+i_{k+1}} \frac{1}{l+1} \left[(t_{k+1} - t_k)^{l+1} - (t - t_k)^{l+1} \right] \\ &\quad \times \left[\binom{i_{k+1}}{l - i_k} (t_{k+1} - t_k)^{i_k+i_{k+1}-l}(-1)^{l-i_k} \right]\end{aligned}$$

So, for B_k in (t_k, t_{k+1})

$$B_k(t) = S_k d_k \sum_{l=i_k}^{i_k+i_{k+1}} \frac{1}{l+1} \left[(t_{k+1} - t_k)^{l+1} - (t - t_k)^{l+1} \right]$$

$$\times \left[\binom{i_{k+1}}{l-i_k} (t_{k+1}-t_k)^{i_k+i_{k+1}-l} (-1)^{l-i_k} \right] \quad (35)$$

holds true, where

$$S_k d_k = \binom{i_k + i_{k+1}}{i_k} \frac{i_k + i_{k+1} + 1}{(t_{k+1}-t_k)^{i_k+i_{k+1}+1}}.$$

Case (i): $t \in (t_{k-1}, t_k)$.

By Lemma 1,

$$B_k(t) = S_{k-1} d_{k-1} \int_{t_{k-1}}^{t} \varphi_{k-1}(\tau) d\tau$$

Evaluation of $\varphi_{k-1}(\tau)$ gives:

$$\begin{aligned}
\varphi_{k-1}(\tau) &= (\tau - t_{k-1})^{i_{k-1}} (t_k - \tau)^{i_k} \\
&= (-1)^{i_k} (\tau - t_k)^{i_k} (\tau - t_{k-1})^{i_{k-1}} \\
&= (-1)^{i_k} (\tau - t_k)^{i_k} [(t_k - t_{k-1}) + (\tau - t_k)]^{i_{k-1}} \\
&= (-1)^{i_k} (\tau - t_k)^{i_k} \sum_{j=0}^{i_{k-1}} \binom{i_{k-1}}{j} (t_k - t_{k-1})^{i_{k-1}-j} (\tau - t_k)^j \\
&= (-1)^{i_k} \sum_{j=0}^{i_{k-1}} \binom{i_{k-1}}{j} (t_k - t_{k-1})^{i_{k-1}-j} (\tau - t_k)^{i_k+j} \\
&= \sum_{j=0}^{i_{k-1}} \binom{i_{k-1}}{j} (t_k - t_{k-1})^{i_{k-1}-j} (\tau - t_k)^{i_k+j} (-1)^{i_k}
\end{aligned}$$

We now make a change of index j again:

$$\begin{aligned}
&i_k + j = l, \qquad l = i_k + 0, \ldots, i_k + j, \quad j = 0, \ldots, i_{k-1}, \\
&j = l - i_k,
\end{aligned}$$

hence,

$$\varphi_k(\tau) = \sum_{l=i_k}^{i_{k-1}+i_k} (\tau - t_k)^l \left[\binom{i_{k-1}}{l-i_k} (t_k - t_{k-1})^{i_{k-1}+i_k-l} (-1)^{i_k} \right]$$

Now integration yields

$$B_k(t) = S_{k-1} d_{k-1} \int_{t_{k-1}}^{t} \varphi_{k-1}(\tau) d\tau$$

$$= S_{k-1}d_{k-1} \int\limits_{t_{k-1}}^{t} \sum_{l=i_k}^{i_{k-1}+i_k} (\tau - t_k)^l \left[\binom{i_{k-1}}{l - i_k} (t_k - t_{k-1})^{i_{k-1}+i_k-l} (-1)^{i_k} \right] d\tau$$

$$= S_{k-1}d_{k-1} \sum_{l=i_k}^{i_{k-1}+i_k} \frac{1}{l+1} \left[(t_k - t_{k-1})^{l+1} - (t - t_k)^{l+1} \right]$$

$$\times \left[\binom{i_{k-1}}{l - i_k} (t_k - t_{k-1})^{i_{k-1}+i_k-l} (-1)^{i_k} \right].$$

Therefore, for B_k in (t_{k-1}, t_k)

$$B_k(t) = S_{k-1}d_{k-1} \sum_{l=i_k}^{i_{k-1}+i_k} \frac{1}{l+1} \left[(t_k - t_{k-1})^{l+1} - (t - t_k)^{l+1} \right]$$

$$\times \left[\binom{i_{k-1}}{l - i_k} (t_k - t_{k-1})^{i_{k-1}+i_k-l} (-1)^{i_k} \right] \quad (36)$$

is fulfilled, where

$$S_{k-1}d_{k-1} = \binom{i_{k-1} + i_k}{i_{k-1}} \frac{i_{k-1} + i_k + 1}{(t_k - t_{k-1})^{i_{k-1}+i_k+1}}.$$

□

4. Concluding Remarks

Since now BFBS are computed in terms a particular polynomial basis between any couple of neighbouring knots, it is already possible to obtain a local representation of BFBS between neighbouring knots in terms of any other polynomial basis spanning the polynomials which have up to the same degree, by using the transformation matrix for change between the local monomial basis, considered here, and the new polynomial basis. However, the nature of BFBS suggests that BFBS admits particularly insightful representation in terms of local Bernstein polynomial bases, because the integrands in the definition of BFBS are local Bernstein polynomials themselves. (A local Bernstein polynomial is a standard Bernstein polynomial on $[0, 1]$ shifted and rescaled for the respective interval between the knots. Compare this to the local monomial bases where the basis functions are monomial shifted (possibly, without rescaling) to the respective knot.) This representation is closely related to the geometric-modelling properties of the linear combinations of BFBS and will be studied in one of the two other papers in the present sequence. On the other hand, for the purposes of operational calculus (which may be useful in the future as a toolbox in studying BFBS-based multiwavelet and other multilevel constructions), it may be of interest to represent BFBS in terms of the same polynomial basis uniformly in all intervals between neighbouring knots, i.e., globally. One natural choice of such a basis is the *monomial basis* $1,\ t,\ t^2,\ \dots\ t^m$, for an appropriate choice of the degree $m \in \mathbb{N}$. This topic will be addressed in the other one of the two remaining papers in this sequence.

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