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WEIGHTED COMPOSITION OPERATORS AND HYPERCYCLICITY

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Abstract: In this paper we give some sufficient conditions for the adjoint of a weighted composition operator on a Banach function space to be hypercyclic.

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1. Introduction

From now on let X be a Banach space of functions analytic on the open unit disc $\mathbb{D}$ such that for each λ in $\mathbb{D}$ the linear functional of evaluation at λ given by $e_\lambda(f) = f(\lambda)$ is a bounded linear functional on X.

Let T be a bounded linear operator on X. For $f \in X$, the orbit of f under T is the set of images of f under the successive iterates of T: $orb(T, f) = \{f, Tf, T^2f, \ldots\}$. The vector f is called hypercyclic for T if $orb(T, f)$ is dense in X. Also a hypercyclic operator is one that has a hypercyclic vector.

A complex-valued function ψ on $\mathbb{D}$ is called a multiplier of X if $\psi X \subset X$. The operator of multiplication by ψ is denoted by M_ψ and is given by $f \longrightarrow \psi f$. By the closed graph theorem M_ψ is bounded.

If w is a multiplier of X and φ is a mapping from $\mathbb{D}$ into G such that $f \circ \varphi \in X$ for all $f \in X$, then $M_w C_\varphi$ defined by $M_w C_\varphi f = w.f \circ \varphi$ is called a weighted composition operator. We define the iterates $\varphi_n = \varphi \circ \varphi \circ \ldots \circ \varphi$

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(n times). In this paper we investigate the property of hypercyclicity criterion for the adjoint of a weighted composition operator acting on a Banach space of analytic functions. For some sources on hypercyclic topics, see [1]-[14].

2. Main Results

The formulation of the hypercyclicity criterion in the following theorem was given by J. Bes in his Ph.D. Thesis (see [1], also [2]).

Theorem 1. (The Hypercyclicity Criterion) *Suppose X is a separable Banach space and $T \in B(X)$. If there exist two dense subsets Y and Z in X and a sequence $\{n_k\}$ such that:*

1. $T^{n_k}y \to 0$ for every $y \in Y$, and

2. There exist functions $S_{n_k} : Z \to X$ such that for every $z \in Z, S_{n_k}z \to 0$, and $T^{n_k}S_{n_k}z \to z$,

then T is hypercyclic.

Throughout this section let $w : \mathbb{D} \to \mathbb{C}$ be a non-constant multiplier of X and φ be an analytic univalent map from $\mathbb{D}$ onto $\mathbb{D}$. By φ_n^{-1} we mean the n-th iterate of φ^{-1}.

Theorem 2. *Suppose that the composition operator C_φ is bounded on X and w is a nonconstant multiplier of X such that the sets $E_m = \{\lambda \in \mathbb{D} : \prod_{i=1}^{\infty} w(\varphi_i^m(\lambda))^m = 0\}$ have limit points in $\mathbb{D}$ for $m = -1, 1$. If for each $\lambda \in E_m$ the sequence $\{K_{\varphi_i^m(\lambda)}\}_i$ is bounded for $m = -1, 1$, then $(M_w C_\varphi)^*$ hypercyclic.*

Proof. Put $A = M_w C_\varphi$ and $\varphi_0 = I$ where I is the identity mapping on $\mathbb{D}$. Then for all $n \in \mathbb{N}$ and all λ in $\mathbb{D}$ we get $(A^*)^n e_\lambda = \left(\prod_{i=0}^{n-1} w(\varphi_i(\lambda))\right) e_{\varphi_n(\lambda)}$. Put $X_{E_m} = span\{e_\lambda : \lambda \in E_m\}$ for $m = -1, 1$. If $\lambda \in E_1$, then we have $\prod_{i=0}^{\infty} w(\varphi_i(\lambda)) = 0$ and so $\lim_n (A^*)^n e_\lambda = 0$. Thus $(A^*)^n \longrightarrow 0$ pointwise on X_{E_1} that is dense in X^* because the zeros of f has limit point in $\mathbb{D}$. First consider the special case where the collection of linear functionals of point evaluations $\{e_\lambda : \lambda \in E_{-1}\}$ is linearly independent. Define $B : X_{E_{-1}} \longrightarrow X^*$ by extending the definition

$$Be_\lambda = (w(\varphi^{-1}(\lambda)))^{-1} e_{\varphi^{-1}(\lambda)} \qquad (\lambda \in E_{-1})$$

linearly to $X_{E_{-1}}$. Clearly $B^n e_\lambda = (\prod_{i=1}^{n}(w(\varphi_i^{-1}(\lambda)))^{-1}).e_{\varphi_n^{-1}(\lambda)}$, where φ_i^{-1} is the i-th iterate of φ^{-1} and $n \in \mathbb{N}$. By the definition of B we have $A^*Be_\lambda =$

$A^*((w(\varphi^{-1}(\lambda)))^{-1}e_{\varphi^{-1}(\lambda)}) = e_{\varphi(\varphi^{-1}(\lambda))} = e_\lambda$ for all λ in E_{-1}. Thus A^*B is identity on the dense subset $X_{E_{-1}}$ of X^*. Note that if $\lambda \in E_{-1}$, then $\lim_n(\prod_{i=1}^n |w(\varphi_i^{-1}(\lambda))|^{-1}).e_{\varphi_n^{-1}(\lambda)} = 0$. This implies that $B^n \longrightarrow 0$ pointwise on $X_{E_{-1}}$ that is dense in X^*. Thus $A^* = (M_wC_\varphi)^*$ satisfies the hypercyclicity criterion and so is hypercyclic.

In the case that linear functionals of point evaluations are not linearly independent, by the same way we can use a standard method as in Theorem 4.5 in [6] to complete the proof: consider a countable dense subset $F_1 = \{\lambda_n : n \geq 1\}$ of E_{-1} and inductively choose a subsequence $\{z_n\}$ as follows. Let $z_1 = \lambda_1$. Now define $F_2 = F_1 \backslash \{\lambda \in F_1 : e_\lambda \in span\{e_{z_1}\}\}$. Denote the first element of F_2 by z_2 and define $F_3 = F_2 \backslash \{\lambda \in F_2 : e_\lambda \in span\{e_{z_1}, e_{z_2}\}\}$. By continuing this manner, we obtain a subset $G = \{z_n\}_n$ of E_{-1} for which the set $X_G = span\{e_\lambda : \lambda \in G\}$ is dense in X^* with linearly independent linear functionals of point evaluations $\{e_\lambda : \lambda \in G\}$. Now for each $n \in \mathbb{N}$, define the mappings $S_n : X_G \longrightarrow X^*$ by extending the definition

$$S_n e_\lambda = (\prod_{i=1}^n (w(\varphi_i^{-1}(\lambda)))^{-1})e_{\varphi_n^{-1}(\lambda)} \qquad (\lambda \in G)$$

linearly to X_G. Note that if we substitute $\varphi_n^{-1}(\lambda)$ instead of λ in the formula obtained earlier for $(A^*)^n e_\lambda$, we get

$$\begin{aligned} (A^*)^n e_{\varphi_n^{-1}(\lambda)} &= \left(\prod_{i=0}^{n-1} w(\varphi_i(\varphi_n^{-1}(\lambda)))\right) e_{\varphi_n(\varphi_n^{-1}(\lambda))} \\ &= \prod_{i=1}^n (w(\varphi_i^{-1}(\lambda)))e_\lambda \end{aligned}$$

for all λ in G. Now by the definition of S_n we have

$$(A^*)^n S_n e_\lambda = (A^*)^n((\prod_{i=1}^n (w(\varphi_i^{-1}(\lambda)))^{-1})e_{\varphi_n^{-1}(\lambda)}) = e_\lambda$$

for all λ in G. Thus for all $n \in \mathbb{N}$, $(A^*)^n S_n$ is identity on the dense subset X_G of X^*. Also, exactly as before it is proved that $B^n \longrightarrow 0$ pointwise on $X_{E_{-1}}$, we can see that $S_n \longrightarrow 0$ pointwise on X_G that is dense in X^*. This completes the proof. □

Corollary 3. *Suppose that h is a nonconstant multiplier of X such that range of h intersects the unit circle. Then the adjoint of the multiplication operator M_h satisfies the hypercyclicity criterion.*

Proof. In Theorem 2, let φ be identity and $w = h$. Then $\varphi_n(\lambda) = \lambda$ and $\varphi_n^{-1}(\lambda) = \lambda$ for all λ in $\mathbb{D}$. Also, we note that the condition $1 \in X$ implies that $h \in X$ and so h is analytic on the open unit disc $\mathbb{D}$. Now by the Open Mapping Theorem $h(\mathbb{D})$ is open. But ran $h = h(\mathbb{D})$ intersects the unit circle, thus for $m = -1, 1$, the sets $F_m = \{\lambda \in \mathbb{D} : |h(\lambda)|^m < 1\}$ are nonempty and open sets in $\mathbb{D}$ and so clearly have limit points in $\mathbb{D}$. But $F_m \subset E_m$ where $E_m = \{\lambda \in \mathbb{D} : \prod_{i=1}^{\infty} h(\lambda)^m = 0\}$ for $m = -1, 1$. Now we can apply the proof of Theorem 2, and so the proof follows immediately. □

From now on we suppose that for some $n \geq 1$, $\varphi_n = \varphi_0$.

Corollary 4. *If ran* $(\prod_{i=0}^{n-1} w \circ \varphi_i)$ *intersects the unit circle, then the adjoint of the operator* $(M_w C_\varphi)^n$ *satisfies the hypercyclicity criterion.*

Proof. Put $A = M_w C_\varphi$. Then for all $n \in \mathbb{N}$ and all λ in $\mathbb{D}$ we get

$$(A^*)^n e_\lambda = \left(\prod_{i=0}^{n-1} \overline{w(\varphi_i(\lambda))}\right) e_{\varphi_n(\lambda)} = \left(\prod_{i=0}^{n-1} \overline{w(\varphi_i(\lambda))}\right) e_\lambda.$$

Put $h = \prod_{i=0}^{n-1} w(\varphi_i(\lambda))$. Thus $(A^*)^n = (M_h)^*$. Since the range of h intersects the unit circle, by Corollary 3, $(M_h)^*$ and so $(A^*)^n$ satisfies the hypercyclicity criterion. □

Corollary 5. *If there exists* $\lambda_0 \in \mathbb{D}$ *such that* $|w \circ \varphi_i(\lambda_0)| = 1$ *for all* $i = 1, ..., n$*, then the adjoint of the weighted composition operator* $(M_w C_\varphi)^n$ *satisfies the hypercyclicity criterion.*

Proof. Note that $\varphi_{-n} = \varphi_0$. For each $m = -1, 1$, the set $\{\lambda \in \mathbb{D} : |w(\varphi_i^m(\lambda))|^m < 1, \quad i = 1, ..., n\}$ contains a nonempty open subset of $\mathbb{D}$. Since for all $\lambda \in \mathbb{D}$, the set $\{K_{\varphi_i^m(\lambda)} : i \geq 0\}$ is finite for $m = -1, 1$, thus by Theorem 2, the proof is complete. □

Lemma 6. *If* $(M_w C_\varphi)^*$ *is hypercyclic, then the closure of ran* $(\prod_{i=0}^{n-1} w \circ \varphi_i)$ *intersects the unit circle.*

Proof. Put $A = M_w C_\varphi$ and note that $(A^*)^n$ is hypercyclic. Clearly, $(A^*)^n = (M_h)^*$ where $h = \prod_{i=0}^{n-1} w(\varphi_i(\lambda)$. Thus $(M_h)^*$ is hypercyclic and so its spectrum intersects the unit circle. This completes the proof. □

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