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A NOTE ON UPPER BOUND FOR d-COVERED TRIANGULATION OF CLOSED SURFACES

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Abstract: A triangulation of a closed surface is said to be d-covered if at least one vertex in each edge has degree d. In [7] the authors have shown that: (a) if a closed surface of non-positive Euler characteristic χ admits a d-covered triangulation then $d \leq 2\left\lfloor\frac{5+\sqrt{49-24\chi}}{2}\right\rfloor$, and (b) the upper bound for d is attained when: $(i)\, \chi = 0$ and for $(ii)\, d = 2(n-1)$, whenever $n = \frac{7+\sqrt{49-24\chi}}{2}$ is a positive integer. In this note we show that the upper bound is attained for $\chi \in S := \{-2, \ldots, -8\} \cup \{-13, -14, -15\}$.

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1. Introduction and Definitions

A closed surface F, is a connected, compact two dimensional manifold without boundary. Let G denote a finite graph which has no loops or multiple edges and each vertex of G has valence ≥ 3 (see [1]). An embedding of G in F such that closure of each component of $F \setminus G$ is a closed 2-cell is called a *map* on F. When each of these closed 2-cells is a triangle such that a non-empty intersection of any two of these triangles is either a common edge or a common vertex, the map is said to be a *triangulation* of F. A triangulation of F in which exactly q triangles meet at each vertex is called a *q-equivelar* triangulation. The number of edges incident at a vertex is called the *degree* of the vertex. The surface of Icosahedron

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is a 5-equivelar triangulation of the 2-sphere. The $\{3,6\}$ tessellation of the Euclidean plane is an example of $6-$equivelar triangulation. In [2], $6-$equivelar triangulations of the torus and Klein bottle on few vertices are presented. To a triangulation of F, we associate an integer called its *Euler characteristic* as follows: let n denote the number of vertices, e the number of edges and f the number of triangles in the triangulation. Then $\chi(F) = n - e + f$ is an integer, known as Euler characteristic of the triangulation (see [6] for details).

In an equivelar triangulation of a closed surface, consider the set of all triangles containing a given vertex v. Let e denote those edges in these triangles which are not incident with v and denote this set by $\mathrm{lk}(v)$. Let $|\mathrm{lk}(v)|$ denote the cycle obtained by joining common vertices of the edges in $\mathrm{lk}(v)$. It is easy to see that $|\mathrm{lk}(v)|$ is a polygon called the *link* of v. Similarly, let $\mathrm{st}(v)$ denote the set of all triangles containing v and $|\mathrm{st}(v)|$ denote the 2-cell formed by joining triangles in $\mathrm{st}(v)$ along their common edges. Then $|\mathrm{st}(v)|$ is called the *star* of v. It is easy to see that $|\mathrm{st}(v)|$ is a closed topological 2-disc with polygonal boundary $|\mathrm{lk}(v)|$. If the triangulation is q-equivelar then $|\mathrm{lk}(v)|$ is a q-gon for each vertex v.

Given an integer χ, consider the set $S_\chi = \{(n,d)\,|\,\chi = n - \frac{nd}{2} + \frac{nd}{3}, d \leq (n-1)\}$. A pair $(n,d) \in S_\chi$ is called an *admissible pair* for χ. If $d = 7$ and $n = -6\chi$ then $S_\chi \neq \emptyset$ for all integers $\chi < -1$. Let F be a closed surface such that $S_{\chi(F)} \neq \emptyset$ and $(n,d) \in S_{\chi(F)}$ be an admissible pair for $\chi(F)$. It is easy to see that $(12,7) \in S_{-2}$, $(9,8) \in S_{-3}$, $(12,8) \in S_{-4}$, $(10,9) \in S_{-5}$, $(12,9) \in S_{-6}$, $(14,9) \in S_{-7}$, $(12,10) \in S_{-8}$, $(13,12) \in S_{-13}$, $(14,12) \in S_{-14}$ and $(15,12) \in S_{-15}$. Various results about equivelar triangulations of surfaces presented in [4] and [9] lead to the following:

Proposition 1. *For each* $\chi \in \{-2,\ldots,-8,-13,-14,-15\}$ *there exist equivelar triangulations of closed surfaces of Euler charateristic* χ. *These triangulations correspond to the pairs* $(n,d) = (12,7)$, $(9,8)$, $(12,8)$, $(10,9)$, $(12,9)$, $(14,9)$, $(12,10)$, $(13,12)$, $(14,12)$ *and* $(15,12)$ *respectively.*

The authors have, in fact, listed the number of equivelar triangulations of surfaces with upto 12 vertices (in [9], pp. 12) and vertex transitive triangulations of closed surfaces with ≤ 15 vertices (in [4], pp. 45-46). These and much more are also available in tabular form from [11]. In [3], 7-equivelar triangulations of double torus on 12 vertices have been constructed.

A triangulation of a closed surface F is said to be d-covered (see [7]), if at least one vertex in each edge has degree d. In other words, we say that edges are covered by vertices of degree d.

For $\chi \leq 0$, in [7], the authors have shown following:

Proposition 2. *If a closed surface F admits a d-covered triangulation then $d \leq 2\lfloor \frac{5+\sqrt{49-24\,\chi(F)}}{2} \rfloor$.*

The authors have also shown that the upper bound in Proposition 2 is attained when (i) $\chi = 0$ and when (ii) $n = \frac{7+\sqrt{49-24\,\chi(F)}}{2}$ is a positive integer and F is not the Klein bottle. The second case is established by considering embeddings of the complete graph K_n in F, see [8]. Here we show:

Theorem 3. *Let F be a closed surface with Euler characteristic $\chi(F) \in S = \{-2, \ldots, -8, -13, -14, -15\}$. Then there exists $d = 2\lfloor \frac{5+\sqrt{49-24\,\chi(F)}}{2} \rfloor$-covered triangulation of F.*

2. Results

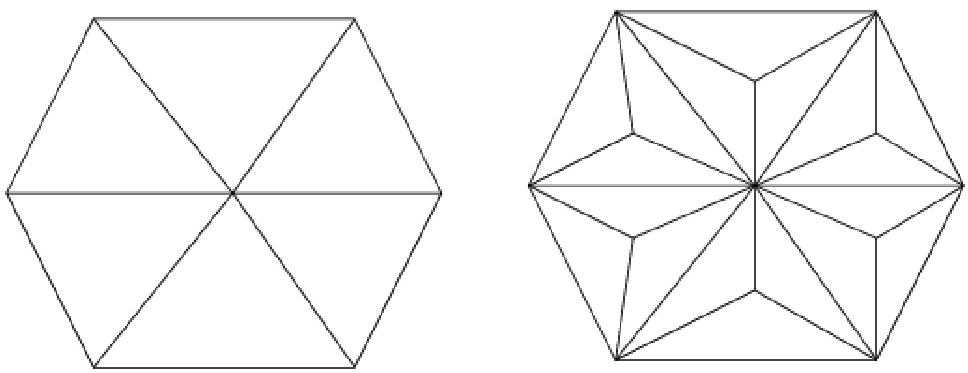
Lemma 4. *Let S be as in Theorem 3 and F be a closed surface with $\chi(F) \in S$ and $q = \lfloor \frac{5+\sqrt{49-24\,\chi(F)}}{2} \rfloor$. Then, there exists $n = \frac{6\,\chi(F)}{6-\lfloor \frac{5+\sqrt{49-24\,\chi(F)}}{2} \rfloor}$ vertex q-equivelar triangulation of F.*

Proof. The proof of lemma follows by observing that the Euler characteristic relation $\chi(F) = n - \frac{nq}{2} + \frac{nq}{3}$ for F can be re-written as $\chi(F) = n\,(\frac{6-q}{6})$. Then, for the given values of χ, i.e., for $\chi \in S$, we have $(n, q) \in \{(12, 7), (9, 8), (12, 8), (10, 9), (12, 9), (14, 9), (12, 10), (13, 12), (14, 12), (15, 12)\}$. The existence of triangulation now follows by Proposition 1. □

Proof. of Theorem 3. Given $\chi \in S$, consider a $q = \lfloor \frac{5+\sqrt{49-24\chi}}{2} \rfloor$- equivelar triangulation of a closed surface of Euler characteristic χ. Following [7], subdivide each triangle of this triangulation into three by introducing a vertex of degree 3 (as shown in example below). Note that the value of χ does not change in this subdivision. Then, the resulting triangulation is $d = 2\,q = 2\lfloor \frac{5+\sqrt{49-24\chi}}{2} \rfloor$ covered. □

3. Example

Example. In [3], classification of equivelar triangulations of double torus is presented. In this case $S_{-2} = \{(12, 7)\}$, i.e. $(n, d) = (12, 7)$ and $\chi = -2$. The figure below shows $|\mathrm{st}(v)|$, star of a vertex v in these triangulations before and after subdivision see Figure 1.



$|st(v)|$ $|st(v)|$ after subdivision

Figure 1

Remark 5. The authors in [7] have also shown that d covered triangulations exists for all $\chi > 0$ and that the upper bound is attained for both the values $\chi = 1$ and 2.

Remark 6. The case of $\chi = -10$ has been covered in [7]. After some simple calculations one observes that for $\chi = -1, -9, -11$ and -12 the values of $q = 6, 10, 11$ and 11 respectively. For these values of q, $n = \frac{6\,\chi(F)}{6 - \lfloor \frac{5+\sqrt{49-24\,\chi(F)}}{2} \rfloor}$ is not an integer. So, the observations and method used in this article do not work for these cases. On the other hand, for $\chi \leq -16$ the values of $d \geq 12$. In [5] existence of such triangulations is shown for $-16 \leq \chi \leq -127$ and whenever $n = \frac{6\,\chi}{(6-d)}$ is a positive integer. In [10] 12-covered triangulations on surface of Euler characteristic -1 has been constructed.

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References

[1] U. Brehm, E. Schulte, Polyhedral maps, In: *Handbook of Discrete*

and Computational Geometry, Second Edition (Ed-s: J.E. Goodman, J. O'Rourke), Chapman and Hall/CRC, Boca Raton, FL (2004), 477-491.

[2] B. Datta, A.K. Upadhyay, Degree-regular triangulations of torus and Klein-bottle, *Proc. Indian Acad. Sci. (Math. Sci.)*, **3**, No. 113 (2005), 279-307.

[3] B. Datta, A.K. Upadhyay, Degree-regular triangulations of the double torus, *Forum Mathematicum*, **18**, No. 6 (2006), 1011-1026.

[4] F.H. Lutz, *Triangulated Manifolds with Few Vertices and Vertex Transitive Group Actions*, Shaker Verlag, Aachen (1999).

[5] F.H. Lutz, T. Sulanke, A.K. Tiwari, A.K. Upadhyay, Equivelar and d-covered triangulations of surfaces – I, http://arxiv.org/abs/1001.2777.

[6] J.R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, California (1984).

[7] S. Negami, A. Nakamoto, Triangulations on closed surfaces covered by vertices of given degree, *Graphs and Combinatorics*, **17** (2001), 529-537.

[8] G. Ringel, *Map Color Theorem*, Springer-Verlag, Berlin (1974).

[9] T. Sulanke, F. Lutz, Isomorphism-free lexicographic enumeration of triangulated surfaces and 3-manifolds, http://arxiv.org/abs/math/0610022.

[10] A.K. Upadhyay, A.K. Tiwari, D. Maity, Some semi-equivelar maps, http://arxiv.org/abs/1101.0671.

[11] http://www.math.tu-berlin.de/diskregeom/stellar/