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COMPARISON RESULTS VIA PICONE-TYPE FORMULAS: TWO OSCILLATION CRITERIA

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Abstract: In this note, the sole use of some Picone-type formulas and some comparison methods allows us to establish some oscillation criteria of some ODE, departing from some known trigonometric functions. Further developments in that direction can be found in the references [1, 2, 3, 4, 5].

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1. Introduction

Because this note concerns oscillatory theory, we start with some definitions and notations.

Definitions 1. Let $f \in C(\mathbb{R}, \mathbb{R})$, Supp$(f)= \{x \in \mathbb{R} \ ; \ f(x) \neq 0 \}$ and $\forall R > 0, \quad D_R := \{x \in \mathbb{R} \ ; \ |x| > R \}$.

a) The function f is said to be:

1) (weakly) oscillatory in $\mathbb{R}$ if $\forall R > 0$, f has a zero in D_R;

2) strongly oscillatory in $\mathbb{R}$ if $\forall R > 0$, D_R contains non empty and bounded connected components of Supp(f). Such a component $\overline{D(f)}$, say, is called a nodal set for f ($f \neq 0$ in $D(f)$ and $f|_{\partial[D(f)]} = 0$).

b) A differential equation will be said to be oscillatory (resp. strongly oscillatory) if any of its non trivial solution is.

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c) A function w defined in $\mathbb{R}$ will then be said non oscillatory if it is eventually non zero, i.e. if $\exists\tau,\ R>0;\ |w|>\tau$ in D_R.

In the sequel, we will use oscillatory for "strongly oscillatory in $\mathbb{R}$". Also the prime denotes the usual differentiation in $\mathbb{R}$.

Two Examples of Oscillatory Equations

$$u'' + k^2 u = 0\ , \quad 0 \neq k \in \mathbb{R}, \tag{1.1}$$

whose general solution is of the form $a\sin kt + b\cos kt\ ;\quad a, b \in \mathbb{R}$.

For $\alpha > 0$ and the function $\phi_\alpha(t) := |t|^{\alpha-1}t$

$$\Big\{\phi_\alpha(u')\Big\}' + \alpha\phi_\alpha(u) = 0, \quad \alpha > 0; \quad \phi_\alpha(t) = |t|^{\alpha-1}t. \tag{1.2}$$

whose solutions are generalized Sine functions $S := S_\alpha$ with the following properties:

$$\begin{cases} |S_\alpha(t)|^{\alpha+1} + |S'_\alpha(t)|^{\alpha+1} = 1\ ; \quad S_\alpha(t+\pi_\alpha) = -S_\alpha(t); \\ \pi_\alpha = \dfrac{2\pi}{(\alpha+1)\sin\{\dfrac{\pi}{\alpha+1}\}}. \end{cases} \tag{1.3}$$

In this note, based on the above examples and some comparison methods by means of Picone-type formulas, we provide some oscillatory criteria for some more general equations. We recall that a (classical) solution of (1.2) is any function $x \in C^2(\mathbb{R})$ which satisfies the equation and $\phi_\alpha(x') \in C^1(\mathbb{R}\ ,\ \mathbb{R})$.

2. Preliminaries

In the sequel the functions f and g are assumed continuous in $\mathbb{R}$.

2.1. Picone-Types Identities

Consider u is an oscillatory solution of (1.1) and v is a non trivial solution of

$$\Big\{a(t)v'\Big\}' + c(t)k^2 v + f(v) + g(t) = 0 \quad \text{in } \mathbb{R}. \tag{2.1}$$

Formally, easy verifications show that: wherever $v \neq 0$

$$\begin{aligned} \left\{uu' - \frac{u^2}{v}a(t)v'\right\}' &= a(t)[u' - \frac{u}{v}v']^2 \\ &+ (1-a(t))|u'|^2 + [c(t)-1]k^2u^2 + \frac{u^2}{v}[f(v)+g(t)]. \end{aligned} \tag{2.2}$$

Similarly if u is a non trivial solution of (1.2) and v that of

$$\left\{a(t)\phi_\alpha(v')\right\}' + c(t)\phi_\alpha(v) + f(v) + g(t) = 0, \tag{2.3}$$

then wherever $v \neq 0$

$$\begin{aligned} \left[u\phi_\alpha(u') - u\phi_\alpha\left(\frac{u}{v}\right)a(t)\phi_\alpha(v')\right]' &= (1-a(t))|u'|^{\alpha+1} + a(t)Z_\alpha(u,v) \\ + (c(t)-\alpha)|u|^{\alpha+1} + \frac{|u|^{\alpha+1}}{\phi_\alpha(v)}\left[f(v)+g(t)\right], \end{aligned} \tag{2.4}$$

where $\forall\alpha > 0$ and $x, y \in C^1(\mathbb{R};\ \mathbb{R})$

$$Z_\alpha(x,y) := |x'|^{\alpha+1} - (\alpha+1)\phi_\alpha\left(\frac{x}{y}y'\right)\frac{x}{y}x'y' + \alpha|\frac{x}{y}y'|^{\alpha+1} \tag{2.5}$$

is non negative and is zero only if $\exists k \in \mathbb{R}\ ;\ x = ky$ (see [2]).

3. Oscillation Criteria

In what follows,

$$\begin{aligned} &(i) \quad a \in C^1(\mathbb{R},\ (0,\ \infty)), \quad c \in C(\mathbb{R},\ (0,\ \infty)) \\ &\qquad \text{and } \ tf(t) > 0\ \forall t \in \mathbb{R}\setminus\{0\}; \\ &(ii) \quad \forall\varepsilon > 0,\ \exists R_\varepsilon > 0\ ;\ |g(t)| < \varepsilon \quad \forall t \in D_{R_\varepsilon}. \end{aligned} \tag{3.1}$$

Theorem 3.1. *(1) If (i) of (3.1) holds with* $0 < a(t) < 1 < c(t)$, $\forall t \in \mathbb{R}$ *then:* $[a(t)v']' + c(t)v + f(v) = 0$ *in* $\mathbb{R}$ *is oscillatory.*

(2) If (i) and (ii) of (3.1) hold then any non trivial solution v $[a(t)v']' + c(t)k^2v + f(v) + g(t) = 0$ *in* $\mathbb{R}$ *is oscillatory unless*

$$\lim_{t\to\infty} \inf\{|v(t)|\} = 0. \tag{3.2}$$

Proof. (1) If we assume that v is such a solution of the equation in (1) and is eventually positive in $\mathbb{R}$, say, then $\exists\rho > 0$; $w > 0$ in D_ρ. Let $D(u) \subset D_\rho$ be such that u is an oscillatory solution of (1.1) with $k = 1$ and $u > 0$ in $D(u)$, $u|_{\partial[D(u)]} = 0$. The integration over $D(u)$ of (2.2) (where $k = 1$ and $g \equiv 0$) leads to a contradiction as the left hand side is zero while the right is strictly positive.

(2) This time for a similar solution v of the equation in (2), assume that $\exists\tau, \rho > 0$; $v > \tau$ in D_ρ and let $D(u) \subset D_\rho$ and u be as above. Then from the integration over $D(u)$ of (2.2)

$$0 = \int_{D(u)} \left\{ a(t)[u' - \frac{u}{v}v']^2 + (1 - a(t))|u'|^2 + [c(t) - 1]k^2u^2 + \frac{u^2}{v}[f(v) + g(t)] \right\} dt.$$

The only integrand which might be non positive is $\frac{u^2}{v}g(t)$ and $\int_{D(u)} \frac{u^2}{v}g(t)dt$ can be made arbitrary small by chosing a much larger ρ as $|\frac{u^2}{v}g(t)| < \frac{|u|^2_\infty}{\tau}\varepsilon'$. The same conclusion holds. ☐

Theorem 3.2. *Assume that (3.1),(i) holds and* $\forall t \in \mathbb{R}$, $0 < a(t) < 1$ *and* $c(t) > \alpha$.

Then if $g \equiv 0$ *(2.3) is oscillatory.*

If in addition (3.1) (ii) holds then any non trivial solution v *of (2.3) is oscillatory unless (3.2) holds.*

The proof follows the same pattern as for Theorem 3.1.

3.1. Concluding Remarks

1) If a perturbation term $h(t, u, u')$ with continuous h is added to the equations, the conclusions stand if either $\forall s \in \mathbb{R} \setminus \{0\} sh(t, s, u') > 0$ or $t \mapsto h(t, u, u')$ satisfies (3.1) (ii) for any fixed u, u'.

2) The oscillation criteria in the theorems above can be "adapted" to multidimensional equations (see [2]).

3)The conclusions of the Theorems are that v must have a zero in those $D(u)$ implying that meas$\{D(u) \bigcap \text{Supp}(v)\} > 0$. Therefore as u, v is also strongly oscillatory.

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112