

THE HODGE-DE RHAM DECOMPOSITION THEOREM
AND AN APPLICATION TO A PARTIAL
DIFFERENTIAL EQUATION

Paul Bracken

Department of Mathematics
University of Texas
Edinburg, TX, 78541-2999, USA
e-mail: bracken@panam.edu

Abstract: The Hodge-de Rham Theorem is introduced and discussed. This result has implications for the general study of several partial differential equations. Some propositions which have applications to the proof of this theorem are used to study some related results concerning a class of partial differential equation in a novel way.

AMS Subject Classification: 58A05, 58A10, 47J05

Key Words: differential forms, harmonic, decomposition, adjoint

*

1. Let M be a compact, orientable Riemannian manifold. Let $E^p(M)$ denote the space of smooth p -forms on M . An inner product can be defined on the vector space $E^p(M)$ of p -forms on M with all the usual properties (see [3], [4]) by writing

$$(\alpha, \beta) = \int_M \alpha \wedge * \beta, \quad (1)$$

for $\alpha, \beta \in E^p(M)$ and $*$ is the Hodge operator, see [4]. The corresponding norm is denoted by $\|\alpha\|$. Suppose that α and β are forms of degree p and $p+1$, respectively. Stokes' Theorem can be used to show that

$$\int_M d\alpha \wedge * \beta = (-1)^{p-1} \int_M \alpha \wedge d * \beta. \quad (2)$$

Using (1), this can be put in the form

$$(d\alpha, \beta) = (\alpha, \delta\beta). \quad (3)$$

The operators d and δ are adjoints or duals of each other, see [1], [2]. Analogously, if β is of degree $p - 1$, then (2) gives $(\alpha, d\beta) = (\delta\alpha, \beta)$. Thus in order that α be closed, it is necessary and sufficient that it be orthogonal to all co-exact forms of degree p . An operator called the Laplace-Beltrami operator is defined in terms of d and δ as

$$\Delta = d\delta + \delta d.$$

The form $\omega \in E^p(M)$ is called harmonic if $\Delta\omega = 0$. Let $H^p(M)$ denote the subspace of $E^p(M)$ of harmonic forms on M . From (3), it is clear that Δ is a self-adjoint operator. It is clear that the harmonic forms of a given degree will form a linear space.

Let β be a p -form on a compact, orientable Riemannian manifold M . If there is a p -form α such that $\Delta\alpha = \beta$, then for a harmonic form $\gamma \in H^p$

$$(\beta, \gamma) = (\Delta\alpha, \gamma) = (\alpha, \Delta\gamma) = 0.$$

Therefore, in order that a form α exist with the property that $\Delta\alpha = \beta$, it is necessary that β be orthogonal to the subspace H^p of $E^p(M)$. This also turns out to be sufficient. This reasoning might be thought to lead to the idea that $E^p(M)$ can be partitioned into three distinct subspaces which will be denoted Λ_d^p and Λ_δ^p and H^p . These subspaces are respectively the ones which consist of those forms which are exact, co-exact and harmonic, respectively. These subspaces should be orthogonal in pairs, so forms belonging to distinct subspaces are orthogonal. This is in fact the case and when formulated more precisely is referred to as the Hodge-de Rham Decomposition Theorem. It is the intention here to introduce this theorem and some associated results used in the proof. These results and techniques are then used to prove some related theorems which are of great interest in a new way.

The Hodge-de Rham Theorem can be stated in a number of ways and two are now presented, see [1, 5, 6].

HdR 1. A regular differential form of degree p may be uniquely decomposed into a sum of the form

$$\alpha = \alpha_d + \alpha_\delta + \alpha_H,$$

where $\alpha_d \in \Lambda_d^p$, $\alpha_\delta \in \Lambda_\delta^p$ and $\alpha_H \in H^p$.

HdR 2. For each integer p with $0 \leq p \leq n$, H^p is finite dimensional and there is the direct sum decomposition of the space $E^p(M)$ of smooth p -forms on M given by

$$E^p(M) = \Delta(E^p) \oplus H^p = d\delta(E^p) \oplus \delta d(E^p) \oplus H^p.$$

Consequently, the equation $\Delta\omega = \alpha$ has a solution in $E^p(M)$ if and only if the p -form α is orthogonal to the space H^p .

2. The Hodge Theorem is connected to the problem of finding necessary and sufficient conditions for there to exist a solution ω for the equation $\Delta\omega = \alpha$. Suppose that ω is a solution of $\Delta\omega = \alpha$, then

$$(\Delta\omega, \vartheta) = (\alpha, \vartheta), \tag{4}$$

for all $\vartheta \in E^p(M)$. Usually no distinction is made between Δ and the adjoint Δ^* , however, this distinction is important in making the following definition. Thus from (4), we can write

$$(\omega, \Delta^*\vartheta) = (\alpha, \vartheta). \tag{5}$$

Equation (5) suggests that a solution of $\Delta\omega = \alpha$ can be regarded as a certain type of linear functional on $E^p(M)$. In fact, ω determines a bounded linear functional L on $E^p(M)$ as follows

$$L(\beta) = (\omega, \beta). \tag{5}$$

Using (5), it follows that L satisfies

$$L(\Delta^*\vartheta) = (\alpha, \vartheta), \tag{7}$$

for all $\vartheta \in E^p(M)$. Such a linear functional is referred to as a *weak* solution of $\Delta\omega = \alpha$, or to state it another way, a weak solution of $\Delta\omega = \alpha$ is a bounded linear functional $L : E^p(M) \rightarrow \mathbb{R}$ such that (7) holds. Each ordinary solution determines a weak solution by (6). In fact, the converse is true, each weak solution determines an ordinary solution. The proposition which follows is fundamental for what will come.

Proposition 1. *Let $\alpha \in E^p(M)$ and suppose L is a weak solution of $\Delta\omega = \alpha$. Then there exists $\omega \in E^p(M)$ such that*

$$L(\beta) = (\omega, \beta) \tag{8}$$

for every $\beta \in E^p(M)$. Consequently $\Delta\omega = \alpha$.

This proposition as well as the next will be used in the sequel.

Proposition 2. *Let $\{\alpha_n\}$ be a sequence of smooth p -forms on M such that $\|\alpha_n\| \leq c$ and $\|\Delta\alpha_n\| \leq c$ for all n and for some constant $c > 0$. Then a subsequence of $\{\alpha_n\}$ is a Cauchy sequence in $E^p(M)$.*

Assuming the Propositions 1 and 2, a number of interesting results can be developed including a proof of (HdR). It is the intention now to use these to carry this out.

Theorem 1. *For each integer p with $0 \leq p \leq n$, the space H^p has finite dimension.*

Proof. If H^p were not finite dimensional, then H^p would contain an infinite orthonormal sequence $\{e_n\}$. Now $\|e_n\| \leq 1$ and $\|\Delta e_n\| = 0$ since $e_n \in H^p$ for all n . By Proposition 2, a subsequence of $\{e_n\}$ is a Cauchy sequence in $E^p(M)$. Extract this sequence and call it simply $\{e_n\}$. Let $\epsilon < 1$ be given, then since this is Cauchy, there exists an $N(\epsilon)$ such that for $n, m > N(\epsilon)$, $\|e_n - e_m\| < \epsilon$. However, this is not possible since the e_n are orthonormal so $\|e_n - e_m\|^2 = 2$. \square

Lemma 1. *Let H denote the projection operator of $E^p(M)$ onto H^p so that $H(\alpha)$ is the harmonic part of α , which will also be denoted α_H .*

(i) *For any $\alpha \in E^p(M)$, there exists a unique p -form $H(\alpha) \in H^p$ with the property that $(\alpha, \gamma) = (H(\alpha), \gamma)$ for all $\gamma \in H^p$.*

(ii) *For any p -form $\alpha \in E^p(M)$, H satisfies $H(H(\alpha)) = H(\alpha)$.*

Proof. From Theorem 1, let e_1, \dots, e_m be an orthonormal basis for H^p . Then an arbitrary form $\alpha \in E^p(M)$ can be written uniquely in the form

$$\alpha = \gamma + \sum_{i=1}^m (\alpha, e_i) e_i, \quad (9)$$

where $\gamma \in (H^p)^\perp$, the subspace of $E^p(M)$ consisting of all elements orthogonal to H^p . Therefore,

$$H(\alpha) = \sum_{i=1}^m (\alpha, e_i) e_i = \alpha - \gamma,$$

and $(H(\alpha), \beta) = (\alpha - \gamma, \beta) = (\alpha, \beta)$ for all $\beta \in H^p$ since $\gamma \in (H^p)^\perp$. Moreover, since H^p is a linear space $H(H(\alpha)) = H(\alpha) - H(\gamma) = H(\alpha)$, since $\gamma \in (H^p)^\perp$. \square

Consequently, Theorem 1 yields an orthogonal direct sum decomposition of $E^p(M)$ which takes the form $E^p(M) = (H^p)^\perp \oplus H^p$. The Hodge-de Rham Theorem then follows from this result by showing that $(H^p)^\perp = \Delta(E^p)$. Clearly, $\Delta(E^p) \subset (H^p)^\perp$ since if $\omega \in E^p(M)$ and $\alpha \in H^p$, then

$$(\Delta\omega, \alpha) = (\omega, \Delta\alpha) = 0.$$

To complete the proof of the Hodge Theorem, the converse of this must be shown, namely $H^p \subset \Delta(E^p)$. To do so along the present lines, the following lemma is required, and it is stated without proof. \square

Lemma 2. *There exists a constant $k > 0$ such that*

$$\|\gamma\| \leq k \|\Delta\gamma\|, \quad \gamma \in (H^p)^\perp. \quad (10)$$

Suppose $\alpha \in (H^p)^\perp$. Define a linear functional L on $\Delta(E^p)$ by setting

$$L(\Delta\gamma) = (\alpha, \gamma), \quad \gamma \in E^p(M).$$

To show that L is well defined, suppose that $\Delta\gamma_1 = \Delta\gamma_2$, then $\gamma_1 - \gamma_2 \in H^p$, which means that $(\alpha, \gamma_1 - \gamma_2) = 0$. Also L is a bounded linear functional on $\Delta(E^p)$, for let $\gamma \in E^p(M)$ and $\psi = \gamma - H(\gamma) \in (H^p)^\perp$. Using (10) in Lemma 2 and Cauchy-Schwarz

$$|L(\Delta\gamma)| = |L(\Delta\psi)| = |(\alpha, \psi)| \leq \|\alpha\| \|\psi\| \leq k \|\alpha\| \|\Delta\psi\| = k \|\alpha\| \|\Delta\gamma\|.$$

By the Hahn-Banach Theorem, L extends to a bounded linear functional on $E^p(M)$, and so L is a weak solution of $\Delta\omega = \alpha$. By Proposition 1, it may be concluded that there exists $\omega \in E^p(M)$ such that $\Delta\omega = \alpha$. Therefore, it may be concluded that $(H^p)^\perp = \Delta(E^p)$, and this gives (HdR2).

3. Consequently, for a given p -form α , there must exist a p -form β which satisfies the equation

$$\Delta\beta = \alpha - H(\alpha). \quad (11)$$

Thus from the Hodge Theorem and Lemma 1, it follows that $\alpha_d + \alpha_\delta = \alpha - H(\alpha)$ and β can be taken such that $\Delta\beta = \alpha_d + \alpha_\delta$. Let β_1 and β_2 be any two forms which satisfy (11). Subtracting the two equations obtained from (11) which contain β_1 and β_2 , there remains the equation $\Delta(\beta_1 - \beta_2) = 0$. This result

implies that $\beta_1 - \beta_2 \in H^p$, and so modulo harmonic forms, there exists a unique solution to (11). This solution will be denoted here as $\beta = G(\alpha)$. From Lemma 1, it is clear that β can be projected down onto $\Delta(E^p)$, and it is only this part that need be substituted for β in (11). Thus it suffices to suppose that $G(\alpha)$ has no harmonic contribution. Substituting β into (11), we obtain

$$\alpha = \Delta G(\alpha) + H(\alpha), \quad (G(\alpha), \sigma) = 0, \quad (12)$$

for all $\sigma \in H^p$. This can now be summarized as Theorem 2.

Theorem 2. *Operator $G : E^p(M) \rightarrow (H^p)^\perp$ is defined by setting $G(\alpha)$ to be the unique solution of (11) in $(H^p)^\perp$. For a given p -form α , there exists a p -form $G(\alpha)$ which satisfies differential equation (11), namely $\Delta G(\alpha) = \alpha - H(\alpha)$, and satisfies the associated condition given in (12).*

It is important to realize that G has important commutation properties which are revealed in the following theorem.

Theorem 3. *G commutes with any linear operator which commutes with the Laplacian Δ .*

Theorem 4. (i) *G is a self-adjoint operator*

$$(G\alpha, \beta) = (\alpha, G\beta), \quad (13)$$

for forms α and β of the same degree p .

(ii) *G is a Hermitean positive operator,*

$$(G\alpha, \alpha) \geq 0, \quad (14)$$

and equality holds if and only if α is harmonic.

Proof. (i) Based on Theorem 2, let α and γ be p -forms such that $\Delta G\alpha = \alpha - \alpha_H$ and $\Delta G\gamma = \gamma - \gamma_H$. From orthogonality, it follows that $(\alpha_H, \gamma) = (\alpha, \gamma_H)$. Consequently, by means of the relations,

$$(\Delta G\alpha, \gamma) = (\alpha - \alpha_H, \gamma) = (\alpha, \gamma) - (\alpha_H, \gamma),$$

$$(\alpha, \Delta G\gamma) = (\alpha, \gamma - \gamma_H) = (\alpha, \gamma) - (\alpha, \gamma_H),$$

it can be concluded that

$$(\Delta G\alpha, \gamma) = (\alpha, \Delta G\gamma).$$

Since Δ and G commute and G is self-adjoint, we can write the two expressions

$$(\Delta G\alpha, \gamma) = (G\alpha, \Delta\gamma), \quad (\alpha, \Delta G\gamma) = (\alpha, G\Delta\gamma).$$

From these, it can be concluded that

$$(G\alpha, \Delta\gamma) = (\alpha, G\Delta\gamma).$$

Set $\beta = \Delta\gamma$ in this result and the required follows $(G\alpha, \beta) = (\alpha, G\beta)$. It suffices to show this much, since the range of G does not intersect the harmonic subspace. If β has a harmonic term, it will not make a contribution, since it projects out giving zero by orthogonality of these subspaces.

(ii) Since $G\alpha$ has no component in H^p , it follows by applying Theorem 2 that

$$(G\alpha, \Delta G\alpha + H(\alpha)) = (G\alpha, \Delta G\alpha) = (\delta G\alpha, \delta G\alpha) + (dG\alpha, dG\alpha) \geq 0,$$

since $(\sigma, \sigma) \geq 0$ for any p -form σ .

Suppose that $\alpha \in H^p$, then $(G\alpha, \alpha) = 0$ by orthogonality. Suppose that $(G\alpha, \alpha) = 0$ with $\alpha, G\alpha$ not zero forms. Then since $G\alpha \in (H^p)^\perp$, α must lie in the orthogonal subspace, $\alpha \in H^p$ as discussed in Lemma 1, so α is purely harmonic. \square

4. Consider the Laplace-Beltrami operator Δ which acts on p -forms for some fixed p . A real number λ for which there is a p -form ω which is not identically zero such that $\Delta\omega = \lambda\omega$ is called an eigenvalue of Δ and the corresponding p -form ω an eigenfunction of Δ corresponding to λ . The eigenfunctions corresponding to a fixed λ form a subspace of $E^p(M)$, namely the eigenspace of λ . Operator G can also be used to discuss some of the properties of the eigenvalues and their forms. It is useful to start by reviewing some basic properties of these objects which will be assumed and used later.

The eigenvalues of Δ are non-negative. Suppose that $\alpha \in E^p(M)$ such that $\Delta\alpha = \lambda\alpha$. It follows that $(\Delta\alpha, \alpha) = \lambda(\alpha, \alpha)$, which can be written in the equivalent form $(\delta\alpha, \delta\alpha) + (d\alpha, d\alpha) = \lambda\|\alpha\|^2$. With the exception of λ , all the terms in this equality are known to be positive, therefore λ must be positive as well.

The eigenspaces of Δ are finite dimensional. Suppose this is not the case, then there exists an eigenspace which contains an infinite orthonormal sequence or basis, which will be called $\{e_n\}$ such that $\|e_n\| = 1$ and $\|\Delta e_n\| \leq \lambda$ for all n . Then as in Theorem 1, this orthonormal sequence would contain a Cauchy subsequence, which is not possible.

The eigenvalues have no finite accumulation point. Suppose the eigenvalues have such a value, μ . Let $\{e_n\}$ be a set of orthonormalized eigenfunctions of Δ associated to the eigenvalues $\{\lambda_j\}$, which accumulate to μ , with one selected each respective eigenspace. Then $\|e_j\| = 1$ is bounded for all j and $\|\Delta e_j\| = \lambda_j$. Since the λ_j accumulate to the value μ , $\sup_j \|\Delta e_j\| \leq C$. Then there is a subsequence of the $\{e_j\}$ which is Cauchy, and again this is not possible.

The eigenfunctions corresponding to distinct eigenvalues are orthogonal. Let $\alpha, \beta \in E^p(M)$ be eigenfunctions with $\Delta\alpha = \lambda\alpha$ and $\Delta\beta = \mu\beta$. Using the first of these, $(\Delta\alpha, \beta) = \lambda(\alpha, \beta)$. Since Δ is self-adjoint, this is equivalent to $(\alpha, \Delta\beta) = \lambda(\alpha, \beta)$. However, using the second equation gives $(\alpha, \Delta\beta) = \mu(\alpha, \beta)$. Combining these, it must be that $\lambda(\alpha, \beta) = \mu(\alpha, \beta)$. If $\lambda \neq \mu$, then it follows that $(\alpha, \beta) = 0$, which means α and β are orthogonal.

5. The operator G can be of use in discussing the existence of eigenvalues and some of their properties. Clearly, zero is an eigenvalue of Δ and only if there are nontrivial harmonic p -forms on M . The eigenspace agrees exactly with the space H^p of harmonic forms on M .

Theorem 5. *The eigenvalues of G acting on $(H^p)^\perp$ are the reciprocals of the nonzero eigenvalues of Δ acting on $(H^p)^\perp$.*

Proof. Consider the restriction $\Delta : (H^p)^\perp \rightarrow (H^p)^\perp$, which matches the domain of G as well. Any $\omega \in (H^p)^\perp$ satisfies $\Delta G\omega = \omega$, which implies $G\Delta\omega = \omega$, since G and Δ commute. Suppose ω is an eigenfunction of δ so $\Delta\omega = \lambda\omega$ for some $\lambda \in \mathbb{R}^+$. By the opening remark, we have $\omega = G\Delta\omega = \lambda G\omega$. Since $\lambda \neq 0$ for such ω , this implies that $G\omega = \lambda^{-1}\omega$. \square

The following theorem ensures that this is not an empty statement.

Theorem 6. *The Laplacian Δ has a positive eigenvalue and in fact a whole sequence of eigenvalues which diverge to $+\infty$.*

Proof. The argument which follows can be applied the first time to conclude the existence of a least positive eigenvalue λ_1 . Iterating this procedure, a sequence of eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n$ is generated with corresponding eigenfunctions representative of each subspace $\omega_1, \omega_2, \dots, \omega_n$ for Δ . These can be orthonormalized as well. The set of eigenfunctions $\{\omega_1, \dots, \omega_n\}$ span a subspace of $(H^p)^\perp$ which will be called V_n . Clearly both G and Δ map $(H^p \oplus V_n)^\perp \rightarrow (H^p \oplus V_n)^\perp$. Define μ_{n+1} to be

$$\mu_{n+1} = \sup_{\substack{\|\beta\|=1 \\ \beta \in (H^p \oplus V_n)^\perp}} \|G\beta\|. \quad (15)$$

It will be shown that $\lambda_{n+1} = \mu_{n+1}^{-1}$ is an eigenvalue of the Laplacian Δ . Moreover,

$$\mu_{n+1} = \sup_{\substack{\|\beta\|=1 \\ \beta \in (H^p \oplus V_n)^\perp}} \|G\beta\| \leq \sup_{\substack{\|\beta\|=1 \\ \beta \in (H^p \oplus V_{n-1})^\perp}} \|G\beta\| = \mu_n, \quad (16)$$

on account of the fact that $(H^p \oplus V_n)^\perp \subset (H^p \oplus V_{n-1})^\perp$. Inequality (16) will be used to conclude that $\lambda_{n+1} \geq \lambda_n$ at the end. Let $\{\beta_j\} \in (H^p \oplus V_n)^\perp$ be a maximizing sequence for μ_{n+1} , in other words $\|\beta_j\| = 1$ and $\|G\beta_j\| \rightarrow \mu_{n+1}$ as $j \rightarrow \infty$. The first claim to be shown that will be subsequently used is that $\|G^2\beta_j - \mu_{n+1}^2\beta_j\| \rightarrow 0$. Since G is self-adjoint, it follows that

$$\begin{aligned} & \|G^2\beta_j - \mu_{n+1}^2\beta_j\|^2 = \|G^2\beta_j\|^2 - 2\mu_{n+1}^2(G^2\beta_j, \beta_j) + \mu_{n+1}^4 \\ & \leq \mu_{n+1}^2 \|G\beta_j\|^2 - 2\mu_{n+1}^2 \|G\beta_j\|^2 + \mu_{n+1}^4 = \mu_{n+1}^4 - \mu_{n+1}^2 \|G\beta_j\|^2 \rightarrow 0, \quad j \rightarrow \infty. \end{aligned} \quad (17)$$

Moreover from (17) it follows that $\|G\beta_j - \mu_{n+1}\beta_j\| \rightarrow 0$. Define $\psi_j = G\beta_j - \mu_{n+1}\beta_j$, then

$$\begin{aligned} & (\psi_j, G^2\beta_j - \mu_{n+1}^2\beta_j) \\ & = (\psi_j, G(\psi_j + \mu_{n+1}\beta_j) - \mu_{n+1}^2\beta_j) = (\psi_j, G\psi_j + \mu_{n+1}(G\beta_j - \mu_{n+1}\beta_j)) \\ & = (\psi_j, G\psi_j + \mu_{n+1}\psi_j) = (\psi_j, G\psi_j) + \mu_{n+1} \|\psi_j\|^2 \geq \mu_{n+1} \|\psi_j\|^2 > 0. \end{aligned} \quad (18)$$

The last inequality follows from the fact that G is a positive operator, as noted in (14). Since the left-hand side of (18) approaches zero as $j \rightarrow \infty$, these inequalities force $\|\psi_j\|^2 \rightarrow 0$ as $j \rightarrow \infty$ as well.

Now $\|G\beta_j\|$ is bounded along with $\|\Delta G\beta_j\| = \|\beta_j\|$. Therefore, by Proposition 2, there must be a subsequence of these β_j , which may as well be called $\{\beta_j\}$, such that $\{G\beta_j\}$ is a Cauchy sequence. Consequently, a linear functional L can be defined on $E^p(M)$ by means of the following limit

$$L(\sigma) = \lim_{j \rightarrow \infty} \mu_{n+1} (G\beta_j, \sigma), \quad \sigma \in E^p(M). \quad (19)$$

Then for $u \in (H^p \oplus V_n)^\perp$, next replace σ in (19) by $(\Delta - \frac{1}{\mu_{n+1}})u$. Since Δ is a self-adjoint operator, the definition in (19) takes the following form

$$\begin{aligned} & L((\Delta - \frac{1}{\mu_{n+1}})u) = \lim_{j \rightarrow \infty} \mu_{n+1} (G\beta_j, \Delta u - \frac{1}{\mu_{n+1}}u) \\ & = \lim_{j \rightarrow \infty} \mu_{n+1} (\Delta G\beta_j, u) - \mu_{n+1} (\frac{1}{\mu_{n+1}}G\beta_j, u) = \lim_{j \rightarrow \infty} \mu_{n+1} (\frac{1}{\mu_{n+1}}(\mu_{n+1}\beta_j - G\beta_j), u) \end{aligned}$$

$$= \lim_{j \rightarrow \infty} ((\mu_{n+1}\beta_j - G\beta_j), u) = 0.$$

Therefore, this L is a non-trivial, weak solution of the equation

$$\left(\Delta - \frac{1}{\mu_{n+1}}\right)u = 0. \quad (20)$$

Setting $\lambda_{n+1} = \mu_{n+1}^{-1}$, Proposition 1 can be applied to this. For $u \in E^p(M)$ and L a weak solution of the equation $\Delta u - \lambda_{n+1}u = 0$, there then exists an $\omega_{n+1} \in E^p(M)$ such that $L(\beta) = (\omega_{n+1}, \beta)$ for every $\beta \in E^p(M)$. Consequently, this ω_{n+1} satisfies the equation $\Delta\omega_{n+1} = \lambda_{n+1}\omega_{n+1}$. Finally, this implies that λ_{n+1} so defined is an eigenvalue of Δ . \square

To finish it might be of interest to formulate a very useful result which takes advantage of many of the facts and results which have been produced up to now.

Theorem 7. *Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of Δ on $E^p(M)$ such that each eigenvalue is included as many times as the dimension of its eigenspace. Let $\{\omega_k\}$ be a corresponding sequence of eigenfunctions. For any $\alpha \in E^p(M)$,*

$$\lim_{n \rightarrow \infty} \left\| \alpha - \sum_{k=1}^n (\alpha, \omega_k) \omega_k \right\| = 0. \quad (21)$$

Proof. Let m be the dimension of the space H^p . There exists $\beta \in (H^p)^\perp$ such that

$$G\beta = \alpha - \sum_{k=1}^m (\alpha, \omega_k) \omega_k,$$

as in Lemma 1. The set $\{\omega_i\}$ can be orthonormalized, which means that $(\omega_i, \omega_j) = 0$ when $i \neq j$ and consequently,

$$\begin{aligned} \left\| \alpha - \sum_{k=1}^n (\alpha, \omega_k) \omega_k \right\| &= \left\| \alpha - \sum_{k=1}^m (\alpha, \omega_k) \omega_k - \sum_{k=m+1}^n (\alpha, \omega_k) \omega_k \right\| \\ &= \left\| G\beta - \sum_{k=m+1}^n (G\beta + \sum_{j=1}^m (\alpha, \omega_j) \omega_j, \omega_k) \omega_k \right\| = \left\| G\beta - \sum_{k=m+1}^n (G\beta, \omega_k) \omega_k \right\| \\ &= \left\| G(\beta - \sum_{k=m+1}^n (\beta, \omega_k) \omega_k) \right\|, \end{aligned}$$

for $n > m$. However, for any $v \in (H^p \oplus V_n)^\perp$, from Theorem 6, it is known that $\|Gv\| \leq \mu_{n+1} \|v\|$. It then follows that

$$\left\| G\left(\beta - \sum_{k=m+1}^n (\beta, \omega_k)\omega_k\right) \right\| \leq \frac{1}{\lambda_{n+1}} \left\| \beta - \sum_{k=m+1}^n (\beta, \omega_k)\omega_k \right\| \leq \frac{1}{\lambda_{n+1}} \|\beta\|.$$

Since $\|\beta\|$ is a fixed number, by the results developed pertaining to the eigenvalues, the right-hand side goes to zero as $n \rightarrow \infty$. \square

References

- [1] S.S. Chern, *Topics in Differential Geometry*, Institute for Advanced Study Notes (1951).
- [2] S.S. Chern, W.H. Chen, K.S. Lam, *Lecture Notes on Differential Geometry*, University Mathematics, Volume 1, World Scientific, Singapore (1999).
- [3] S. Goldberg, *Curvature and Homology*, Dover, NY (1970).
- [4] S. Morita, *Geometry of Differential Forms*, AMS Translations of Mathematical Monographs, Volume 209, Providence (2001).
- [5] G. de Rham, *Variétés Différentiables*, Hermann (1955).
- [6] F. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer, NY (1983).

28