International Journal of Pure and Applied Mathematics
Volume 66 No. 2 2011, 121-136

SOME NEW AXIOMS FOR SET THEORY

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Abstract: It is suggested that new axioms can be added to ZFC by proceeding cautiously. Some examples are given, together with justifications. A discussion of $V = L$ is also given.

AMS Subject Classification: 03E55
Key Words: axioms of set theory, collecting the universe, inaccessible cardinals, Mahlo cardinals

1. Introduction

There has been much interest in the possibility of extending ZFC; e.g., see [1], [8], [9], [13], [15]. Here, it is argued that this can be done cautiously, adding axioms for which there are compelling arguments that they are true. While such extensions are of little interest to mathematics in general, they are of interest to set theory. They represent the adoption of facts which are held to be true, and provide a tool for further research. For example they may be helpful in considering yet more powerful axioms; this will be considered in a subsequent paper [7].

The simplest example of such an axiom is “there exists an inaccessible cardinal” (by “inaccessible” will be meant “strongly inaccessible” throughout). This axiom could have been adopted long ago. Arguments in its favor which will be given here are refinements of arguments already given by set theorists, including (in reverse temporal order) Godel, Hausdorff, and Cantor.

This paper provides “more details of the justification” of assuming the existence of Mahlo cardinals, which was noted would be desirable in [6]. Section 2 discusses the principle of collecting the universe. Section 3 gives axioms for

Received: September 25, 2010

existence of higher inaccessible cardinals. Section 4 gives the definition and some properties of schemes. Section 5 proves from the axioms of Section 3 that the universe has the Mahlo property. Section 6 gives axioms for existence of higher Mahlo cardinals. Section 7 gives a general axiom subsuming those of Sections 3 and 6. Section 8 discusses the greatly Mahlo cardinals. Section 9 discusses an application. Section 10 discusses relativization to L. Section 11 discusses $V = L$.

The following notation will be used.

- Ord denotes the class of ordinals.
- Inac denotes the class of strongly inaccessible cardinals.
- $\triangle$ denotes diagonal intersection.

2. Collecting the Universe

The principle of collecting the universe states that, given a sufficiently well-described universe of sets, it may be collected into a set, thereby adding a level to the cumulative hierarchy. This is an example of what Shoenfield calls a "vague principle" in [14]; his example is, "there is a stage after all the stages in S provided we can imagine a situation in which all of the stages in S have been completed".

As in [14], some particular methods may be singled out as qualifying. Shoenfield gives some which are used to justify the axioms of ZFC. He also mentions adding another, which would justify the axiom asserting the existence of an inaccessible cardinal.

The particular methods to be considered here will make use of the system of axioms called BGC in [12]. BGC should be taken as a true system of axioms. The justification of ZFC given in [14] applies equally well to BGC; and BGC is a conservative extension of ZFC. Class variables are quite useful in discussions of collecting the universe. Further, BGC already blurs the distinction between the universe and a suitable level of the cumulative hierarchy.

An application of the principle of collecting the universe proceeds in two steps. Step 1 is to argue that V has some property; in a variety of cases of interest this property may be stated in Π^1_1 form. Step 2 is to argue that reflection to V_κ for some κ holds. If V were assumed to be weakly compact then this would be automatic. However, one goal of these methods is to argue via the principle of collecting the universe that V is weakly compact, so initially at least a specific justification must be given in Step 2. In many cases of interest, the property states some "closure conditions" on V. One must argue that these conditions are met at some level of the cumulative hierarchy.

In the case of an inaccessible cardinal, for example, that Ord behaves like an inaccessible cardinal follows from the axioms of BGC. Thus, V is a collection which behaves like V_κ where κ is an inaccessible cardinal. But this collection is well-prescribed, and should be collectible into a set. The distinction between sets and proper classes must be blurred at the point in the cumulative hierarchy where the axioms of BGC first become satisfied (the "initial" universe).

Other arguments in favor of making this "leap of faith" include the following.

- ZFC does not give the "whole story" regarding adding stages to the cumulative hierarchy. Axioms should be added which allow blurring the distinction between sets and classes.

- That the blurring of the distinction should be taken as a principle of set theory is further illustrated by extensions of ZFC more powerful than BGC, for example the system ZFV of [6].

- BGC "describes" certain levels of the cumulative hierarchy, namely those closed under the operations, which are "mini-universes", in that the universe is closed.

- The "truncation" of the cumulative hierarchy at the first level where BGC becomes satisfied is clearly absurd, once one supposes that there are such levels. Satisfying second order replacement is no reason to stop adding levels.

- "There exists an inaccessible cardinal" is not that much stronger a statement than "ZFC is consistent".

3. Four Axioms

This section will consider four axioms, and give justifications for adding them to BGC, in terms of the principle of collecting the universe. To begin with, some abbreviations are adopted.

- "$\alpha \in \mathrm{Lim}(X)$", $X \subseteq \mathrm{Ord}$, for "$\forall\beta < \alpha\exists\gamma < \alpha(\beta \leq \gamma \wedge \gamma \in X)$".

- "$Y = \mathrm{Lim}(X)$" for "$\forall\alpha(Y(\alpha) \Leftrightarrow \alpha \in \mathrm{Lim}(X)$". $\mathrm{Lim}(X)$ may be used in formulas as a term, by standard methods.
- "$Y = \mathrm{LimI}(X)$" for "$Y = \mathrm{Lim}(X) \cap \mathrm{Inac}$".
- "X is Y-closed", $X, Y \subseteq \mathrm{Ord}$, for "$\mathrm{Lim}(X) \cap Y \subseteq X$".
- "X is Y-club", $X, Y \subseteq \mathrm{Ord}$, for "$X$ is Y-closed and unbounded".
- "I-closed" for Inac-closed.

Lemma 1. *Suppose $X, X_\xi, Y \subseteq Ord$.*

1. *Y is Y-closed*
2. *If X is Y-closed then $Lim(X) \cap Y$ is Y-closed.*
3. *If $\eta \in Ord$ and $\langle X_\xi : \xi < \eta\rangle$ is a sequence (coded as a class) of Y-closed classes then $\cap_{\xi<\eta} X_\xi$ is Y-closed.*
4. *If $\langle X_\xi : \xi \in Ord\rangle$ is a sequence (coded as a class) of Y-closed classes then $\triangle_{\xi \in Ord} X_\xi$ is Y-closed.*

Proof. Part 1 follows trivially. Suppose $\mathrm{Lim}(X) \cap Y \subseteq X$, $\mathrm{Lim}(X_\xi) \cap Y \subseteq X_\xi$. For part 2, $\mathrm{Lim}(\mathrm{Lim}(X) \cap X) \subseteq \mathrm{Lim}(X)$, and $\mathrm{Lim}(\mathrm{Lim}(X) \cap X) \cap Y \subseteq \mathrm{Lim}(X) \cap Y \subseteq X$. For part 3, $\mathrm{Lim}(\cap_\xi X_\xi) \cap Y \subseteq \mathrm{Lim}(X_\xi) \cap Y \subseteq X_\xi$ for any ξ. For part 4, suppose $\alpha \in \mathrm{Lim}(\triangle_\xi X_\xi) \cap Y$. Let α_η be a sequence in $\triangle_\xi X_{\xi \cap Y}$ converging to α. If $\xi < \alpha$ then some suffix of the sequence converges in X_ξ to α, so $\alpha \in X_\xi$. But this shows that $\alpha \in \triangle_\xi X_\xi$. □

The four axioms are as follows, where $X, X_\xi \subseteq \mathrm{Ord}$.

I1. Inac is I-club.

I2. If X is I-club then $\mathrm{LimI}(X)$ is I-club.

I3. If $\eta \in \mathrm{Ord}$ and $\langle X_\xi : \xi < \eta\rangle$ is a sequence (coded as a class) of I-club classes then $\cap_{\xi<\eta} X_\xi$ is I-club.

I4. If $\langle X_\xi : \xi \in \mathrm{Ord}\rangle$ is a sequence (coded as a class) of I-club classes then $\triangle_{\xi \in \mathrm{Ord}} X_\xi$ is I-club.

To justify Axiom I it suffices to argue that Y is unbounded, where Y equals Inac, $\mathrm{LimI}(X)$, $\cap_{\xi<\eta} X_\xi$, or $\triangle_\xi X_\xi$ when I is 1, 2, 3, 4 respectively. Suppose α is the largest element (this might be $\emptyset$).

For Axiom I1, V is obtained from V_α by applying the powerset operation Ord times. The resulting universe is sufficiently well-described that it can be collected, contradicting the claim that α is the largest inaccessible. In summary, V would not be sufficiently well-behaved with respect to the principle of collecting the universe.

For Axiom I2, we may choose a sequence of Ord elements of X, starting at α, to arrive at a universe which can be collected, resulting in V_β. β is an element of $\mathrm{LimI}(X)$, so α cannot be the largest such.

For Axiom I3, for $\gamma \in \mathrm{Ord}$, at stage $\eta\cdot\gamma+\xi$ choose an element of C_ξ greater than the elements chosen so far (greater than α at stage 0). This arrives at a universe which can be collected, resulting in V_β. $\beta \in \mathrm{Lim}(X_\xi) \cap \mathrm{Inac}$ for each ξ, so $\beta \in X_\xi$ for each ξ, so β is an element of $\cap_\xi C_\xi$. Thus, α cannot be the largest such.

For Axiom I4, for $\gamma \in \mathrm{Ord}$, at stage ξ choose β_ξ greater than the elements chosen so far (greater than α at stage 0), with $\beta_\xi \in \cap_{\zeta<\xi} X_\zeta$. This arrives at a universe which can be collected, resulting in V_β where $\beta = \sup \beta_\xi$ and $\beta \in \mathrm{Inac}$. For any ξ, $\beta_\zeta \in X_\xi$ for $\zeta > \xi$, so $\beta \in \mathrm{Lim}(X_\xi)$, so $\beta \in X_\xi$. Thus, β is an element of $\triangle_\xi X_\xi$. Thus, α cannot be the largest such.

4. Schemes

Schemes were defined in [5], both in the universe and in V_κ for $\kappa \in \mathrm{Inac}$ (they were called systems of operations), as a tool for simplifying methods used by Gaifman in [10]. In a (class) well-order, a sequence x_α is said to approach x if the sequence is increasing and is unbounded below x. By a (class) scheme is meant a class which codes the following:

- A well order W on Ord, with a largest element.
- For each limit point x of W whose cofinality is less than Ord, a sequence indexed by some ordinal which approaches x.
- For each limit point x of W whose cofinality equals Ord, a sequence indexed by Ord which approaches x.

This definition can readily be transformed into a definition of a scheme in V_κ for $\kappa \in \mathrm{Inac}$.

Suppose F is a definable function on classes of ordinals. Given a scheme Σ and a class $X \subseteq \mathrm{Ord}$, the predicate $Y = F^\Sigma(X)$ may be defined, with the following properties.

- For each point x of W there is a class X_x.
- $X_0 = X$.
- If x is the successor of y in W then $X_x = F(X_y)$.
- If x is approached by y_ξ for $\xi < \eta$ where $\eta \in \mathrm{Ord}$ then $X_x = \cap_{\xi<\eta} X_{y_\xi}$.
- If x is approached by y_ξ for $\xi < \mathrm{Ord}$ then $X_x = \triangle_\xi X_{y_\xi}$.
- $Y = X_x$ where x is the largest element.

F^Σ is function on classes of ordinals.

5. Ord is Mahlo

Theorem 2. *Suppose Axioms I1-I4 hold. For any scheme Σ, $LimI^\Sigma(Inac)$ is I-club.*

Proof. By induction in the well-order of Σ, X_x is an I-club for each x. □

This property of Ord can be shown to be equivalent to a property which is commonly called the Mahlo property. A class X is said to be club if $\mathrm{Lim}(X) \subseteq X$ and X is unbounded. A class X is said to be stationary if it has nonempty intersection with every club class. The Mahlo property for Ord is the statement that Inac is stationary. For the remainder of this section, facts will be stated for classes; corresponding facts hold in V_κ for $\kappa \in \mathrm{Inac}$. Some facts will be stated in greater generality than required for Theorem 6.

Lemma 3. *Suppose $X, Y \subseteq Ord$, Y is stationary, and X is Y-club. Then X is stationary.*

Proof. Since X is unbounded it is easy to see that $\mathrm{Lim}(X)$ is club. Given a club class C, $\mathrm{Lim}(X) \cap C$ is club, so $\mathrm{Lim}(X) \cap Y \cap C$ is nonempty, so $X \cap C$ is nonempty. □

Lemma 4. *If $Y \subseteq Inac$ is stationary then for any scheme Σ, $LimI^\Sigma(Y)$ is Y-club.*

Proof. The proof of the lemma is by induction on the point x of the well-order. The basis is trivial. Suppose X is Y-club. Then $\mathrm{Lim}(X) \cap Y \subseteq X$, and it follows that $\mathrm{Lim}(\mathrm{Lim}(X) \cap \mathrm{Inac}) \cap Y \subseteq \mathrm{lim}(X) \cap \mathrm{Inac}$, i.e., that $\mathrm{LimI}(X)$ is Y-closed. Since Y is stationary $\mathrm{Lim}(X) \cap Y$ is unbounded, whence $\mathrm{LimI}(X) = \mathrm{Lim}(X) \cap \mathrm{Inac}$ is unbounded. Suppose X_ξ is Y-closed for $\xi < \eta$ where $\eta \in \mathrm{Ord}$. That $\cap_{\xi<\eta} X_\xi$ is Y-closed follows by Lemma 1. Further, $Z = \cap_{\xi<\eta} \mathrm{Lim}(X_\xi)$ is club, so $Y \cap Z$ is unbounded; and $Y \cap Z \subseteq \cap_{\xi<\eta} X_\xi$. Suppose X_ξ is Y-closed for $\xi \in \mathrm{Ord}$. That $\triangle_\xi X_\xi$ is Y-closed follows by Lemma 1. Further, $Z = \triangle_\xi \mathrm{Lim}(X_\xi)$ is club, so $Y \cap Z$ is unbounded; and $Y \cap Z \subseteq \triangle_\xi X_\xi$. □

Lemma 5. *If $Y \subseteq \mathrm{Ord}$ is not stationary then for some scheme Σ, $\mathrm{LimI}^\Sigma(Y) = \emptyset$.*

Proof. Let $Z \subseteq \mathrm{Ord}$ be a club class disjoint from Y. Enumerate Z in natural order as $\langle \alpha_\gamma : \gamma \in \mathrm{Ord} \rangle$. Let Σ be the scheme with points $\mathrm{Ord} \cup \{\infty\}$, with the sequence approaching a limit ordinal α being the identity function, and the sequence approaching ∞ being $\langle \alpha_\gamma \rangle$. By induction $Y_\alpha \cap \alpha = \emptyset$ for $\alpha \in \mathrm{Ord}$. It follows that $\alpha \notin Y_\infty$ for limit ordinals α, whence $\mathrm{LimI}(Y_\kappa) = \emptyset$. □

Theorem 6. *For $Y \subseteq \mathrm{Ord}$ the following are equivalent:*

a. *Y is stationary.*

b. *For any scheme Σ, $\mathrm{LimI}^\Sigma(Y)$ is Y-club.*

c. *For any scheme Σ, $\mathrm{LimI}^\Sigma(Y)$ is stationary.*

d. *For any scheme Σ, $\mathrm{LimI}^\Sigma(Y) \neq \emptyset$.*

Proof. b follows from a by Lemma 4. c then follows by Lemma 3. d follows from c immediately. a follows from d by Lemma 5. □

Suppose Axioms I1-I4 hold; then by Theorems 2 and 6, Inac is stationary, that is, Ord has the Mahlo property.

6. Some Higher Mahlo Cardinals

An inaccessible cardinal κ is said to be (strongly) Mahlo iff $\mathrm{Inac} \cap \kappa$ is a stationary subset of κ. By results of the previous section it is easy to see that an inaccessible cardinal κ is Mahlo iff Axioms I1-I4 hold in V_κ.

From what has been seen so far, the principle of collecting the universe justifies extending the cumulative hierarchy to the next level where Axioms I1-I4 are satisfied. Here we hold that this universe is sufficiently well-prescribed that it may be collected. That is, the principle of collecting the universe may be strengthened, to hold that the universe may be collected at stages where Axioms I1-I4 become satisfied.

Let "Mahl" denote the class of Mahlo cardinals, and write "M-club" for "Mahl-club" and "LimM(X)" for "$\mathrm{Lim}(X) \cap \mathrm{Mahl}$". Four further axioms may be considered, as follows, where $X, X_\xi \subseteq \mathrm{Ord}$.

M1. Mahl is M-club.

M2. If X is M-club then LimM(X) is M-club.

M3. If $\eta \in$ Ord and $\langle X_\xi : \xi < \eta\rangle$ is a sequence (coded as a class) of M-club classes then $\cap_{\xi<\eta} X_\xi$ is M-club.

M4. If $\langle X_\xi : \xi \in \mathrm{Ord}\rangle$ is a sequence (coded as a class) of M-club classes then $\triangle_\xi X_\xi$ is M-club.

These may be justified as Axioms I1-I4, by appealing to the strengthened principle of collecting the universe. By Lemma 1 the classes are all M-closed, so it suffices to argue that they are unbounded. For Axiom M1, the "next universe" containing α satisfies Axioms I1-I4, and can be collected. For Axiom M2, let β be as in the argument for Axiom I2; then $\beta \in \mathrm{LimM}(X)$. The modifications for Axioms M3 and M4 are similar.

7. A General Axiom

An axiom can be given which iterates the transition from the I-axioms to the M-axioms, through schemes.

Merely for the sake of notation, let $\mathcal{F}_1$ be the functions from Pow(Inac) to Pow(Inac). Let $\mathcal{F}_1^L$ be those $F \in \mathcal{F}_1$ such that for any $\kappa \in$ Inac, $F(X) \cap \kappa = F(Y) \cap \kappa$ whenever $X \cap \kappa = Y \cap \kappa$. Such an F defines a function from Pow(Inac $\cap\, \kappa$) to Pow(Inac $\cap\, \kappa$) for any $\kappa \in$ Inac, and $F(X \cap \kappa) = F(X) \cap \kappa$, where the notation is abused by letting F denote both the function in Ord, and the function in κ.

For $F \in \mathcal{F}_1^L$ and a subset $X \subseteq$ Inac, let $F^*(X)$ be the set of $\kappa \in \mathrm{Inac} \cap X$ such that $F^\Sigma(X \cap \kappa)$ is stationary for all schemes Σ in κ. A version of this operation may be found in [10].

Suppose Σ is a scheme. Define classes M_x for points x of the well-order of Σ by recursion on x, as follows.

- $M_0 =$ Inac.
- If x is the successor of y let F be the operation $\mathrm{Lim}(X) \cap M_y$. Let $M_x = F^*(M_y)$.
- If x is approached by y_ξ for $\xi < \eta$ where $\eta \in$ Ord then $M_x = \cap_{\xi<\eta} M_{y_\xi}$.
- If x is approached by y_ξ for $\xi <$ Ord then $M_x = \triangle_\xi M_{y_\xi}$.

M_Σ denotes M_x where x is the largest element.

Write "$\mathrm{LimM}_\Sigma(X)$" for "$\mathrm{Lim}(X) \cap M_\Sigma$". Let Axiom G be as follows. For any scheme Σ,

1. M_Σ is M_Σ-club;
2. if X is M_Σ-club then $\mathrm{LimM}_\Sigma(X)$ is M_Σ-club;
3. if $\eta \in \mathrm{Ord}$ and $\langle X_\xi : \xi < \eta \rangle$ is a sequence (coded as a class) of M_Σ-club classes then $\cap_{\xi<\eta} X_\xi$ is M_Σ-club; and
4. if $\langle X_\xi : \xi \in \mathrm{Ord} \rangle$ is a sequence (coded as a class) of M_Σ-club classes then $\triangle_\xi X_\xi$ is M_Σ-club.

The justification of this axiom is similar to the justification of M1-M4, by induction on Σ. There are two new cases, property 1 when Σ ends with an intersection, and when Σ ends with a diagonal intersection. Let Σ_ξ be the prefix of Σ ending with the point x_ξ, where $\langle x_\xi \rangle$ is the sequence approaching the end point of Σ. M_{Σ_ξ} is unbounded, so the universe is M_{Σ_ξ} for all ξ, so the universe is M_Σ, so M_Σ is unbounded.

The preceding justification needs to be improved, in that the statement "the universe is M_{Σ_ξ}" is vague. Cases such as Σ being $\omega + 1$ with the identity sequence approaching the end point are fairly clear, though.

For one example where care is required in the use of schemes, let κ be the least element of $\mathrm{Inac} \cap \mathrm{Lim}(\mathrm{Mahl})$. Then κ cannot be Mahlo, else since $\mathrm{Lim}(\mathrm{Mahl}) \cap \kappa$ is club, $\mathrm{Inac} \cap \mathrm{Lim}(\mathrm{Mahl}) \cap \kappa$ is nonempty.

Thus, $\mathrm{LimI}^\Sigma(\mathrm{Inac} \cap \kappa)$ cannot be I-club in κ for all schemes Σ in κ. On the other hand, this can be shown to hold for various Σ. For example, if $\alpha < \kappa$ then for $\lambda \in \mathrm{Mahl} \cap \kappa$ with $\lambda > \alpha$, λ is α-inaccessible. It is a question of interest, what is the smallest length of a scheme Σ such that $\mathrm{LimI}^\Sigma(\mathrm{Inac} \cap \kappa)$ is not I-club.

8. Greatly Mahlo Cardinals

It is easy to see that, for $\kappa \in \mathrm{Inac}$, $\mathrm{LimI}^\Sigma(X \cap \kappa) \neq 0$ for all schemes Σ in V_κ iff $X \cap \kappa$ is a stationary subset of κ.

Club subsets of α are defined for any limit ordinal α, and the collection of such has additional properties if α is an uncountable cardinal. More generally, many properties hold when α is a limit ordinal of uncountable cofinality. For a subset X of a cardinal $\kappa \in \mathrm{Inac}$ let $\mathrm{Sta}(X)$ denote the set of such α, such that $X \cap \alpha$ is a stationary subset of α.

Let H_0 denote the function on subsets of an inaccessible cardinal κ, where $\mathrm{H}_0(X) = \mathrm{Sta}(X) \cap \mathrm{Inac}$. By remarks above, this is the same function as LimI^*, where F^* is defined in the previous section. Let H_1 denote the function where $\mathrm{H}_1(X) = \mathrm{Sta}(X) \cap \mathrm{Inac} \cap X$.

Call a function F on subsets of an inaccessible cardinal κ a D-function if it has the following properties:

1. if $X \subseteq Y$ then $F(X) \subseteq F(Y)$,
2. $F(X \cup Y) = F(X) \cup F(Y)$, and
3. $F(F(X)) \subseteq F(X)$.

Say that X is F-regular if $F(X) \subseteq X$. It is readily verified that the family of F-regular sets is closed under F and intersections of length $< \kappa$; and that $F(X)$ is F-regular for any X.

It is well-known that Sta is a D-function (Exercise 8.11 of [12]). By modifying the argument, H_0 may be seen to be a D-function as well. It is not difficult to show that the Sta-regular sets are closed under $\triangle$. A slight modification to the argument shows that the H_0-regular sets also are closed under $\triangle$. Finally, it follows that if Σ is a scheme in κ and X is H_0-regular then $\mathrm{H}_1^{\Sigma}(X) = \mathrm{H}_0^{\Sigma}(X)$. We let H (the "Mahlo operation") be H_1.

An inaccessible cardinal κ will be said to be greatly Mahlo iff there is a κ-complete normal proper filter of subsets of κ, containing $\mathrm{Inac} \cap \kappa$ and closed under H. The term "κ^+-Mahlo" is also used. It is readily seen that this is so iff, for any scheme Σ in κ, $\mathrm{H}^{\Sigma}(\mathrm{Inac} \cap \kappa) \neq \emptyset$. It should be noted that some authors define the greatly Mahlo cardinals using Sta; however for considerations of collecting the universe, H is preferable.

For $\kappa \in \mathrm{Inac}$, if V_κ satisfies Axiom G then it is readily seen to be greatly Mahlo. It seems likely that the converse holds, but this will be omitted here.

In [6], the "number of times" H may be iterated, starting with Inac, is considered as a measure of the size of a large cardinal. Axiom G suggests that this is at least Ord^+, so that the universe has the "greatly Mahlo" or "κ^+-Mahlo", property.

H is more convenient for measuring the size, whereas strengthening the operator at successive iterations as in Axiom G is more convenient for justifying existence.

9. An Application

Even the system BGC+I1-I4+M1-M4 settles independent questions. A notable example is the "exotic cases of Boolean Relation Theory" (see [3]).

10. Relativizing to L

To relativize axioms as in preceding sections to L, a definition of a constructible class is needed. Following [16], define an R-structure for the language of set theory to be a structure $\langle M, E\rangle$ where M is a proper class, which satisfies foundation (i.e., any subset of M contains an E-minimal element), extensionality, and the axioms stating closure under the Godel operations. An R-structure $\langle M, E\rangle$ is called a R^+-structure iff it satisfies σ, where σ is any of several well-known sentences stating that $V = L$.

Given an R^+-structure $\langle M, E\rangle$ and a $z \in M$ let $X_{M,E,z}$ be the class $\{x \in L$ such that for some $y \in M$, yEz and the transitive closure of x (in the universe) is isomorphic to the transitive closure of y (in $\langle M, E\rangle)\}$. The constructible classes are those of the form $X_{M,E,z}$ for some M, E, z.

See Theorem 3 of [6] for a proof that this definition is in accordance with what one expects if V is taken as V_κ for some $\kappa \in \text{Inac}$.

Suppose Axioms I1-I4 hold in V. Since Lemma 1 holds in L, to show that Axioms I1-I4 hold in L it suffices to show that Y as in the remarks following the statement of these axioms is unbounded.

For Axiom I1, that Inac^L is unbounded follows because Inac is.

For the remaining axioms, suppose X is a constructible class which is I-closed in L. Then $\text{Lim}(X) \cap \text{Inac}^L \subseteq X$, so $\text{Lim}(X) \cap \text{Inac} \subseteq X$, so X is I-closed. Thus, if X is I-club in L then X is I-club.

For Axiom I2, suppose that X is I-club in L. Then X is I-club, so $\text{Lim}(X) \cap \text{Inac}$ is I-club, in particular unbounded.

For Axiom I3, suppose that X_ξ is I-club in L for $\xi < \eta$. Then X_ξ is I-club for $\xi < \eta$, so $\cap_\xi X_\xi$ is I-club, so unbounded.

Axiom I4 is similar.

The preceding discussion should be better formalized and generalized; further discussion is omitted here.

11. $V = L$

The results of the preceding section suggest that "cautious" large cardinal properties P are down-absolute for L, that is, if $P(\kappa)$ holds then it holds in L. This in turn can be seen as evidence that perhaps $V = L$, and "cautious" large cardinal theory might lead to further substantiation that this might be the case.

Other evidence that the possibility that $V = L$ should be taken seriously exists. Some remarks may be found in [6]. A review will be given here.

After results of Jensen and others, interest in the possible truth of the hypothesis of constructibility (usually abbreviated $V = L$) intensified; to quote [4], "since the axiom of constructibility has applications in several areas of mathematics now, it certainly merits attention".

Subsequently, however, the majority of set theorists concluded that, although $V = L$ settles a great number of important independent questions, it does so in a way which they find unsatisfactory. In particular, it implies:

1. GCH is true;
2. 0# does not exist;
3. PD is false (this follows from 2);
4. measurable cardinals do not exist (this follows from 2).

In a recent survey [2], none of the 31 respondents believed that $V = L$ should be accepted as an axiom.

It should be noted that there has been debate on whether independent questions have a truth value at all. The classical position would be that any statement in the language of set theory has a truth value. Informal metamathematical arguments supporting the classical position can be given. For example, a statement such as CH concerns very basic sets, and that it has a truth value is a consequence of assumptions that are routinely made in model theory. Similarly, GCH has a truth value because it has one at any cardinal (although quantifying over all cardinals is logically more complex than quantifying over a set).

The classical position does lead to a logical conundrum, in that even though the statement has a truth value, we may never be sure what it is, which in turn raises questions about the character of truth in this setting. Mathematics is free to ignore such questions, and routinely has done so.

From hereon, we suppose that any statement of set theory, including statements known to be independent of ZFC, has a truth value. Determining this for various questions, notably CH, has been and remains a mysterious problem of mathematics. This question of interest to philosophers of mathematics as well as mathematicians.

Philosophically, mathematical existence seems to be of a different character than physical existence. Mathematicians assume the existence of mathematical objects as a convenience; not only do sets "exist" in some sense, but they are well-described by the notion of the cumulative hierarchy, and the axioms of ZFC set theory. Philosophers (perhaps rightly so) find it necessary to "quibble" over various aspects of the situation. The classical position is, that these questions are a distraction to mathematics, which should admit that statements in the language of set theory have a truth value in the universe of sets. The latter concept encounters philosophical complexities, and even mathematical ones; but determining the truth value of statements increases understanding of the universe.

It should be a topic of interest in mathematics what new axioms might be adopted, and why. The author's view, which might be called the "minimalist" view, has been that these should be, $V = L$, and axioms asserting the existence of large cardinals which can be justified by mathematical theories formalizing the principle of collecting the universe. Implicit in this view is the conviction that no large cardinal contradicting $V = L$ can be "built up" in this manner. In fact, this conviction is evidence for $V = L$.

Other points of view have been under intensive consideration; the reader is referred to the literature. From hereon, only the minimalist view will be considered. From this view, axioms that contradict $V = L$ are "pathological".

To begin with, it is worth considering the axiom of choice. The axiom of choice states that, given a collection of non-empty sets, there is a function mapping each set in the collection to an element ("representative") of the set. Arguments in favor of it include the following.

- It is self-evident.
- Mathematicians use it constantly.
- It can be justified by informal arguments involving the cumulative hierarchy; see [14].

A critical discussion of these and other topics may be found in [11].

Not only should the axiom of choice be taken as true, but its character can be seen as a consideration in the evaluation of other principles. In particular,

$V = L$ can be seen as a "glorified" extension of it, asserting the existence of "construction sequences" and not just the more mundane "choice functions".

After AC, the "next" independent question is CH. In [4] it is stated that "there is no intuitive reason for taking GCH as an axiom of set theory" (this being so one needs to justify $V = L$ to derive GCH). There is however a strong argument in favor of CH, and the same argument generalizes to GCH.

CH is true iff there is a bijection of $\aleph_1$ with $P(\omega)$. CH is false iff there is an injection of $\aleph_2$ into $P(\omega)$. One of these existence principles is false; and the other is merely independent of ZFC. One can well suspect that it is the second principle which is false, in that if it were true then it would be possible to provide from first principles a construction of an embedding. When Godel showed that CH was consistent with ZFC, he showed that no such construction is possible.

Further, the independence of the first principle is not surprising. It is independent of ZF whether $P(\omega)$ can be well-ordered at all. Even assuming the axiom of choice, we still cannot construct a well-ordering by $\aleph_1$; however the evidence against an embedding of $\aleph_2$ leads one to conclude that such a well-ordering exists.

The author discovered this argument in 1985, and it appears in print in [5]. Recently, it has appeared in a Web posting. In view of the basic nature of the observation, it is not surprising that other authors have made it; indeed, it is surprising that it is not more widely considered.

The injection from $\aleph_1$ to $P(\omega)$ is of interest. It consists of the composition of a choice function which maps a countable ordinal to a binary relation on ω which is a well-order, and a standard injection from the binary relations on ω to $P(\omega)$. One can see that there are various difficulties which arise in attempting to obtain an injection from $\aleph_2$.

Of course, a bijection of $\aleph_1$ with $P(\omega)$ cannot be obtained either. But the difficulties here are due to an inability to proceed. It is reported that Cantor tried to prove CH; perhaps he attempted to find such a method.

Further remarks include the following.

- $V = L$ is not terribly much stronger than GCH; not only does a well-order exist, but one may be obtained in a natural manner.

- Since L is a model of ZFC, it is plausible that it is all the sets.

- Independently of GCH, it can be argued that $\omega_1^L = \omega_1$. In ω_1 steps, the constructibility process should produce ω_1 bijections of ω with an ordinal. This claim of interest in itself, in that it refutes $0\#$. It is more direct than the claim that every real is constructed by stage ω_1, which follows if every real is constructible, by condensation.

A better understanding of "building up" large cardinals might very well shed further light on the question of whether $V = L$.

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