OUTPUT OBSERVABILITY OF GENERALIZED LINEAR SYSTEMS

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Abstract: In this paper we consider finite-dimensional generalized linear discrete-time-invariant systems in the form \( E \dot{x}(k+1) = Ax(k) + Bu(k) \), \( y(k) = Cx(k) \) where \( E, A \in M_n(C) \), \( B \in M_{n \times m}(C) \), \( C \in M_{p \times n}(C) \), describing convolutional codes. The notion of output observability is analyzed and a characterization of output observable systems is obtained.

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1. Introduction

Let us consider a discrete finite-dimensional generalized linear time-invariant system \( E \dot{x}(k+1) = Ax(k) + Bu(k) \), \( y(k) = Cx(k) \), where \( E, A \in M_n(C) \), \( B \in M_{n \times m}(C) \), \( C \in M_{p \times n}(C) \), describing convolutional codes. For simplicity, we denote the systems as a quadruples of matrices \( (E, A, B, C) \) and we denote by \( \mathcal{M} \) the set of this kind of systems. In the case where \( E = I_n \) the system is standard and we denote merely, as a triple \( (A, B, C) \).

For simplicity but without loss of generality, we consider that matrix \( B \) has column full rank and rank \( B = m \) and \( C \) has row full rank and rank \( C = p \), so \( 0 < p, m \leq n \).

It is well known that there is a close connection between linear systems and convolutional codes and there is a large literature about that as for example [2], [5], [6], [8].

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Ch. Fragouli and R.D. Wesel in [1], give the following definition of output observability for standard systems.

**Definition 1.** The standard system \((A, B, C)\) is said to be output observable if the state sequence \(\{x_0, x_1, \ldots, x_{n-1}\}\) is uniquely determined by the knowledge of the output sequence \(\{y_0, y_1, \ldots, y_{n-1}\}\) for a finite number of steps \(n-1\).

Output observability is characterized by the following proposition.

**Proposition 1.** (see [1]) The system \((A, B, C)\) ∈ \(M\) is output observable if and only if the following matrix

\[
M(A, B, C) = \begin{pmatrix}
C & 0 & 0 & \ldots & 0 \\
CA & CB & 0 & \ldots & \vdots \\
CA^2 & CAB & CB & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
CA^{n-1} & CA^{n-2}B & CA^{n-3}B & \ldots & CB
\end{pmatrix} \in M_{pn \times (n+(n-1)m)}(C)
\]

has full rank.

In this paper, we generalize the definition of output observability given for standard linear systems to the generalized linear systems, and we give a characterization of the set of output observable systems. In the last section a characterization of output controlable systems is also presented.

### 2. Preliminaries

We consider quadruples of matrices \((E, A, B, C)\) ∈ \(M\), representing generalized discrete time invariant linear systems, a manner to understand the properties of the system is treating it by purely algebraic techniques. The main aspect of this approach is defining an equivalence relation preserving these properties, many interesting and useful equivalence relations between generalized systems have been defined. We deal with the equivalence relation accepting one or more, of the following transformations: basis change in the state space, input space, output space, operations of state and derivative feedback, state and derivative output injection and to premultiply the first equation of the system, by an invertible matrix.

**Definition 2.** Two quadruples \((E_i, A_i, B_i, C_i)\) ∈ \(M\), \(i = 1, 2\), are equivalent if and only if there exist matrices \(P ∈ Gl(n; C), Q ∈ Gl(p; C), R ∈ Gl(m; C),\)
S ∈ Gl(q; C), F_{BE}, F_{BA} ∈ M_{m×n}(C), F_{CE}, F_{CA} ∈ M_{p×q}(C) such that
\begin{align*}
E_2 &= QE_1P + QB_1F_{BE}^T + F_{CE}C_1P, \\
A_2 &= QA_1P + QB_1F_{BE}^T + F_{CE}C_1P, \\
B_2 &= QB_1R, \\
C_2 &= SC_1P.
\end{align*}

(1)

If we restrict the relation to standard systems, this is by keeping \( E = I_n \) then \( Q = P^{-1} \), \( F_{BE} = 0 \) and \( F_{CE} = 0 \).

Given a quadruple of matrices \((E, A, B, C) \in \mathcal{M}\), we can associate the following matrix pencil

\[
H(\lambda) = \begin{pmatrix}
\lambda E + A & \lambda B & B \\
\lambda C & 0 & 0 \\
C & 0 & 0
\end{pmatrix},
\]

Restricted to standard systems we associate the pencil

\[
H(\lambda) = \begin{pmatrix}
\lambda I + A & B \\
C & 0
\end{pmatrix},
\]

and we have

**Proposition 2.** Two quadruples are equivalent under equivalent relation considered if and only if the associated matrix pencils are strictly equivalent.

So, we can apply Kronecker’s theory of singular pencils (see [2], for more details).

### 3. Output- Observability

In this section we generalize the output observability condition for generalized systems.

**Definition 3.** The generalized system \((E, A, B, C)\) is said to be output observable if the state sequence \(\{x_0, x_1, \ldots, x_{n-1}\}\) is uniquely determined by the knowledge of the output sequence \(\{y_0, y_1, \ldots, y_{n-1}\}\) for a finite number of steps \(n - 1\).

Output observability is characterized by the following proposition.

**Proposition 3.** The system \((E, A, B, C) \in \mathcal{M}\) is output observable if and only if the following matrix

\[
M(E, A, B, C) = \begin{pmatrix}
A & B & E & 0 & 0 \\
C & 0 & 0 & 0 & 0 \\
0 & 0 & A & B & E \\
0 & 0 & C & 0 & 0 \\
& & & & \ddots \\
& & & & & A & B & E \\
& & & & & C & 0 & 0 \\
& & & & & 0 & 0 & C
\end{pmatrix} \in M_{x×y}(C),
\]
\[ x = (n - 1)n + np, \quad y = n^2 + (n - 1)m, \text{ has full row rank.} \]

**Proof.** It suffices to observe that

\[
\begin{pmatrix}
A & B & 0 & E & 0 & 0 \\
C & 0 & 0 & 0 & 0 & A & B & E \\
0 & 0 & C & 0 & 0 & 0 & 0 & 0 & \ddots
\end{pmatrix}
\begin{pmatrix}
x(0) \\
u(0) \\
x(1) \\
u(1) \\
x(n-2) \\
u(n-2) \\
x(n-1)
\end{pmatrix}
= \begin{pmatrix}
0 \\
y(0) \\
0 \\
y(1) \\
0 \\
y(n-2) \\
y(n-1)
\end{pmatrix}.
\]

In the case where the system \((E, A, B, C)\) is standard (i.e. \(E = I_n\)), making elementary transformations to the matrix \(M\) we obtain

\[
\text{rank } M(I_n, A, B, C) = n(n - 1) + \text{rank } \begin{pmatrix}
C & 0 & 0 & \ldots & 0 \\
CA & CB & 0 & \ddots & \vdots \\
CA^2 & CAB & CB & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
CA^{n-1} & CA^{n-2}B & CA^{n-3}B & \ldots & CB
\end{pmatrix} = n(n - 1) + \text{rank } M(A, B, C).
\]

So, the definition generalizes the definition given for standard systems.

**Proposition 4.** Let \((E, A, B, C)\) be a system in \(M\). Then, the rank of the matrix \(M(E, A, B, C)\) is invariant under equivalence relation considered.

**Proof.** Let \((E_1, A_1, B_1, C_1)\) be an equivalent system, then there exist matrices \(P \in \text{Gl}(n; C), \ Q \in \text{Gl}(p; C), \ R \in \text{Gl}(m; C), \ S \in \text{Gl}(q; C), \ F_E, F_A, F_B \in \text{M}_{m \times n}(C), \ F_E^C, F_A^C \in \text{M}_{p \times q}(C)\) such that (1) is verified.

Denoting by

\[
Q = \begin{pmatrix}
Q & F_E^C & 0 & F_E^C \\
0 & S & 0 & 0 \\
0 & 0 & Q & F_A^C \\
0 & 0 & 0 & S \\
\ddots & \ddots & \ddots & \ddots \\
Q & F_E^C & F_E^C \\
0 & S & 0 & 0 \\
0 & 0 & S
\end{pmatrix}
\]

and

\[
P = \begin{pmatrix}
P & 0 & 0 & 0 & 0 \\
F_A^B & R & F_E^B & 0 & 0 \\
0 & 0 & P & 0 & 0 \\
0 & 0 & F_A^B & R & F_E^B \\
0 & 0 & 0 & 0 & P \\
& \ddots & \ddots & \ddots & \ddots \\
P & 0 & 0 & 0 & 0 \\
F_A^B & R & F_E^B & 0 & 0
\end{pmatrix}.
\]
We have,

\[
\begin{pmatrix}
A & B & E & 0 & 0 \\
C & 0 & 0 & 0 & 0 \\
0 & A & B & E & 0 \\
0 & 0 & C & 0 & 0
\end{pmatrix} \cdot \begin{pmatrix}
P \end{pmatrix} = \text{rank} Q = \begin{pmatrix}
A_1 & B_1 & E_1 & 0 & 0 \\
C_1 & 0 & 0 & 0 & 0 \\
0 & 0 & A_1 & B_1 & E_1 \\
0 & 0 & C_1 & 0 & 0
\end{pmatrix}.
\]

4. Qualitative Properties of the Systems

In this section we will go to analyze the qualitative properties characterizing output observable systems in the case of standard systems.

Having defined an equivalence relation, the standard procedure then is to look for a canonical form, that is to say to look for a standard system which is equivalent to a given system and which has a simple form from which we can directly read off the properties and invariants of the corresponding generalized system. In this case it is well known the following proposition.

**Proposition 5.** Let \((A, B, C)\) be a standard system, then it is equivalent to \((A_c, B_c, C_c)\) with

\[
A_c = \begin{pmatrix}
N_1 & N_2 & N_3 \\
& & J
\end{pmatrix}, \quad B_c = \begin{pmatrix}
B_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & B_2 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad C_c = \begin{pmatrix}
0 & C_1 & 0 & 0 \\
0 & 0 & C_2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

with

- \(N_1 = \text{diag} (N_1^1, \ldots, N_1^l) \in M_{n_1} (\mathbb{C}),\)
- \(N_2 = \text{diag} (N_2^1, \ldots, N_2^l) \in M_{n_2} (\mathbb{C}),\)
- \(N_3 = \text{diag} (N_3^1, \ldots, N_3^l) \in M_{n_3} (\mathbb{C}),\)
- \(J = \text{diag} (J_{11}, \ldots, J_{44}) \in M_{n_4} (\mathbb{C}),\)
- \(B_1 = \text{diag} (B_1^1, \ldots, B_1^l) \in M_{n_1 \times m_1} (\mathbb{C}),\)
- \(B_2 = \text{diag} (B_2^1, \ldots, B_2^l) \in M_{n_2 \times m_2} (\mathbb{C}),\)
- \(C_1 = \text{diag} (C_1^1, \ldots, C_1^l) \in M_{p_1 \times n_1} (\mathbb{C}),\)
- \(C_2 = \text{diag} (C_2^1, \ldots, C_2^l) \in M_{p_2 \times n_4} (\mathbb{C}),\)
- \(N_i^1 = \begin{pmatrix} 0 & 0 \\ 0 & I_{k_i-1} \end{pmatrix} \in M_{k_i} (\mathbb{C}), \quad 1 \leq i \leq r,\)
- \(B_i^1 = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix} \in M_{k_i \times 1} (\mathbb{C}), \quad 1 \leq i \leq r,\)
- \(N_i^2 = \begin{pmatrix} 0 & I_{l_i-1} \\ 0 & 0 \end{pmatrix} \in M_{l_i} (\mathbb{C}), \quad 1 \leq i \leq s,\)
$C^1_i = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \in M_{1 \times l_i}(\mathbb{C})$, $1 \leq i \leq s$,
$N^3_i = \begin{pmatrix} 0 & I_{m_i - 1} & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \in M_{m_i}(\mathbb{C})$, $1 \leq i \leq t$,
$B^2_i = \begin{pmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix} \in M_{m_i \times 1}(\mathbb{C})$, $1 \leq i \leq t$,
$C^2_i = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \in M_{1 \times m_i}(\mathbb{C})$, $1 \leq i \leq t$,
$J_{l_i}$ is the endomorphism in its Jordan form.

The canonical form can be obtained directly from the initial system, without knowing transformations that permit us to reduce the system to its reduced form (see [3], [4]).

**Theorem 1.** The system $(A, B, C)$ is output observable if and only if

1. if $p \leq m \leq n$, the system has not observable non controllable part, that is to say the reduced form is in the form $\left( \begin{pmatrix} N_1 \\ N_3 \end{pmatrix}, \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \begin{pmatrix} 0 & C_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$.

2. if $m < p \leq n$, the system has only controllable and observable part and observable non controllable part with $C = \begin{pmatrix} C_1 \\ t_{p_2} \end{pmatrix}$, $A = \begin{pmatrix} N_2 \\ t_{p_2} \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ t_{p_2} \end{pmatrix}$.

**Proof.** Proposition 4 permits us to consider the system in its reduced form and for study output observability. So

$$M(A, B, C) = \begin{pmatrix} 0 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & C_3 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\in M_{pm \times (n+1)(m_0)}(\mathbb{C})$$

and

$$\text{rank} \left( \begin{array}{cccccccc} C & 0 & 0 & \ldots & 0 \\ CA & CB & 0 & \ldots & \vdots \\ CA^2 & CAB & CB & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{n-1} & CA^{n-2}B & CA^{n-3}B & \ldots & CB \end{array} \right) = \text{rank} \left( \begin{array}{cccccccc} C_2 & 0 & 0 & \ldots & 0 \\ C_2N_3 & C_2B_2 & 0 & \ldots & \vdots \\ C_2N_3^2 & C_2N_3B_2 & C_2B_2 & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_2N_3^{n-1} & C_2N_3^{n-2}B_2 & C_2N_3^{n-3}B_2 & \ldots & C_2B_2 \end{array} \right) + \text{rank} \left( \begin{array}{c} C_1 \\ C_1N_2 \\ \vdots \\ C_1N_2^{n-1} \end{array} \right).$$
Calling
\[
M_1 = \begin{pmatrix}
C_1 \\
\vdots \\
C_1 N_n^{-1}
\end{pmatrix},
\]
and
\[
M_2 = \begin{pmatrix}
C_2 & 0 & 0 & \ldots & 0 \\
C_2 N_3 & C_2 B_2 & 0 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_2 N_n^{n-1} & C_2 N_n^{n-2} B_2 & C_2 B_2 & \ldots & C_2 B_2
\end{pmatrix},
\]
we have that
\[
\text{rank } M(A, B, C) = \text{rank } M_1 + \text{rank } M_2.
\]
Matrices \(M_i\) have always full rank with rank \(M_1 = n_2\) and rank \(M_2 = p_2 n\) (observe that \(p_2 n \leq n_3 + (n - 1)m_2\)).

Consequently, matrix \(M\) has full rank if and only if \(\min (pn, n + (n - 1)m) = n_2 + np_2\).

1. If \(\min (pn, n + (n - 1)m) = pn\), then \((p_1 + p_2)n = n_2 + np_2\) and \(p_1 n = n_2\), and that is only possible if \(p_2 = p_1 = 0\).

2. If \(r = \min (pn, n + (n - 1)m) = n + (n - 1)m\), observing matrix (2) we have that \(n_1 = n_4 = 0\), notice that if \(n_1 = 0\) then \(m_1 = 0\) and \(m_2 = p_2 \neq 0\). If \(p_1 = 0\) then \(pn < n + (n - 1)m\), consequently \(p_1 \neq 0\).

With all these data we have that the system is output observable if and only if \(n_2 + n_3 + (n_2 + n_3 - 1)m_2 = n_2 + p_2(n_2 + n_3)\), equivalently \(n_3 - m_2 = 0\).

**Example 1.**

1. The system \((A, B, C)\) with \(A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n \end{pmatrix}\), \(B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) and \(C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\), is output observable.

2. The system \((A, B, C)\) with \(A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\), \(B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) and \(C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\), is not output observable.

3. The system \((A, B, C)\) with \(A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\), \(B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\) and \(C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\) is output observable.
4. The system \((A, B, C)\) with \(A = 0, B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\) and \(C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\) is not output observable.

5. Output Controllability

Another important concept for standard systems \((A, B, C)\) is output controllability that is the related notion for the output of the system; the output controllability describes the ability of an external input to move the output from any initial condition to any final condition in a finite time interval. A controllable system is not necessarily output controllable, and an output controllable system is not necessarily controllable.

The output controllability can be characterized as follows

**Proposition 5.** A standard system is output controllability if and only if the controllability matrix

\[
(CB \quad CAB \quad \ldots \quad CA^{n-1}B)
\]

must have full row rank.

**Theorem 2.** The output controllability character is invariant under equivalence relation considered.

**Proof.** Suppose \((A_1, B_1, C_1) = (P^{-1}AP, P^{-1}BR, SCP)\), then

\[
\text{rank} \left( \begin{array}{cccc}
C_1B_1 & C_1A_1B_1 & \ldots & C_1A_1^{n-1}B_1 \\
\end{array} \right) = \text{rank} S \left( \begin{array}{cccc}
CBR & CAB & \ldots & CA^{n-1}B \\
\end{array} \right) R = \text{rank} \left( \begin{array}{cccc}
CB & CAB & \ldots & CA^{n-1}B \\
\end{array} \right).
\]

Now, consider \((A_1, B_1, C_1) = (A + BF_A^B, C)\), it is easy to compute \(A_1^k\) obtaining

\[
CA_1^kB = CA^kB + \sum_{0 \leq \ell \leq k-1} CA^{k-\ell-1}BF_A^B A_1^\ell B.
\]

Making the following column elementary transformations

\[
c_j + c_{j-1}F_E^B B + c_{j-2}F_E^B A_1 B + \ldots + c_1 F_E^B A - 1^{j-2} B,
\]

where \(c_\ell\) indicates the \(\ell\) column of the output controllability matrix of \((A, B, C)\), we obtain the output controllability matrix for \((A_1, B_1, C_1)\).

Finally, we consider \((A_1, B_1, C_1) = (A + F_E^C, B, C)\), in this case it suffices to consider \((A_1^t, C_1^t, B_1^t)\), and the proof is concluded. \(\square\)
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References


