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INTERESTING PROBABILITY CALCULATIONS USING CARDS TO WIN THE RED OR BLACK GAME

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Abstract: This article introduces an interesting probability game that uses playing cards. The game places 52 cards, 26 red and 26 black, face down. The object is to predict the color of a card before turning it over. It would seem that a player should average 26 correct answers per game, but this article describes a strategy that results in an average of 30 correct answers. The course of the game is formulated as a probability problem, and a mathematical proof is given using advanced mathematics such as convolutions of generating functions and the use of Taylor expansions.

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1. 86% Odds of Winning a 50-50 Game

There is a well-known Japanese card game called “Flip the Monk” that is played something like the following. One hundred cards are placed faced down, and players alternate turning them over one at a time. The cards have one of three types of illustrations on them: a prince, a princess, or a Buddhist monk. A player who turns over a card with a prince on it keeps that card. A player that turns over a card with a monk on it discards all held cards into a discard pile.

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Revealing a card depicting a princess allows players to add the entire discard pile to their hand. The game is fun to play, despite its reliance on sheer luck to win, probably because of the dynamic way in which the cards change hands when a monk or a princess card is revealed.

Next, consider the following card game, “Red or Black”. All 52 cards in a standard deck are shuffled and placed face down on a table, and players take turns predicting the color (red or black) of a card before turning it over. Since there are 26 red and 26 black cards, one would predict an average of 26 correct guesses per game, assuming an even distribution of possibilities between the extreme cases where every prediction is wrong and where every prediction is correct.

Since there is an equal probability of revealing a red or a black card, it would seem that this game could not be as interesting as Flip the Monk. Changing the way in which you make predictions, however, can raise your average number of correct guesses from 26 to 30. The interesting details of this topic in probability were published about 20 years ago in two papers by Kenneth Levasseur and Don Zagier (see [2], [3]). Levasseur gave a mathematical proof of a particular method for using random guesses to average 30.007 correct answers. Extending Levasseur's work, Zagier showed how to average 86% correct answers versus an opponent who makes random guesses. Let us take a look at their methods to see what is going on.

As shown in Figure 1, starting with n red cards and n black cards we can consider the Red or Black game as a descending path drawn from the point (n, n) to the origin. A given coordinate (r, b) on the path indicates the existence of r remaining red cards and b black cards. If the next revealed card is red then the path moves to $(r-1, b)$, and if black then to $(r, b-1)$. The path that the game progresses along is called the *game path*.

2. Visiting the Diagonal is Worth 0.5 Points

When such a path is created using a binary result set such as the red/black color of a card or the heads/tails result of a coin flip, it is often called a random walk, or sometimes a Bernoulli trial after the first mathematician to consider it (James Bernoulli, 1713). A Bernoulli trial normally uses an infinite sequence of head/tail coin flips, but we can consider a trial using 26 red and black cards as a limited, conditional Bernoulli trial.

Let us next consider the game path from (n, n) to the origin. In combi-

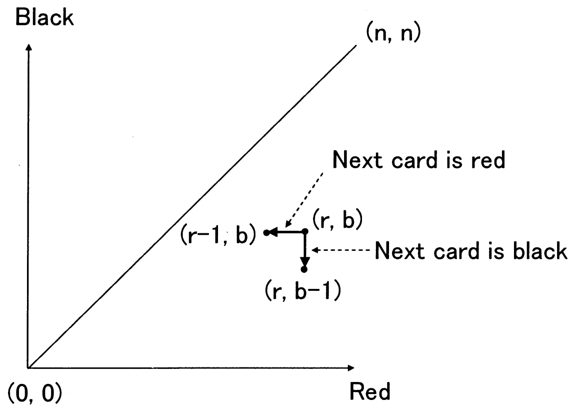

Figure 1: The game path

natorial terms there are a total of ${}_{2n}C_n$ possible paths, but we will show that for each of these paths visiting the diagonal significantly affects the current prediction. Namely, each visit to the diagonal is worth 0.5 points.

Starting from point (n, n), assume that the first card is black. This result is in the lower right triangular area in Figure 1, and knowledge of this first card contributes 0.5 points to our final score. Let us next suppose that the game path visits the next point on the diagonal, at the point $(n-k, n-k)$. For a zigzagging path such as this, immediately after an initial show of black we should predict that the next card is red, simply because for this path there will be more red cards than black ones on the path back to the next diagonal point $(n-k, n-k)$.

$$\text{Number of red cards} > \text{Number of black cards}$$

We will draw $2k$ cards along the path from (n, n) to $(n-k, n-k)$, but since one of those cards was already black, then of the remaining $2k-1$ cards, k of those cards are red and $k-1$ of them are black. If, therefore, we guess "red" for each of those $2k-1$ cards then we will be correct k times and incorrect $k-1$ times. The probability that our guess for the first card was correct was 0.5, so our total predicted score is $k+0.5$ out of a possible $2k$ points.

If, on the other hand, we were simply to use random numbers for our predictions then our chances of being correct would be $1/2$, meaning k out of $2k$ correct answers along the path between points (n, n) and $(n-k, n-k)$. The difference between k and $k+0.5$ is 0.5, so each time we visit the diagonal we

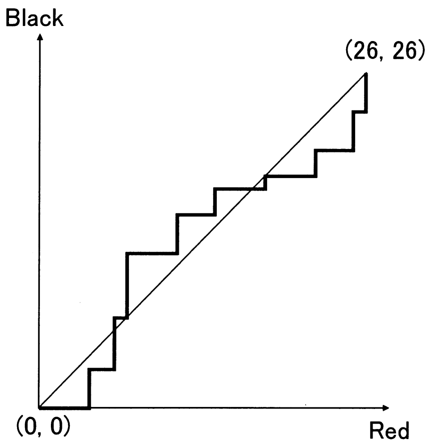

Figure 2: Each visit to the diagonal adds 0.5 points

are rewarded with a 0.5 point bonus to our predicted score.

Taking the game path in Figure 2 as an example, the diagonal was visited five times (including arrival at the origin) along the path between (26, 26) and (0, 0). Since each visit is worth 0.5 points, adding that to the average score of 26 we get a predicted score of $26 + 0.5 \times 5 = 28.5$.

Let $v(p)$ be the number of times that a given game path p visits the diagonal. The initial point (n, n) lies on the diagonal, but we do not count the initial state so the actual number of visits will be $v(p) - 1$. Since each visit adds 0.5 points to the score, in total we will receive a $0.5(v(p) - 1)$ point bonus. The average score for random guesses would be n, so our predicted score would therefore be $n = 0.5(v(p) - 1)$. This discussion applies only to a particular path p, but we can take the summation of all paths P and divide that by the total number of paths ${}_{2n}C_n$, thereby calculating our predicted score $S(n)$ as follows:

$$S(n) = \sum_{p \in P} (n + 0.5(v(p) - 1))/C(2n, n)$$

$$= n + 0.5((\sum_{p \in P} v(p)/C(2n, n)) - 1). \quad (1)$$

Next, we will examine how $S(n)$ is calculated.

3. Total Visits to the Diagonal, $V(n)$

First, let us consider how we might find the total number of visits to the diagonal $V(n)$ for a game path $p \in P$.

Define the variable $\chi(p,m)$ as 1 when the path p meets the diagonal at point (m,m), and 0 otherwise. There are n points on the diagonal between $(0,0)$ and (n,n), and because they must be calculated for each game path $p \in P$ we obtain the equation below. Note that we can change the order of summation, and that equation gives the number of paths that meet the diagonal at point (m,m), the summation from 0 to n.

Combining the above we have the following, rewriting the combinatorial ${}_{2m}C_m$ in the form $C(2m,m)$.

$$\begin{aligned} V(n) &= \sum_{p\in P}\left[\sum_{m=0}^{n}\chi(p,m)\right] = \sum_{m=0}^{n}\left[\sum_{p\in P}\chi(p,m)\right] \\ &= \sum_{m=0}^{n} \text{Total number of paths meeting point } (m,m) \\ &= \sum_{m=0}^{n} C(2m,m)C(2(n-m),n-m). \end{aligned}$$

Rewriting the last equation, we obtain equation

$$V(n) = \sum_{m=0}^{n} C(2m,m)C(2(n-m),n-m). \tag{2}$$

Here, note that the terms in the summation are the combinatorial function $C(2m,m)$ and a convolution of $C(2m,m)$.

4. Generating Functions and Convolutions

The previous equation (2) is sufficient to calculate a game path's total number of visits to the diagonal $V(n)$. The value can easily be calculated using a modern PC spreadsheet, such as with the *COMBIN* function in *Microsoft Excel*. Believe it or not, however, Levasseur's paper simplifies equation (2) to the following:

$$V(n) = 4^n. \tag{3}$$

According to this simplification, $V(2) = 16, V(3) = 64, V(4) = 256$, and so on. Readers should verify that the equation is correct.

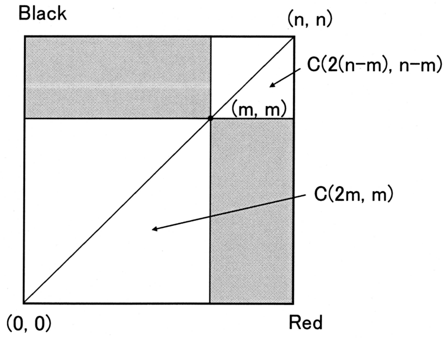

Figure 3: Finding $V(n)$

We cannot jump immediately into introducing equation (3) from equation (2), however. Doing so requires some knowledge of Taylor expansions, generating functions, and convolutions. Let us examine each of these in turn.

The key to the proof is a Taylor expansion of the term $1/\sqrt{1-4z}$. Doing so gives the result

$$\sum_{n=0}^{\infty} C(2n,n)z^n.$$

Taking the function to expand as $f(z)$, we have

$$f(z) = 1/\sqrt{1-4z} = (1-4z)^{-\frac{1}{2}},$$

$$f(0) = 1.$$

Applying the well-known Taylor expansion $f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots$ to $f(z)$ and its derivative functions $f'(z), f''(z), f^{(3)}(z), \cdots$, and substituting $z = 0$, we get

$$f(z) = 1 + 2z + 6z^2 + 20z^3 + 70z^4 + 252z^5 + \cdots.$$

Examining the coefficients of this equation, we see that

$f(z) = C(0,0)z^0 + C(2,1)z^1 + C(4,2)z^2 + C(6,3)z^3 + C(8,4)z^4 + C(10,5)z^5 + \cdots.$

From this, we derive the proof that

$$\frac{1}{\sqrt{1-4z}} = \sum_{n=0}^{\infty} C(2n,n)z^n. \tag{4}$$

The concept of generating functions is famous for their use by de Moivre in finding a general formulation for the Fibonacci sequence (1730).

Given a sequence $A = \{a, ar, ar^2, \cdots, ar^n\}$, its generating function $G(A; z)$ is written as

$$G(A; z) = a + arz + ar^2z^2 + ar^3z^3 + \cdots,$$

an infinite geometric progression with initial term a and geometric ratio rz. The sum of the sequence is found by the following:

$$G(A; z) = \frac{a}{1 - rz}.$$

Given a sequence D, and supposing that its generating function is written as

$$G(D; z) = \frac{1}{\sqrt{1 - 4z}},$$

then the generating function for the sequence $D * D$ is

$$G(D * D; z) = \frac{1}{1 - 4z}.$$

We shall use this relationship in our proof. Here, $D * D$ is what is referred to as a convolution, and since convolutions are necessary to our goal. Let us briefly discuss them.

In problems of probability, it is often necessary to calculate some new sequence $\{c_k\}$ from two other sequences $\{a_k\}, \{b_k\}$. A new sequence $\{c_k\}$ defined as

$$c_k = a_0b_k + a_1b_{k-1} + a_2b_{k-2} + \cdots + a_{k-1}b_1 + a_kb_0$$

is called a convolution of $\{a_k\}$ and $\{b_k\}$ and is written as

$$\{c_k\} = \{a_k\} * \{b_k\}.$$

It is particularly important to note the subscripts in these sequences. The terms of the sequence are not simply combined in order starting from the first term, but rather, as the term "convolution" might imply, "woven" amongst each other.

It is also possible to create a convolution of generating functions, as follows. Let $G(A; z)$ be the generating function of a sequence $A = \{a_k\}$, and $G(B; z)$ the generating function of a sequence $B = \{b_k\}$. Then

$$G(A; z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n,$$

$$G(B; z) = b_0 + b_1z + b_2z^2 + \cdots + b_nz^n,$$

and so,

$$G(A; z) \times G(B; z)$$
$$= (a_0 + a_1z + a_2z^2 + \cdots + a_nz^n) \times (b_0 + b_1z + b_2z^2 + \cdots + b_nz^n)$$

$$= a_0b_0+(a_0b_1+a_1b_0)z+(a_0b_2+a_1b_1+a_2b_0)z^2+\cdots+(a_0b_n+a_1b_{n-1}+\cdots+a_nb_0)z^n$$
$$+ (a_1b_n + a_2b_{n-1} + \cdots + a_nb_1)z^{n+1} + \cdots + a_nb_nz^{2n}.$$

Examining the coefficients of the terms of the generating function, we see that they form a convolution of sequence A and sequence B, and so show that it is possible to form a convolution of generating functions.

From equation (4), taking a sequence $D = \{{}_{2k}C_k\} = \{C(2k, k)\}$ and letting $D * D$ be the convolution of D and D, then we have

$$G(D * D; z) = G(D; z)^2 = \frac{1}{1 - 4z}. \tag{5}$$

Letting $D * D = V$, then

$$V(n) = 4^n.$$

The reasoning behind this is that the generating function $G(V; z)$ for the sequence $V = \{4^0, 4^1, 4^2, 4^3, \cdots, 4^n\}$ can easily be confirmed to be

$$G(V; z) = 4^0 + 4^1z + 4^2z^2 + 4^3z^3 + \cdots = \frac{4^0}{1 - 4z} = \frac{1}{1 - 4z}.$$

From this we can replace equation (1) as

$$S(n) = n + 0.5((4^n/C(2n, n)) - 1).$$

Applying Stirling's formula

$$n! = \sqrt{2\pi n}(n/e)^n + O(\sqrt{2\pi/n}(n/e)^n)$$

to the combinatorial, yields

$$S(n) = n + 0.5(\sqrt{n\pi} - 1) + O(1/\sqrt{n}) \tag{6}$$

(Note: Interested readers should consult [1] for a proof of Stirling's formula).

In this equation, substituting $n = 26$ for the case of using playing cards, we get $S(26) = 30.007$. In other words, it is possible to predict correctly the color of approximately 30 out of 52 cards, four more than the standard value of 26. For $n = 100, S(100) = 108.290$.

Figure 4 shows a composition of the probability distributions for scores using random guesses versus the scores expected using equation (6) for $n = 26$. The minimum score in the distribution is 26, and the median value is around 30.

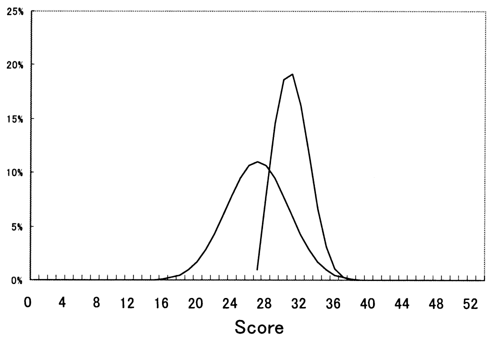

Figure 4: Random versus strategic predictions

5. Implementation

Equation (6) was our goal, but the value of $S(n)$ can easily be calculated using a PC. Doing so, however, requires use of the Catalan number

$$C(n) = \frac{{}_{2n}C_n}{n+1} = {}_{2n}C_n - {}_{2n}C_{n-1}$$

and the repeated combination

$${}_nH_r = {}_{n+r-1}C_r.$$

We should be suitably impressed with the minds of mathematicians who used only deduction and mathematical expansions to derive equation (6) before the widespread use of personal computers. Well done, humans!

It is sufficient to note that each visit to the diagonal gives one a theoretical bonus of 0.5 points, but when implementing this theory with actual cards, simply stick to the following procedure:

(1) Call black for the first card.

(2) If the overturned card was red, there are now more black cards than red cards available, so continue calling black until the number of red and black cards is the same.

(3) If the overturned card was black, there are now more red cards than black cards available, so continue calling red until the number of red and black cards is the same.

Crossing the diagonal will gain you one point. Figure 5 shows a case where the diagonal was crossed twice, resulting in a final score of 28 (26+2=28).

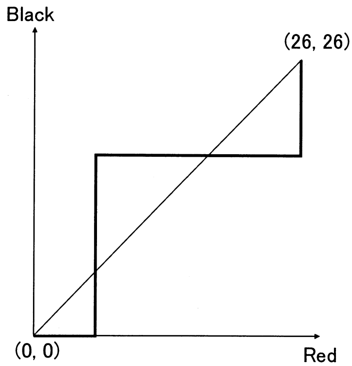

Figure 5: Each cross over the diagonal gains one point.

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