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INVERSE PROBLEMS IN NONCLASSICAL STATEMENTS

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Abstract: The inverse problems which cannot be solved in the classical framework are investigated in this article. They are as follows: Krylov inverse problem, early diagnostics of a rotor unbalance, the most probable solution of inverse problem. For obtaining the steady solutions of these problems some algorithms based on the method of Tikhonov regularization are offered. Krylov inverse problem in various statements has been considered and numerical calculation based on real measurements has been executed. Nonclassical statements of inverse problems extend of regularization method possibilities.

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1. Introduction

Intensive development of methods of the solution of inverse problems had begun after fundamental works of A.N. Tikhonov and members of scientific schools which were created by him (see [18], [17]). It is difficult to find an area of scientific researches where the inverse problems would not be examined. The essential progress are available in tomography (see [6]), in processing of the images (see [5]), in the non-destroying control problems (see [1]), in finance (see [19]), in inverse problems applied to the wave equations (see [2]) and many other areas (see [11]). In analysis of similar progress arise the inverse problems

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which cannot be investigated in classical statement. We shall consider a wide class of inverse problems the solutions of which are reduced to the solution of the linear equation

$$A_p z = u_\delta, \tag{1}$$

where $z \in Z, u_\delta \in U; Z, U$ are functional spaces with norm, $A_p : Z \to U$.

The operator A_p is assumed as completely continuous. The function u_δ is obtained with the help of an experimental measurements with the known error δ:

$$\| u_\delta - u_{ex} \|_U \leq \delta, \tag{2}$$

where u_{ex} is exact right part which corresponds to absolutely exact experimental measurements.

Let us designate z_{ex} the solution of the equation

$$A_{ex} z_{ex} = u_{ex},$$

where A_{ex} is the exact operator.

Let us consider set of functions $Q_{\delta,p}$ each of which after substitution in the equation (1) gives inaccuracy which does not exceed an error of initial data. Further we shall name this set as set of the possible solutions. This set has the form

$$Q_{\delta,p} = z : \| A_p z - u_\delta \|_U \leq \delta. \tag{3}$$

The set $Q_{\delta,p}$ is an unbounded set in norm of space Z as A_p is completely continuous operator (see [18]).

However the solution z_{ex} does not get in this set with guarantee as the operator A_p is given approximately.

It is required in many inverse problems to estimate of the exact solution z_{ex} of the equation (1). In such a way the set of the possible solutions should be constructed so that exact solution z_{ex} would belong to set $Q_{\delta,p}$ with guarantee. For this purpose it is necessary to take into account an inaccuracy of the operator A_p.

Let us suppose that the error of the operator A_p is given and so:

$$\| A_{ex} - A_p \|_{Z \to U} \leq h. \tag{4}$$

The set of possible solutions of equation (1) with account of the inaccuracy of operator A_p has the form:

$$Q_{h,\delta} = \{z : z \in Z, \| A_p z - u_\delta \|_U \leq \delta + h \| z \|_Z\}.$$

The set $Q_{h,\delta}$ is an unbounded set in norm of space Z as well (see [18]).

At calculation of the value of h there exist principal difficulties as far as the exact operator A_{ex} cannot be constructed in principle. In the process of solution of practical problems it is possible to obtain only rough rating from above of this size (see [3], [7]).

The theoretical questions of the solution of inverse problems with the approximately given operator by a regularization method are investigated in works [18], [5], [4], [10], [8]. Thus, the initial problem of the solution of the equation (1) is replaced by the following extreme value problem on set of the possible solutions $Q_{h,\delta}$:

$$\Omega[z^0] = \inf_{z \in Q_{h,\delta}} \Omega[z], \tag{5}$$

where $\Omega[z]$ is a stabilizing functional defined on Z_1, Z_1 - an everywhere dense set in Z.

It is possible to interpret this solution as the lowest estimation of the exact solution in the sense of the chosen stabilizing functional $\Omega[z]$. But at the solution of practical problems the regularized solution z_0 coincides with the trivial solution even by with small size of h in such statement (see [7], [3], [8]).

The success in removing the specified lacks resulted in development of a method of special mathematical models (see [7], [8]). Such approach has allowed considerably to increase accuracy of approximate regularized solution of problems with the approximately given operator, and also has allowed to exclude from calculations the size h. Briefly, the method of special mathematical models can be described as follows.

It is assumed that all operators A_p in equation (1) have identical structures and depend continuously on vector parameters of mathematical model $p = (p_1, p_2, ...p_N)^T$ of researched process $p \in R^N$ ($(.)^T$ is the mark of transposition). It is supposed that the parameters of mathematical model are determined inexactly with some error and by virtue of it they can accept values in the known limits $p_i^0 \le p_i \le p_i^1, i = 1, 2, 3, ..., N$. Therefore, the vector parameters p has not been defined inexactly and that it can accept values in some closed area $p \in D \subset R^N$. The operator A_p in (1) will correspond to any vector parameters $p \in D$ and they form some class of operators $K_A = \{A_p\}$.

According accepted assumptions all operators A_p are completely continuous operators. The exact operator A_{ex} has the same structure as operators A_p and corresponds to some vector $p_{ex} \in D$.

Let us consider the union of sets $Q_{p,\delta}$ on all vectors $p \in D$

$$Q^U = \bigcup_{p \in D} Q_{p,\delta}$$

($\bigcup$ is the union).

This set belongs to set $Q_{h,\delta}$ for any $h > 0$ and any $\delta > 0$.

Further, the approached solution z_0 of an inverse problem (1) is found on the set Q^U (see [9], [16]):

$$\Omega[z_0] = \inf_{z \in Q^U \cap Z_1} \Omega[z] = \inf_{A_p \in K_A} \inf_{z \in Q_{p,\delta} \cap Z_1} \Omega[z],$$

where $\Omega[z]$ is a stabilizing functional, defined on Z_1 (Z_1 is an everywhere dense set in Z).

In some cases among the sets $Q_{p,\delta}$ it is possible to select the set (mathematical description or operator $A_{p_0} \in K_A$) with special properties (see [9], [16]). The use in calculations of such set allows to replace the set of the possible solutions Q^U with set $Q_{p_0,\delta}$. In other words, such approach allows to reduce a problem with approached operator to a problem with the fixed operator (see [9], [16]):

$$\Omega[z_0] = \inf_{z \in Q^U \cap Z_1} \Omega[z] = \inf_{A_p \in K_A} \inf_{z \in Q_{p,\delta} \cap Z_1} \Omega[z] = \inf_{z \in Q_{p_0,\delta} \cap Z_1} \Omega[z]. \tag{6}$$

The history of the solution of the first inverse problems (Leverier problem, Krylov problem (see [10]), a problem of Newton about opening of the law of the world gravitation) shows that they were solved without the account of an error of the mathematical description of real physical process that is inadmissible. This contradiction can be removed if we assume that in the solution of the first inverse problems, probably intuitively, the special mathematical models were used. This method can be used at solution of an inverse problems in traditional statement also.

2. Minimax Statement of Inverse Problems

In the process of solution of some inverse problems we face a situation when approximate regularized solution z_0 in statement (5) or (6) does not correspond to ultimate goals of research. One of such problems is the inverse Krylov's problem related to definition of the real pressure in compressors of ship guns [10]. But solution of this problem with help of classical method has not resulted in success.

Detailed description of this inverse problem and method of its solution are given in work (see [10]). Let us to introduce in brief the content of this problem which represents the certain methodological interest.

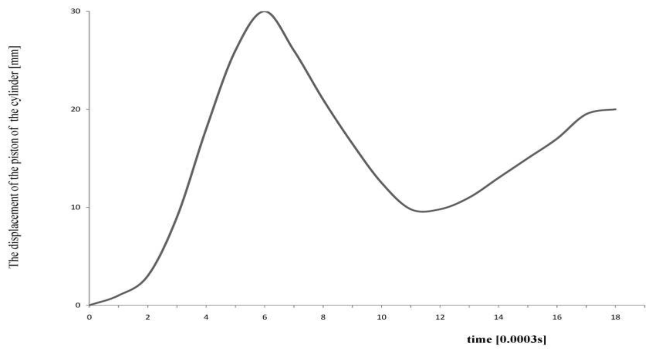

Figure 1: The displacement of the piston of the cylinder during the test

In January 1914 during the test of ready ship guns the diagrams of pressure in the cylinder of the compressor were recorded by a special Vikkers indicator (see [10]). According to the records of this instrument the maximal pressure in the cylinder of the compressor surpassed 45 MPa with the permissible pressure 25 MPa. The replacement of a set of ready guns by new ones would require additional expenses of 2.5 millions gold roubles and the term of readiness of the ships would be delayed significantly. The research carried out by Krylov showed that Vikkers indicator worked during the tests under conditions when the records of the instrument differed considerably from the real ones. During this research the following inverse problem was originally considered by Krylov: to determine the real pressure in the cylinder of the compressor using the curve of piston motion of Vikkers indicator and the equation of mathematical model motion of the indicator (mass on a spring) (see [10]).

The dependence of a piston motion of the indicator in time (Figure 1) was approximated by Krylov by means of three terms. This approximation was done so well, that the error in the uniform metrics did not surpass the thickness of a pencil line on the diagram of motion of the piston of the indicator. Using this information Krylov originally considered the following inverse problem: to determine the real pressure in the cylinder of the compressor $P(t)$, assuming that the known mathematical model of the moving piston on a spring performs the program of motion. As a result of the solution of such an inverse problem was obtained the discontinuous function of pressure (see [10]). Such result does not correspond to physical sense of the problem and Krylov rejected such a method of its solution.

Let us consider this problem from the position of the theory of unstable (incorrect) problems (see [10]).

Mathematical model of piston motion on spring was chosen as following:

$$m\ddot{x} + b\dot{x} + cx = z(t),$$

where m is the mass of piston, b is the coefficient of friction, c is the stiffness of the spring.

Its solution can be reduced to the solution of the integral Volterra equation of the first kind such as (1):

$$\int_0^t \sin\omega_1(t-\tau)\exp(-b(t-\tau))z(\tau)d\tau = A_p z = u_\delta(t) = B_p x_\delta, \qquad (7)$$

where

$$u_\delta(t) = \omega_1\{x_\delta(t) - \exp(-bt)[x_\delta(0)\cos\omega_1 t + \frac{1}{\omega_1}(-bx_\delta(0) + \dot{x}_\delta(0))\sin\omega_1 t]\},$$

$u_\delta(t) \in U; x_\delta(t) \in X$ is the function known from experiment; $\omega_1 = \sqrt{\omega^2 - b^2}$; A_p – linear operator which depends continuously on vector parameters of mathematical model p; $A_p : Z \to U$; B_p – linear operator which depends continuously on vector parameters of mathematical model p too; $B_p : X \to U; z \in Z$.

Let us assume, that $Z = C[0,T]$ ($[0,T]$ is the interval of time, on which the behavior of function $z(t)$ is being investigated), $U = L_2[0,T], X = L_2[0,T]$.

The inaccuracy of the experimentally measured function $x_\delta(t)$ in relation to exact function $x_{ex}(t)$ is given and is equal

$$\| x_\delta(t) - x_{ex}(t) \|_X \leq 0.0011 = \delta.$$

The possible variations of parameters of mathematical model of the piston motion on a spring are the following:

$$m^0 \leq m \leq \hat{m}, \quad b^0 \leq b \leq \hat{b}, \quad c^0 \leq c \leq \hat{c}.$$

Thus the vector parameters which corresponded to the mathematical model of process has the form: $p = (m, b, c)^T$.

Let us assume that

$$m^0 = m = \hat{m} = 1, \quad b^0 = 0.0, \quad \hat{b} = 2, \quad c^0 = \hat{c} = 103.$$

The size of the possible scattering of parameters determines the maximal size of an error of the operator A_p.

Let $p_m = (0.5(m^0 + \hat{m}), 0.5(b^0 + \hat{b}), 0.5(c^0 + \hat{c}))^T = (m_m, b_m, c_m)^T \in D$.

It is supposed that the exact operators A_{ex} and B_{ex} in the equation (7) satisfy the inequalities:

$$\| A_{ex} - A_{p_m} \|_{C \to L_2} \leq \sup_{p \in D} \| A_p - A_{p_m} \|_{C \to L_2} \leq h; \tag{8}$$

$$\| B_{ex} - B_{p_m} \|_{C \to L_2} \leq \sup_{p \in D} \| B_p - B_{p_m} \|_{C \to L_2} \leq d.$$

Let us estimate the maximal size of an inaccuracy of the operators A_p and B_p:

$$\| A_{ex} - A_{p_m} \|_{C \to L_2} \leq \sup_{\|z\|_C \leq 1} \| A_{ex} z - A_{p_m} z \|_{L_2}$$

$$\leq \sup_{p \in D} \sup_{\|z\|_C \leq 1} \{ \int_0^T [\int_0^t K_1(t-\tau) z(\tau) d\tau]^2 dt \}^{0.5}$$

$$\leq \sup_{p \in D} \{ \int_0^T [\int_0^t \mid K_1(t-\tau) \mid d\tau]^2 dt \}^{0.5} = h,$$

where

$$K_1(t-\tau) = \exp(-b_m(t-\tau)) \sin \omega_{1m}(t-\tau) - \exp(-b(t-\tau)) \sin \omega_1(t-\tau),$$

$$\omega_{1m} = \sqrt{\frac{c_m}{m_m} - b_m^2}, \quad \omega_1 = \sqrt{\frac{c}{m} - b^2};$$

$$\| B_{ex} - B_{p_m} \|_{C \to L_2} \leq \sup_{\|x\|_C \leq 1} \| B_{ex} x - B_{p_m x} \|_{L_2}$$

$$\leq \sup_{p \in D} \sup_{\|x\|_C \leq 1} \| B_{ex} x - B_{p_m} x \|_{L_2}$$

$$\leq \sup_{p \in D} \sup_{\|x\|_C \leq 1} \{ \int_0^T [K_2(t-\tau) + K_3(t-\tau)]^2 dt \}^{0.5} = d,$$

where

$$K_2(t-\tau) = \Delta\omega_1 x(t) - \exp(-b_m(t))(-b_m x(0) + \dot{x}(0) \sin \omega_{1m} t + \omega_{1m} x(0) \cos \omega_{1m} t),$$

$$K_3(t-\tau) = \exp(-bt)(-bx(0) + \dot{x}(0)) \sin \omega_1 t + \omega_1 x(0) \cos \omega_1 t;$$

$$\Delta\omega_1 = \omega_{1m} - \omega_1.$$

Then

$$\omega_1 = 10.15, \quad b_m = 1, \quad \omega_{1m} = 10.05.$$

The calculation of h and d in the given problem was carried out by numerical methods with account that the maximal value of vector p is reached in a corner point of area D when $p = p^1$, and also when $x(0) = \dot{x}(0) = 0$. As a result we obtain that $h = 0.09333$, $d = 0.1$. In the given problem we try to find the real pressure on the piston, which should be examined with account of an error of the operators A_p and B_p (see [18], [11], [7]). According to ideology of such problems the set of the possible solutions $Q_{h,d,\delta}$ of the equation (7) is defined in view of an error of the operators A_p and B_p (having in mind that p is accepted as p_m):

$$Q_{h,d,\delta} = \{z : z \in Z, \|A_{p_m} z - B_{p_m} x_\delta\|_U \leq \delta_0 b_0 + d \parallel x_\delta \parallel_C + h \parallel z \parallel_C\},$$

where

$$b_0 = \sup_{p \in D} \parallel B_p \parallel, \quad \parallel A_{ex} - A_{p_m} \parallel_{C \to L_2} \leq h, \quad \parallel B_{ex} - B_{p_m} \parallel_{C \to L_2} \leq d.$$

The exact solution z_{ex} of the equation (7) belongs to the set $Q_{h,d,\delta}$ with guarantee.

The functional $\Omega[z]$ is chosen as follows

$$\Omega[z^0] = \parallel z \parallel^2_{W_2^1[0,T]} = \int_0^T (\dot{z}^2 + z^2) dt. \tag{9}$$

The solution z^0 of an extreme value problem (5) with the set of possible solutions $Q_{h,d,\delta}$ has a norm in $C[0,T]$ which equals 0.59 MPa. Such a result of the solution of an inverse problem cannot give an answer to the question – whether it is necessary to accept or reject the ready ship guns?

The reason is, that in set $Q_{h,d,\delta}$ there are functions which do not give piston motion coinciding with the experimentally measured one with accuracy δ.

To eliminate the negative influence of such way of the account of inaccuracy size of the operators A_p, B_p in the equation (7) here is offered to use at calculations the special mathematical model of object (see [7], [3], [16]).

Let us introduce into consideration the sets

$$X_\delta = \{x : x \in X, \parallel x_\delta - x \parallel_C \leq \delta\};$$

$$U_p = \{u : u \in U, u = B_p x, x \in X_\delta\};$$

$$Q_{\delta,p} = \{z : z \in Z; A_p z \in U_p\};$$

$$Q^* = \cup Q_{\delta,p},$$

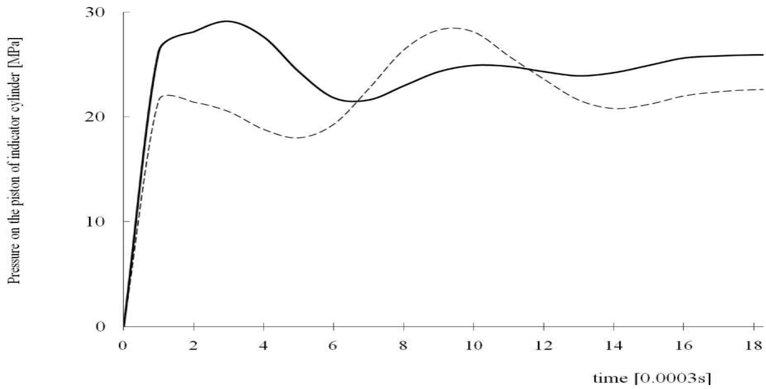

Figure 2: Regularizing solutions of Krylov's inverse problem

where $\cup$ is the union for all $p \in D, z_{ex} \in Q^*$.

It is obvious that $Q^* \subset Q_{h,d,\delta}$ for anyone $\delta > 0$, $d > 0$ and $h > 0$.

To increase the accuracy of the approximate solution it is offered to use an extreme value problem (5) which will replace extreme value problem suggested in [11], [7], [8]:

$$\Omega[z^*] = \inf_{z \in Q^* \cap W_2^1} \Omega[z] = \inf_{p \in D} \inf_{z \in Q_{\delta,p} \cap W_2^1} \Omega[z], \tag{10}$$

where $W_2^1[0,T]$ is the Sobolev functional space.

The statement of Krylov's inverse problem as an extreme value problem (10) is more suitable. In this case each function from the set of the possible solutions Q^* under substitution into the equation (7) gives discrepancy, which does not surpass δ with some possible parameters of mathematical model.

The solution $[z^*]$ of an extreme value problem (10) with the set of possible solutions Q^* has 27MPa in the norm of $C[0,T]$ (dotted line on Figure 2).

However there is no guarantee that the function z^* corresponds to the most favorable parameters of mathematical model. It is quite possible that there exist vector parameters $p \in D$ and the set $Q_{\delta,p}$, where there will be the function which has the lowest norm in $C[0,T]$ larger than 30 MPa. In this case the set of ready ship guns should be rejected.

With this purpose a statement of the following extreme value problem will be more valid:

$$\Omega[z^s] = \sup_{p \in D} \inf_{z \in Q_{\delta,p} \cap W_2^1} \Omega[z]. \tag{11}$$

If the function z^s has the norm in $C[0,T]$ smaller than 30MPa then there is no objective base for the rejection of ship guns. With any set of parameters of mathematical model in a set of possible solutions Q^* there will be a function having the lowest amplitude smaller than 30MPa. In other words, with the most adverse variant of parameters of mathematical model in a set of possible solutions there will be a function with the minimal amplitude smaller than 30MPa.

Theorem 1. *The solution z^s of an extreme value problem* (11) *with functional $\Omega[z]$ such as* (9) *always exists.*

Proof. It is known that the solution of an extreme value problem (11) exists for any vector $p \in D$ if the functional $\Omega[z]$ is stabilizing (see [18]). The functional $\Omega[z]$ is the continuous function of a vector parameters p at the fixed function $x(t) \in X$, i.e. $\Omega[z_p] = \Omega[p]$. By Weierstrass Theorem a least upper bound of function $\Omega[p]$ is reached on closed limited finite-dimensional set D for a vector $p^1 \in D$. Then the function z^s will give the solution of an extreme value problem (11). □

Theorem 2. *If the functional $\Omega[z]$ is stabilizing and if the equation $A_{ex}z = B_{ex}x_{ex}$ has unique solution z_{ex}, then at $h \to 0, d \to 0, \delta \to 0, x^s \overset{C[0,T]}{\to} z_{ex}$.*

Proof. Let $\eta_k = (h_k, d_k, \delta_k)$ is any sequence converging to zero ((h_k, d_k, δ_k) independently converge to zero at $k \to 0$). To every η_k an element corresponds $z_k \in W_2^1[0,T]$. Let $z_k \in Q_{p_0} \subset Q^*$. The set z_k is bounded in $W_2^1[0,T]$. Indeed, $\|z_k\|_{W_2^1[0,T]} \le \|z_{ex}\|_{W_2^1[0,T]} = \Delta$. As far as the operator of imbedding $W_2^1[0,T]$ into $C[0,T]$ is completely continuous, the sequence z_k belongs to the compact set N in $C[0,T]$ which is defined as: $N = \{z : \|z\|_{W_2^1[0,T]} \le \Delta\} \subset C[0,T]$. Hence, from z_k it is possible to choose converging subsequence z_{k_l} such, that $z_{k_l} \overset{C[0,T]}{\to} z_0 \in C[0,T]$ at $l \to \infty$. For convenience we shall save for elements of this sequence the same designations as for an initial sequence, i.e. let z_k converges on norm $C[0,T]$ to z_0. We have

$$\|A_{ex}z_0 - B_{ex}x_{ex}\|_U \le \|A_{ex}z_0 - A_{p_m}x_0\|_U + \|A_{p_m}z_0 - A_{p_m}z_k\|_U$$

$$+ \|A_{p_m}z_k - B_{p_m}x_\delta\|_U + \|B_{p_m}x_\delta - B_p x_\delta\|_U$$

$$+ \|B_p x_\delta - B_p x_{ex}\|_U + \|B_p x_{ex} - B_{ex}x_{ex}\|_U \le \|A_{ex} - A_{p_m}\| \, \|z_0\|_Z$$

$$+ \|A_{p_m}\| \, \|z_0 - z_k\|_Z + \|A_{p_m}z_k - A_{p_0}z_k\|_U + \|A_{p_0}z_k - B_{p_0}x_\delta\|_U$$

$$+ \|B_{p_0}x_\delta - B_{p_m}x_\delta\|_U + \|B_p - B_{p_m}\| \, \|x_\delta\|_X + \|B_p\| \, \delta_k$$

$$+\|B_p - B_{ex}\| \, \|x_{ex}\|_X \le h_k \|z_0\|_Z + \|A_{p_m}\| \, \|z_0 - z_k\|_Z$$
$$+h_k \|z_k\|_Z + 2b_0\delta_k + 2d_k \|x_\delta\|_X + d_k \|x_\delta\|_X$$
$$\le 2h_k\Delta + \|A_{p_m}\| \, \|z_0 - z_k\|_Z + 2b_0\delta_k + 3d_k \|x_{ex}\|_X + d_k\delta_k.$$

From here, in view of continuity and bounded of the operator A_{p_m} at any h_k, convergence $\eta_k \overset{k\to\infty}{\to} 0$ and strong convergence $z_k \overset{k\to\infty}{\to} z_0$, we receive

$$\|A_{ex}z_0 - B_{ex}x_{ex}\|_U = 0.$$

By virtue of prospective uniqueness of the solution of the equation

$$A_{ex}z_{ex} = u_{ex} = B_{ex}x_{ex},$$

we have that $z_0 = z_{ex}$.

But $z_{ex} \in W_2^1[0,T] \subset C[0,T]$. So $z_k \overset{C[0,T]}{\to} z_{ex}$ at $k \to \infty$. As all terms of an initial sequence z_k have the bounded norm in $C[0,T]$ ($\|z_k\|_{C[0,T]} \le \|z_k\|_{W_2^1[0,T]} \le \|z_{ex}\|_{W_2^1[0,T]} = \Delta$), the initial sequence z_k also converges to z_{ex} at $k \to \infty$ in the metrics $C[0,T]$. □

Analogously to see [3], [4], or [10], it is possible to show, that $z_k \overset{W_2^1[0,T]}{\to} z_{ex}$ as $k \to \infty$. Thus, the specified algorithm of the solution of an extreme value problem (11) is regularized (see [18]). For the solution of an extreme value problem (11) the method of special mathematical models is suggested. For this purpose it is necessary to choose among all possible mathematical models of system (operators A_p and B_p in (7)) the mathematical model (operators A_{p^1}, B_{p^1}) for which the inequality is carried out

$$\Omega[A_{p^1}^{-1}B_{p^1}x] \ge \Omega[A_p^{-1}B_px],$$

for any admissible function $x \in X_\delta$ and any vector $p \in D$ (A_p^{-1} is the inverse operator to A_p). We have in mind as admissible function such a function $x(t)$, at which $z(t) \in Z$. Mathematical model for the vector parameters $p^1 \in D$ we shall name as a special maximal mathematical model at the given problem.

If the special maximal mathematical model exists then the problem (11) can be replaced by following more simple extreme value problem

$$\Omega[z^s] = \inf_{z \in Q_{\delta,p^1} \cap Z_1} \Omega[z]. \tag{12}$$

Theorem 3. *The solution of an extreme value problem* (12) *coincides with one of the solutions of an extreme value problem* (11).

The proof of this theorem is given in [11], [8].

Theorem 4. *In the inverse Krylov problem the special maximal mathematical model exists for any admissible function $x \in X_\delta$.*

Proof. Using the general results of the regularization method, we can affirm that the extreme value problem

$$\Omega[z_{\delta,p}] = \inf_{z \in Q_{\delta,p} \cap W_2^1[0,T]} \Omega[z]$$

has a solution for any vector $p \in D$, including $p^1 \in D$ (see [18]). For a fixed admissible function $x \in X$, the functional $\Omega[z]$ is a continuous function of the parameter p:

$$\Omega[\ddot{x}(t) + 2h\dot{x}(t) + \omega^2 x(t)] = \Omega[p].$$

Furthermore,

$$\frac{d\Omega}{dp} = 4\int_0^T (z\dot{x} + \dot{z}\ddot{x})dt = 4p\int_0^T (\dot{x}^2 + \ddot{x}^2)dt$$

$$+2\{\dot{x}^2(T) - \dot{x}^2(0) + \ddot{x}^2(0) + \omega^2[x^2(T) - x^2(0) + \dot{x}^2(T) - \dot{x}^2(0)]\}$$

and the conditions $x(0) = \dot{x}(0) = \ddot{x}^2 = 0$ hold. Therefore, $\frac{d\Omega}{dp} > 0$ for any admissible function $x(t)$. Obviously, the function $\Omega[p]$ has the global maximum at the point $p = p^1$. □

The solution of the extreme value problem (11) has the norm in $C[0,T]$ which equals to 28.9 MPa (Figure 2, dotted line). The solution of the extreme value problem (12) has the norm in $C[0,T]$ which equals to 29.8 MPa (Figure 2, continuous line). Thus there are no objective reasons for the rejection of guns as defective. The minimax statement of inverse problems as in (12) is admitted practically for any inverse problems.

As the second example of such type of a inverse problem we shall consider an inverse problem of an estimation of a rotor unbalance by a method of identification (see [14], [15]). The motion of a rotor in two non rigid supports is described by a system of the ordinary differential equations of 18-th order (see [14], [15]). By analytical transformations three similar integral equations concerning three required characteristics of unbalance $z_1(t), z_2(t), z_3(t)$ are obtained ($z_1(t) = m_r r\dot{\varphi}^2 \sin(\theta + \varphi)$, $z_2(t) = m_r r\dot{\varphi}^2 \cos(\theta + \varphi)$, $z_3(t) = h m_r r\dot{\varphi}^2 \sin(\theta + \varphi)$, r is the radius of rotor, m_r is the mass of unbalance reduced to a surface of rotor, $\dot{\varphi}$ is the angular velocity of rotation, h is unbalance arm, θ is angular deviation of the factor of unbalance with respect to correction plane):

$$\int_0^t K_i(t-\tau)z_i(\tau)d\tau = u_{i,\delta}(t), \quad t \in [0,T]$$

or

$$Az_i = u_{i,\delta} = B_{i,p}x_\delta, \quad z_i \in Z, \quad u_\delta \in U, \quad x_\delta \in X, \quad i = 1,2,3, \tag{13}$$

where Z, U, X are functional spaces, $Z = C[0,T], X = U = L_2[0,T]$; $B_{i,p} : X \to U$.

The vector-function x_δ is obtained from the experiment with the known error δ:

$$\|x_{ex} - x_\delta\|_X \leq \delta,$$

where x_{ex} is an exact response of object to real external load (or unbalance).

It is important to note, that for inverse problems of the investigated type it is necessary to take into account an error of the operator in (13). If this error is not taken into account, the solution of the inverse problems will have another meaning.

Let us assume that the operators $B_{i,p}$ depend on the vector parameters of the mathematical model $p = (p_1, p_2, ...p_n)^T, p \in R^n$. It is supposed that the parameters of mathematical model are determined inexactly with some error and by virtue of it they can accept values in the known limits $p_i^0 \leq p_i \leq \hat{p}_i, i = 1,2,3,...,n$. Therefore, the vector parameter p has not been defined precisely and that it can accept values in some closed area $p \in D \subset R^n$. The operators $B_{i,p}$ in (13) will correspond to any vector parameters $p \in D$ and they form some class of operators $K_{i,B} = \{B_{i,p}\}$. Let us designate by d_i the sizes of the maximal deviation of the operators $B_{i,p}$ from $K_{i,B}$:

$$\|B_{i,ex} - B_{i,p}\|_{X\to U} \leq d_i.$$

It is supposed that the exact operators $B_{i,ex}$ have the structure similar to the structure of $B_{i,p}$ and that vector parameters of $B_{i,ex}$ also belong to the domain D.

In this cases the sets of the possible solutions have the forms:

$$Q_{d_i,\delta} = \{z : z \in Z, \| Az - B_{i,p}x_\delta \|_U \leq \delta_0 b_{i,0} + d_i \| x_\delta \|_Z\},$$

where

$$b_{i,0} = \sup_{p\in D} \|B_{i,p}\|_{X\to U}, \quad i = 1,2,3.$$

The sets $Q_{d_i,\delta}$ are an unbounded sets in the norm of the space Z when the operator A is a completely continues operator (see [18]). Further, the method of Tikhonov regularization for equation with inexactly given operator is a possible way to obtain the steady solution of problem (13) (see [18]). Let $\Omega[z]$ be a stabilizing functional, defined on Z_1 (Z_1 is an everywhere dense set in Z). Let us denote by $z_{i,p}$ the regularized solutions of equations (13):

$$\Omega[z_{i,p}] = \inf_{z \in Q_{d_i,\delta} \cap Z_1} \Omega[z]. \tag{14}$$

In some cases the problem (14) can be transformed to solution of the following problem (see [9], [16])

$$\Omega[z^0_{i,p}] = \inf_{B_{i,p} \in K_{i,B}} \inf_{z \in Q_{i,\delta,p} \cap Z_1} \Omega[z], \tag{15}$$

where $Q_{i,\delta,p} = \{z : z \in Z, \|Az - B_{i,p}x_\delta\|_U \leq \delta b_{i,0}\}$.

It is possible to interpret this solution as the lowest estimation of the exact solution in the sense of the chosen stabilizing functional $\Omega[z]$. However, in some inverse problems such interpretation of the approached solution has no sense. For example, the real unbalance characteristic z_i can distinct from zero and its estimation from below $z_{i,p}$ will be equal to zero. Let us consider minimax statement of an inverse problem of estimation of unbalance characteristics of a rotor. Instead of the extreme value problem solution (15) we shall examine the solutions of the following extreme value problems:

$$\Omega[z^s_{i,p}] = \sup_{B_{i,p} \in K_{i,B}} \inf_{z \in Q_{i,\delta,p} \cap Z_1} \Omega[z], \quad i = 1, 2, 3. \tag{16}$$

It was shown in [13] that for am inverse problem of the unbalance definition there exist maximal special mathematical models for $\Omega[z]$ kind (9) and by $Z_1 = W^1_2[0,T]$. On the basis of this the extreme value problem (16) can be replaced by more simple extreme value problems (12) with special mathematical models.

If the functions $z^s_{i,p} \neq 0$ then the real unbalance is probably distinct from zero. If all functions $z^s_{i,p} = 0$, then the guaranteed conclusion about the size of the real unbalance cannot be taken (see [13]). By the traditional way such an answer cannot be received in principle. The result of its solution is the function which allows to carry out some early diagnostics of rotor unbalance. If this function is equal to zero then the unbalance of a rotor is absent with absolute guarantee. A rotor unbalance probably exists but without any guarantee if this function differs from zero solution. Probably such a statement of inverse problems has sense for the problems of early technical, medical or other diagnostics.

3. The Most Probable Solution of Inverse Problems

Besides, there is a sufficiently wide class of inverse problems which differ from the problems given in (10), (11). For example, the equation (1) of the inverse problem of finding the most probable solution can be considered in the situation when all operators from the class K_A are equivalent. The following inverse problem in this case can be considered (see [12]): to find a function z_{tr} for which the following equality is valid

$$\|A_{a_{tr}} z_{tr} - u_\delta\|_U = \inf_{z_p \in Q_{D,\delta}} \sup_{A_a \in K_A} \|A_a z_p - u_\delta\|_U, \quad a \in D, \tag{17}$$

where

$$Q_{D,\delta} = \{z_p : \Omega[z_p] = \inf_{z \in Q_{p,\delta} \cap Z_1} \Omega[z]\}.$$

The function z_{tr} gives the lowest deviation of the system response from the experiment for all operators simultaneously. So it can be considered as the most probable solution of an inverse problem.

Theorem 5. *The function z_{tr} exists and becoms steady with respect to small variations of initial data if $\Omega[z]$ is a stabilizing functional and the Frecher derivative of $\Omega[z]$ differs from zero.*

The proof of this theorem is given in [12].

Extreme problem (17) can be considered as a problem of synthesis of the model external load for a class operators also (see [11], [16], [12]). In this case a class K_A can consist of final number of the operators $K_A = A_1, A_2, ..., A_N = A_i (i = 1, 2, , 3, ..., N)$. Then the extreme problem (17) is being reduced to following problem:

$$\inf_{z_j \in \hat{Q}_{D,\delta}} \sup_{A_i \in K_A} \|A_i z_j - u_\delta\|_U = \|A_{un} z_{un} - u_\delta\|_U$$

$$= \min_j \max_i \|A_i z_j - u_\delta\|_U,$$

where

$$\hat{Q}_{D,\delta} = \{z_j : \Omega[z_j] = \inf_{z \in Q_{j,\delta} \cap Z_1} \Omega[z]\}, Q_{j,\delta} = \{z : \|A_j z - u_\delta\|_U \leq \delta\}.$$

Function z_{un} was named as unitary model of the external load for a class of operators (see [11]).

It is obvious, that the inequality is correct

$$\delta < \|A_{un} z_{un} - u_\delta\|_U.$$

As an example of the problem of synthesis of the unitary model for class of operators was examined the problem of synthesis of model of the technological resistance moment on the part of metal on a working barrels of rolling mill (see [14], [15]).

In this case functional spaces Z, U are chosen as the spaces of continuous functions with the uniform metrics ($Z[0,T] = U[0,T] = C[0,T]$). The size δ is defined by an error of the measuring equipment and $\delta = 0.0665$ MNm. The operator A_p in this case looks like

$$A_p = \int_0^t \sin\omega(t-\tau)\exp(-b_1(t-\tau))z(\tau)d\tau, \tag{18}$$

where $\omega = \sqrt{\frac{c}{m} - \frac{b^2}{4m}}; b_1 = \frac{-b}{2m}; c, m, b$ are parameters of the mathematical model of the object (c is rigidity on twisting, m is moment of inertia, b is coefficient of friction). Let the class of the operators K_A consists of three operators A_1, A_2, A_3, which have identical structure (18) and are defined by three sets of parameters $p_1 = (c_1, m_1, b_1)^T, p_2 = (c_2, m_2, b_2)^T, p_3 = (c_3, m_3, b_3)^T$. For the chosen structure of the mathematical model of a rolling mill these parameters are equal:

$$c_1 = 737MNm^2/s^2, \quad c_2 = 755MNm^2/s^2, \quad c_3 = 774MNm^2/s^2,$$

$$m_1 = 323KNm^2, \quad m_2 = 336KNm^2, \quad m_3 = 349KNm^2,$$

$$b_1 = 194KNm^2/s, \quad b_2 = 217KNm^2/s, \quad b_3 = 245KNm^2/s.$$

The maximal deviation of the operators $A_p \in K_A$ from one another is defined by an error of parameters of the mathematical model of the rolling mill. The size of the maximal deviation of the operators $A_i \in K_A$ was obtained by numerical methods and it is appeared to be equal $h = 0.121$. As the characteristic of the stability of the solution the such functional $\Omega[z]$ is accepted

$$\Omega[z] = \int_0^T (z^2 + \dot{z}^2)dt.$$

In Figure 3 the diagram of function z_{tr} for a typical case of rolling on a smooth working barrels (top) is submitted (see [14], [15]). For comparison the model of external load for a class of models K_A on the set of the possible solutions $Q_{h,\delta}$ is given. The function, which is the solution of inverse problem in this case has the maximal deviation from zero equal 0.04 MNm. Such function does not represent interest as far as it practically coincides with trivial function.

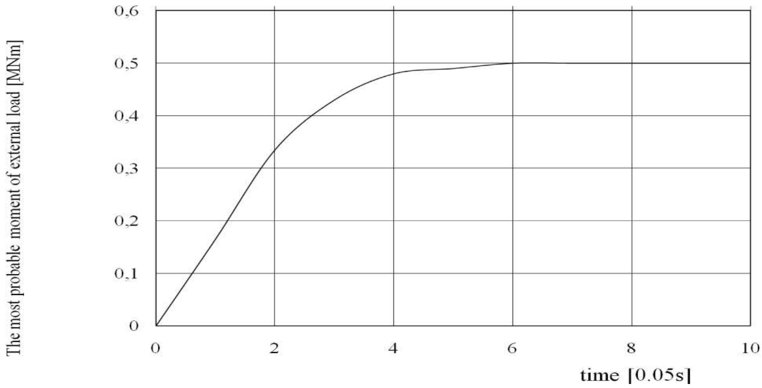

Figure 3: The diagram of the most probable function z_{tr} of the moment of technological resistance on rolling-mill

4. Conclutions

Nonclassical statements of the inverse problems permit us to solve new practical problems and also to extend the possibilities of the regularization method.

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