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A HILBERT-TYPE INEQUALITY WITH VARIATION OF PARAMETERS

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Abstract: By applying the method of the weight function, a new Hilbert-type inequality with multi-parameter and a best constant factor is established, and the best possible range of the parameters which made the inequality valid is considered. Meanwhile, we obtain generalizations of the classic Hilbert's inequality and some Hilbert-type inequalities.

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Key Words: Hilbert-type inequality, weight function, Hölder's inequality

1. Introduction

In 1908, D. Hilbert established the following well known Hilbert's inequality (see [2]): If $f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)dx, \int_0^\infty g^2(x)dx < \infty$, then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible.

Inequality (1.1) is important in analysis and its applications (see [10]). By introducing parameter, a lot of generalizations of the Hilbert's inequality appeared in the literature and inequality (1.1) had been generalized into (see [11]-[13]):

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$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy < \frac{\pi}{\lambda} \left\{ \int_0^\infty x^{1-\lambda} f^2(x)dx \int_0^\infty x^{1-\lambda} g^2(x)dx \right\}^{\frac{1}{2}}, \quad (1.2)$$

where the constant factor $\frac{\pi}{\lambda}$ ($\lambda > 0$) is the best possible. Under the same conditions of (1.1), we have the following basic Hilbert-type inequality (see [2]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < 4 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}. \quad (1.3)$$

We are interested in some new inequality generated by the kernel operation of the basic Hilbert-type inequalities. The following Hilbert-type inequality has been established (see [9], [14])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dxdy < c \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}, \quad (1.4)$$

$$\int_0^\infty \int_0^\infty \frac{|x-y| f(x)g(y)}{(\max\{x,y\})^2} dxdy < \frac{8}{3} \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}, \quad (1.5)$$

where the constant factors $c = \sqrt{2}(\pi - 2\arctan\sqrt{2})$ and $\frac{8}{3}$ are the best possible.

In this article, by applying the weight function and introducing parameters p, q, a, b, λ, contacting with (1.1)-(1.4), a Hilbert-type inequality with a best constant factor is established. Meanwhile, the best possible range of the parameters a, b which made the inequality valid is considered.

2. Some Lemmas

For each $\lambda > 0$, we define the positive functions $A(x)$ and $C_\lambda(a,b)$ as

$$A(x) := \begin{cases} \frac{2}{\lambda\sqrt{-x}} \arctan\sqrt{-x}, & x < 0, \\ \frac{2}{\lambda}, & x = 0, \\ \frac{1}{\lambda\sqrt{x}} \ln\frac{1+\sqrt{x}}{1-\sqrt{x}}, & 0 < x < 1, \end{cases} \quad (2.1)$$

$$C_\lambda(a,b) := \frac{1}{|a+1|} A(\frac{a-b}{a+1}) + \frac{1}{|a+b|} A(\frac{a-1}{a+b}), \quad (2.2)$$

where $(a,b) \in D = \{(a,b)|(1+a)(1+b) > 0, (1+a)(a+b) > 0\} = D_1 \cup D_2$, and $D_1 = \{(a,b)|1+a > 0, 1+b > 0, a+b > 0\}$, $D_2 = \{(a,b)|1+a < 0, 1+b < 0\}$.

Lemma 2.1. *Suppose $\lambda > 0, (x,y) \in (0,\infty) \times (0,\infty), (a,b) \in R^2$. Define the weight functions as follows*

$$\omega_1(y;\lambda,a,b) := \int_0^\infty \frac{y^{\frac{\lambda}{2}} x^{\frac{\lambda}{2}-1}}{|a|x^\lambda - y^\lambda| + bx^\lambda + y^\lambda|} dx;$$

$$\omega_2(x;\lambda,a,b) := \int_0^\infty \frac{x^{\frac{\lambda}{2}} y^{\frac{\lambda}{2}-1}}{|a|x^\lambda - y^\lambda| + bx^\lambda + y^\lambda|} dy. \tag{2.3}$$

(i) Noting the function $\varphi_\lambda(x,y) = a|x^\lambda - y^\lambda| + bx^\lambda + y^\lambda$, then when $(a,b) \in D_1$, we have $\varphi_\lambda(x,y) > 0$; when $(a,b) \in D_2$, we have $\varphi_\lambda(x,y) < 0$.

(ii) When $(a,b) \in D$, the two integrals of (2.3) are convergent. Moreover, we get

$$\omega_1(y;\lambda,a,b) = \omega_2(x;\lambda,a,b) = C_\lambda(a,b). \tag{2.4}$$

Proof. (i) When $0 \le t \le 1$, $\varphi_1(t,1) = (b-a)t + 1 + a$, by monotonicity

$$\min\{1+a, 1+b\} \le \varphi_1(t,1) \le \max\{1+a, 1+b\},$$

when $(a,b) \in D_1$, we get $\varphi_1(t,1) > 0$; when $(a,b) \in D_2$, we get $\varphi_1(t,1) < 0$.

When $t > 1$, $\varphi_1(t,1) = (b+a)t + 1 - a$; when $(a,b) \in D_1$, we get $\varphi_1(t,1) > \varphi_1(1,1) = 1 + b > 0$; when $(a,b) \in D_2$, we get $\varphi_1(t,1) < \varphi_1(1,1) = 1 + b < 0$.

Moreover, $\varphi_\lambda(x,y) = y^\lambda \varphi_1(\frac{x^\lambda}{y^\lambda}, 1)$. Hence the conclusion (i) is valid.

(ii) Let $I := \frac{2}{\lambda} \int_0^\infty \frac{1}{a|u^2-1|+bu^2+1} du ((a,b) \in D)$. By $\varphi_2(u,1) = a|u^2 - 1| + bu^2 + 1$ and the conclusion (i), we have

$$|I| = \frac{2}{\lambda} \int_0^\infty \frac{1}{|a|u^2 - 1| + bu^2 + 1|} du. \tag{2.5}$$

Setting $u = (\frac{x}{y})^{2\lambda}$, by simple calculating, the two integrals of (2.3) turn into

$$\omega_1(y;\lambda,a,b) = \omega_2(x;\lambda,a,b) = \frac{2}{\lambda} \int_0^\infty \frac{1}{|a|u^2 - 1| + bu^2 + 1|} du. \tag{2.6}$$

If the following integrals are correct for $(a,b) \in D_1 \bigcup D_2$,

$$I_1 := \frac{2}{\lambda} \int_0^1 \frac{1}{(b-a)u^2 + 1 + a} du = \frac{1}{a+1} A(\frac{a-b}{a+1});$$

$$I_2 := \frac{2}{\lambda} \int_1^\infty \frac{1}{(b+a)u^2 + 1 - a} du = \frac{1}{a+b} A(\frac{a-1}{a+b}), \tag{2.7}$$

then $I = I_1 + I_2$ is convergent. Furthermore, $a+1$ and $a+b$ have the same sign and $A(x) > 0$, we have $|I| = \frac{1}{|a+1|}A(\frac{a-b}{a+1}) + \frac{1}{|a+b|}A(\frac{a-1}{a+b}) = C_\lambda(a,b)$. Hence by (2.5) and (2.6), (2.4) is correct.

Below, we will prove (2.7). First, we calculate the integral as follows:

(i) When $|1+a| > |b-a| > 0$,

$$\begin{aligned} I_{11} := & \frac{2}{\lambda}\int_0^1 \frac{1}{|b-a|u^2 - |1+a|}du \\ = & \frac{2}{\lambda}\cdot\frac{1}{2\sqrt{|(b-a)(1+a)|}}\ln\frac{\sqrt{|1+a|}-\sqrt{|b-a|}}{\sqrt{|1+a|}+\sqrt{|b-a|}} = \frac{-1}{|1+a|}A(|\frac{a-b}{1+a}|); \end{aligned}$$

(ii) when $1+a \neq 0, b-a \neq 0$,

$$\begin{aligned} I_{12} := & \frac{2}{\lambda}\int_0^1 \frac{1}{|b-a|u^2 + |1+a|}du \\ = & \frac{2}{\lambda}\cdot\frac{1}{\sqrt{|(b-a)(1+a)|}}\arctan(\sqrt{|\frac{b-a}{1+a}|}) = \frac{1}{|1+a|}A(-|\frac{a-b}{1+a}|); \end{aligned}$$

(iii) when $1+a \neq 0, I_{13} := \frac{2}{\lambda}\int_0^1 \frac{1}{|1+a|}du = \frac{2}{\lambda|1+a|}$;

(iv) when $b+a \neq 0, 1-a \neq 0$, in view of $\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}$, we have

$$\begin{aligned} I_{21} := & \frac{2}{\lambda}\int_1^\infty \frac{1}{|b+a|u^2 + |1-a|}du \\ = & \frac{2}{\lambda\sqrt{|(b+a)(1-a)|}}(\frac{\pi}{2} - \arctan(\sqrt{|\frac{b+a}{1-a}|})) \\ = & \frac{2}{\lambda\sqrt{|(b+a)(1-a)|}}\arctan\sqrt{|\frac{1-a}{b+a}|} = \frac{1}{|b+a|}A(-|\frac{1-a}{b+a}|); \end{aligned}$$

(v) when $|b+a| > |1-a| > 0$, we have

$$\begin{aligned} I_{22} := & \frac{2}{\lambda}\int_1^\infty \frac{1}{|b+a|u^2 - |1-a|}du \\ = & \frac{2}{\lambda}\cdot\frac{1}{2\sqrt{|(b+a)(1-a)|}}\ln\frac{\sqrt{|b+a|}u-\sqrt{|1-a|}}{\sqrt{|b+a|}u+\sqrt{|1-a|}}|_1^\infty \\ = & \frac{1}{\lambda\sqrt{|(b+a)(1-a)|}}\ln\frac{\sqrt{|b+a|}+\sqrt{|1-a|}}{\sqrt{|b+a|}-\sqrt{|1-a|}} = \frac{1}{|b+a|}A(|\frac{1-a}{b+a}|); \end{aligned}$$

(vi) when $1+b\neq 0$,

$$I_{23} := \frac{2}{\lambda}\int_1^{\infty}\frac{1}{|b+1|u^2}du = \frac{2}{\lambda|1+b|}.$$

Hence when $(a,b)\in D_1$, we have

$$I_1 = \frac{2}{\lambda}\int_0^1\frac{1}{(b-a)u^2+1+a}du = \begin{cases} -I_{11}, & b-a<0, \\ I_{12}, & b-a>0, \\ I_{13}, & b=a, \end{cases} = \frac{1}{a+1}A(\frac{a-b}{a+1}),$$

$$I_2 = \frac{2}{\lambda}\int_1^{\infty}\frac{1}{(b+a)u^2+1-a}du = \begin{cases} I_{21}, & a<1, \\ I_{22}, & a>1, \\ I_{23}, & a=1, \end{cases} = \frac{1}{a+b}A(\frac{a-1}{a+b});$$

when $(a,b)\in D_2$, we have $0<1-a<-b-a$, i.e. $|1-a|<|b+a|$, and

$$I_1 = \frac{2}{\lambda}\int_0^1\frac{1}{(b-a)u^2+1+a}du = \begin{cases} -I_{12}, & b-a<0, \\ I_{11}, & b-a>0, \\ -I_{13}, & b=a, \end{cases} = \frac{1}{a+1}A(\frac{a-b}{a+1}),$$

$$I_2 = \frac{2}{\lambda}\int_1^{\infty}\frac{1}{(b+a)u^2+1-a}du = -I_{22} = \frac{1}{a+b}A(\frac{a-1}{a+b}).$$

So when $(a,b)\in D_1\cup D_2$, (2.7) is valid. The lemma is proved. □

Lemma 2.2. *Setting* $\lambda>0, p>1, (1/p)+(1/q)=1, n\in N, n>\frac{2}{\lambda p}$, $C_\lambda(a,b)$ *is taken as the definition of (2.2), and letting*

$$f_n(x) = \begin{cases} x^{\frac{\lambda}{2}-1-\frac{1}{np}}, & x\in[1,\infty), \\ 0, & x\in[0,1), \end{cases} \quad g_n(x) = \begin{cases} x^{\frac{\lambda}{2}-1-\frac{1}{nq}}, & x\in[1,\infty), \\ 0, & x\in[0,1), \end{cases} \tag{2.8}$$

when $(a,b)\in D$, *we have*

$$J_n := \frac{1}{n}\int_0^{\infty}\int_0^{\infty}\frac{f_n(x)g_n(y)}{|a|x^\lambda-y^\lambda|+bx^\lambda+y^\lambda|}dxdy \to C_\lambda(a,b)\ (n\to\infty), \tag{2.9}$$

and when $(a,b)\notin D$, J_n *is divergent.*

Proof. Setting $\varphi_\lambda(x,y)$ is taken as the definition of Lemma 2.1 and

$$J_{n1} := \frac{2}{\lambda}\int_1^{\infty}\frac{u^{-\frac{2}{\lambda pn}}}{|\varphi_2(u,1)|}du; \quad J_{n2} := \frac{2}{\lambda}\int_0^1\frac{u^{\frac{2}{\lambda qn}}}{|\varphi_2(u,1)|}du. \tag{2.10}$$

And letting $u = (\frac{x}{y})^{\lambda/2}$, we have $\varphi_\lambda(x,y) = y^\lambda \varphi_2(u,1)$ and

$$J_n = \frac{1}{n}[\int\int_{1\le y\le x} \frac{f_n(x)g_n(y)}{|\varphi_\lambda(x,y)|}dxdy + \int\int_{1\le x\le y} \frac{f_n(x)g_n(y)}{|\varphi_\lambda(x,y)|}dxdy]$$
$$= \frac{1}{n}\{\int_1^\infty y^{\frac{\lambda-2}{2}-\frac{1}{qn}}[\int_y^\infty \frac{x^{\frac{\lambda-2}{2}-\frac{1}{pn}}}{|\varphi_\lambda(x,y)|}dx]dy + \int_1^\infty x^{\frac{\lambda-2}{2}-\frac{1}{pn}}[\int_x^\infty \frac{y^{\frac{\lambda-2}{2}-\frac{1}{qn}}}{|\varphi_\lambda(x,y)|}dy]dx\}$$
$$= \frac{1}{n}\int_1^\infty y^{-1-\frac{1}{n}}J_{n1}dy + \frac{1}{n}\int_1^\infty x^{-1-\frac{1}{n}}J_{n2}dx = J_{n1} + J_{n2}. \quad (2.11)$$

Since $u^{-\frac{2}{\lambda pn}}(u > 1)$ and $u^{\frac{2}{\lambda qn}}(0 < u < 1)$ are both increasing for n, by the Levi's Monotone Convergence Theorem (see [6]) and (2.6) and (2.4), so we have $J_{n1} + J_{n2} \to C_\lambda(a,b)$ $(n \to \infty)$, we get (2.9) by (2.11).

Noting $G_1 = \{(a,b)|1+a \le 0, 1+b \ge 0\}$, $G_2 = \{(a,b)|1+a \ge 0, 1+b \le 0\}$, $G_3 = \{(a,b)|a+b \le 0, 1+a > 0, 1+b > 0\}$, then $R^2 - D = \bigcup_{i=1}^{3} G_i$, and when $(a,b) \in G_1$, we have $b - a \ge -(1+a) \ge 0$, then $J_{n2} = \frac{2}{\lambda}\int_0^1 \frac{u^{\frac{2}{\lambda qn}}}{|(b-a)u^2+(1+a)|}du$ is divergent; when $(a,b) \in G_2$, we have $-(b-a) \ge 1+a \ge 0$, then J_{n2} is divergent too; when $(a,b) \in G_3$, we have $1-a > -(b+a) \ge 0$, then $J_{n1} = \frac{2}{\lambda}\int_1^\infty \frac{u^{-\frac{2}{\lambda pn}}}{|(b+a)u^2+(1-a)|}du$ $(0 < \frac{2}{\lambda pn} < 1)$ is divergent. Thus when $(a,b) \in R^2 - D$, J_n is divergent. □

3. Main Results and Applications

Theorem 3.1. *Suppose that* $\lambda > 0, p > 1, (1/p) + (1/q) = 1$, $C_\lambda(a,b)$ *is taken as the definition of (2.2),* $f(x), g(x) \ge 0$ *are real measurable functions such that* $0 < \int_0^\infty x^{p(1-\frac{\lambda}{2})-1}f^p(x)dx < \infty$ *and* $0 < \int_0^\infty x^{q(1-\frac{\lambda}{2})-1}g^q(x)dx < \infty$, *when* $(a,b) \in D = \{(a,b)|(1+a)(1+b) > 0, (1+a)(a+b) > 0\}$, *we have*

$$J := \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{|a|x^\lambda - y^\lambda| + bx^\lambda + y^\lambda|}dxdy$$
$$< C_\lambda(a,b)\{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1}g^q(x)dx\}^{\frac{1}{q}}, \quad (3.1)$$

where the constant factor $C_\lambda(a,b)$ *is the best possible. When* $(a,b) \notin D$, *(3.1) is not valid.*

Proof. By Hölder's inequality with the weight (see [7]) and (2.3)-(2.4), we obtain

$$J=\int_0^\infty\int_0^\infty \frac{1}{|a|x^\lambda-y^\lambda|+bx^\lambda+y^\lambda|}[\frac{x^{(1-\frac{\lambda}{2})/q}}{y^{(1-\frac{\lambda}{2})/p}}f(x)][\frac{y^{(1-\frac{\lambda}{2})/p}}{x^{(1-\frac{\lambda}{2})/q}}g(y)]dxdy$$

$$\leq\{\int_0^\infty\int_0^\infty \frac{1}{|a|x^\lambda-y^\lambda|+bx^\lambda+y^\lambda|}\frac{x^{(1-\frac{\lambda}{2})(p-1)}}{y^{(1-\frac{\lambda}{2})}}f^p(x)dxdy\}^{1/p}$$

$$\times\{\int_0^\infty\int_0^\infty \frac{1}{|a|x^\lambda-y^\lambda|+bx^\lambda+y^\lambda|}\frac{y^{(1-\frac{\lambda}{2})(q-1)}}{x^{(1-\frac{\lambda}{2})}}g^q(y)dxdy\}^{1/q}$$

$$=\{\int_0^\infty \omega_2(x;\lambda,a,b)x^{p(1-\frac{\lambda}{2})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty \omega_1(y;\lambda,a,b)y^{q(1-\frac{\lambda}{2})-1}g^q(y)dy\}^{\frac{1}{q}}$$

$$=C_\lambda(a,b)\{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1}g^q(x)dx\}^{\frac{1}{q}}. \quad (3.2)$$

If (3.2) takes the form of equality, then there exists constants A and B, which are not all zero (without loss of the generality, suppose $A\neq 0$), such that (see [7]):

$$A\frac{x^{(1-\lambda/2)(p-1)}}{y^{1-\lambda/2}}f^p(x)=B\frac{y^{(1-\lambda/2)(q-1)}}{x^{1-\lambda/2}}g^q(y) \text{ a.e. on } (0,\infty)\times(0,\infty).$$

Hence there exists $y\in(0,\infty)$, such that $x^{p(1-\lambda/2)-1}f^p(x)=y^{q(1-\lambda/2)-1}g^q(y)\frac{B}{Ax}$ a.e. on $(0,\infty)$, which contradicts the fact that $0<\int_0^\infty x^{p(1-\lambda/2)-1}f^p(x)dx<\infty$. Hence (3.2) takes the form of strict inequality, we obtain (3.1).

Suppose there exists a positive number $K\leq C_\lambda(a,b)$, such that (3.1) is still valid that $C_\lambda(a,b)$ is instead of K. In particular, (3.1) is valid for the function $f_n(x),g_n(y)$ which is defined by (2.8). By (2.9), we have

$$J_n<\frac{K}{n}\{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1}f_n^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty y^{q(1-\frac{\lambda}{2})-1}g_n^q(x)dx\}^{\frac{1}{q}}=K,$$

thus $C_\lambda(a,b)\leq K$ for $n\to\infty$. Hence $K=C_\lambda(a,b)$ and the constant factor $C_\lambda(a,b)$ in (3.1) is the best possible.

By Lemma 2.2, when $(a,b)\notin D$, J_n is divergent, (3.1) is not valid. □

Corollary 3.1. *Suppose that $\lambda>0,p,q,f(x),g(y)$ satisfy the condition of Theorem 3.1, and $(a,b,c)\in\widetilde{D}=\{(a,b,c)|(a+c)(c+b)>0,(a+c)(a+b)>0\}$, then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{|a|x^\lambda-y^\lambda|+bx^\lambda+cy^\lambda|}dxdy$$

$$< \widetilde{C}_\lambda(a,b,c)\{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1} g^q(x)dx\}^{\frac{1}{q}}, \tag{3.3}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|a|x^\lambda - y^\lambda| + cx^\lambda + by^\lambda|} dxdy$$

$$< \widetilde{C}_\lambda(a,b,c)\{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1} g^q(x)dx\}^{\frac{1}{q}}, \tag{3.4}$$

where the constant factor $\widetilde{C}_\lambda(a,b,c) := \frac{1}{|a+c|}A(\frac{a-b}{a+c}) + \frac{1}{|a+b|}A(\frac{a-c}{a+b})$ *is the best possible. In particular:*

(i) taking $a = 0, b, c > 0$*, then* $\widetilde{C}_\lambda(0,b,c) = \frac{\pi}{\lambda\sqrt{bc}}$*, we have*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{bx^\lambda + cy^\lambda} dxdy$$

$$< \frac{\pi}{\lambda\sqrt{bc}}\{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1} g^q(x)dx\}^{\frac{1}{q}}; \tag{3.5}$$

(ii) taking $b = c, ab + b^2 > 0$*, we have*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|a|x^\lambda - y^\lambda| + b(x^\lambda + y^\lambda)|} dxdy$$

$$< \frac{2}{|a+b|}A(\frac{a-b}{a+b})\{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1} g^q(x)dx\}^{\frac{1}{q}}. \tag{3.6}$$

Proof. In view of $(a,b,c) \in \widetilde{D} \Leftrightarrow (a,c,b) \in \widetilde{D}$, and $\widetilde{C}_\lambda(a,b,c) = \widetilde{C}_\lambda(a,c,b)$, interchanging b with c in (3.3) or (3.4), obviously, (3.3) is equivalent to (3.4).

For $c \neq 0$, replacing a, b with $\frac{a}{c}, \frac{b}{c}$ in (3.1) respectively, then D turns into $\widetilde{D}$, (3.1) becomes into (3.3). Interchanging b with c in (3.3) ($c \neq 0$), then (3.4) ($b \neq 0$) is valid. If $c = 0$ and $(a,b,0) \in \widetilde{D}$, then $b \neq 0$, by (3.4), we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|a|x^\lambda - y^\lambda| + by^\lambda|} dxdy$$

$$< \widetilde{C}_\lambda(a,b,0)\{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1} g^q(x)dx\}^{\frac{1}{q}}, \tag{3.7}$$

interchanging $p, f(x), g(y)$ with $q, g(x), f(y)$ in the above inequality respectively, then interchanging x with y of the left hand side of the inequality, we get that (3.3) is valid for $c = 0$. This completes the proof. □

Remark 3.1. (3.3) is equivalent to (3.1). D is the best area for parameters a, b which make (3.1) valid; and $\widetilde{D}$ is best area for parameters a, b, c which make (3.3) valid.

Remark 3.2. (i) Taking $\lambda = 1, p = q = 2, a = 0, b = 1$ in (3.1), we have (1.1); (ii) taking $p = q = 2, b = c = 1$ in (3.5), we get (1.2); (iii) in view of $|x-y| + x + y = 2\max\{x, y\}$, taking $p = q = 2, \lambda = a = b = 1$ in (3.1), we get (1.3); (iv) taking $p = q = 2, \lambda = b = 1, a = \frac{1}{3}$ in (3.6), then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)dxdy}{|x-y| + 3(x+y)} < (\sqrt{2}\arctan\frac{1}{\sqrt{2}})\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\}^{\frac{1}{2}},$$

in view of $|x-y|+3(x+y) = 2[(x+y)+\max\{x, y\}]$, $\arctan\frac{1}{\sqrt{2}} = \frac{\pi}{2} - \arctan\sqrt{2}$, the above inequality comes into (1.4). Hence (3.1) or (3.3) is the generalization with best constant factors of (1.1)-(1.4).

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