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MULTIPLE WEIGHTED COMPOSITION OPERATORS AND SUPERCYCLICITY

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Abstract: In this paper we give some sufficient conditions for the adjoint of the multiple weighted composition operators acting on some Banach function spaces supercyclic.

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1. Introduction

Let T be a bounded linear operator on a Banach space X. For $x \in X$, the orbit of x under T is the set of images of x under the successive iterates of T: $orb(T, x) = \{x, Tx, T^2x, \ldots\}$. The vector x is called hypercyclic for T if $orb(T, x)$ is dense in X. A hypercyclic operator is one that has a hypercyclic vector. The vector x is called supercyclic for T if $\mathbb{C}orb(T, x)$ is dense in X.

Supercyclicity was introduced by Hilden and Wallen [5]. They showed that all unilateral backward weighted shifts are supercyclic, but there is no vector that is supercyclic vector for all the unilateral backward weighted shifts. H. Salas [6] give a condition for supercyclicity in Frëchet spaces which is called the supercyclicity criterion.

Theorem 1.1. (Supercyclicity Criterion) *Let X be a separable Banach space and T be a continuous linear mapping on X. Suppose that there exist*

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two dense subsets Y and Z in X, a sequence $\{n_k\}$ of positive integers, and also there exist mappings $S_{n_k} : Z \to X$ such that:

1) $T^{n_k} S_{n_k} z \to z$ for every $z \in Z$,

2) $||T^{n_k} y||\ ||S_{n_k} z|| \to 0$ for every $y \in Y$ and every $z \in Z$.

Then T is supercyclic.

If an operator T holds in the assumptions of Theorem 2.1, then we say that T satisfies the supercyclicity criterion.

The holomorphic self maps of the open unit disc $\mathbb{D}$ are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in $\mathbb{D}$. It is well known that this map is conjugate to a rotation $z \to \lambda z$ for some complex number λ with $|\lambda| = 1$. The maps of that are not elliptic are called of non-elliptic type. The iterate of a non-elliptic map can be characterized by the Denjoy-Wolff Iteration Theorem ([7]).

A complex-valued function ψ on $\mathbb{D}$ is called a multiplier of X if $\psi X \subset X$. The operator of multiplication by ψ is denoted by M_ψ and is given by $f \longrightarrow \psi f$.

If w is a multiplier of X and φ is a mapping from $\mathbb{D}$ into $\mathbb{D}$ such that $f \circ \varphi \in X$ for all $f \in X$, then C_φ (defined on X by $C_\varphi f = f \circ \varphi$) and $M_w C_\varphi$ are called composition and weighted composition operator respectively. We define the iterates $\varphi_n = \varphi \circ \varphi \circ \ldots \circ \varphi$ (n times). For some topics see [1]-[13].

2. Main Results

From now on let X be a Banach space of functions analytic on the open unit disc $\mathbb{D}$ such that for each λ in $\mathbb{D}$ the linear functional of evaluation at λ given by $e_\lambda(f) = f(\lambda)$ is a bounded linear functional on X. Also, let $w_i : \mathbb{D} \to \mathbb{C}$ be non-constant multipliers of H for $i = 1, 2$, and φ be an analytic univalent map from $\mathbb{D}$ onto $\mathbb{D}$. By φ_n^{-1} we mean the nth iterate of φ^{-1}.

Theorem 2.1. *Let $\varphi(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi]$ and every $z \in \mathbb{D}$. Also, let $w_i : \mathbb{D} \to \mathbb{C}$ be such that the sets*

$$F_1 = \{\lambda \in \mathbb{D} : \lim_n \prod_{j=0}^{n-1} w_1(e^{(2j+1)i\theta}\lambda).w_2(e^{2ji\theta}\lambda) = 0\}$$

and

$$F_{-1} = \{\lambda \in \mathbb{D} : \{\left(\prod_{j=1}^{n} w_1(e^{-(2j-1)i\theta}\lambda).w_2(e^{-2ji\theta}\lambda)\right)^{-1} \}_n$$

$$\textit{is a bounded sequence}\}$$

have limit points in $\mathbb{D}$. *Then* $T = (M_{w_2}C_\varphi M_{w_1}C_\varphi)^*$ *is supercyclic.*

Proof. For all $n \in \mathbb{N}$ and all λ in $\mathbb{D}$ we can see that

$$T^n E_\lambda = \left(\prod_{j=0}^{n-1} w_1(e^{(2j+1)i\theta}\lambda).w_2(e^{2ji\theta}\lambda)\right) E_{e^{2ni\theta}\lambda}.$$

Put

$$X_{F_m} = span\{E_\lambda : \lambda \in F_m\}$$

for $m = -1, 1$. Since the sets F_m have limit points in $\mathbb{D}$, thus the sets X_{F_m} are dense in X^* for $m = -1, 1$. Note that for each λ in $\mathbb{D}$, the sequence $\{e^{ijm\theta}\lambda\}_j$ is a subset of the compact set $\{z : |z| = |\lambda|\}$ for $m = -1, 1$. Now if $f \in X$, then the set $\{f(e^{ijm\theta}\lambda) : j \in \mathbb{D}\}$ is bounded and so by the Banach-Steinhaus Theorem the sequence $\{E_{e^{ijm\theta}\lambda}\}_j$ is bounded for $m = -1, 1$. Hence, $T^n \longrightarrow 0$ point wise on X_{F_1}. Define $S_n : X_{F_{-1}} \longrightarrow X^*$ by extending the definition

$$S_n E_\lambda = \left(\prod_{j=1}^{n} w_1(e^{-(2j-1)i\theta}\lambda).w_2(e^{-2ji\theta}\lambda)\right)^{-1} E_{e^{-2ni\theta}\lambda},$$

where $\lambda \in F_{-1}$ and $n \in \mathbb{N}$. Now, clearly we can see that $T^n S_n$ is identity on the dense subset $X_{F_{-1}}$ of X^*. Note that if $\lambda \in F_{-1}$, then

$$\{\left(\prod_{j=1}^{n} w_1(e^{-(2j-1)i\theta}\lambda).w_2(e^{-2ji\theta}\lambda)\right)^{-1} E_{e^{-2ni\theta}\lambda}\}_n$$

is a bounded sequence. This implies that $||T^n y||\ ||S_n z|| \to 0$ for every $y \in F_1$ and every $z \in F_{-1}$. Thus T satisfies the supercyclicity criterion and so it is supercyclic. □

For the proof of the following theorem, we will use a method that we have used in [11].

Theorem 2.2. *Let φ be an elliptic automorphism with interior fixed point p and $w_i : \mathbb{D} \to \mathbb{C}$ satisfies the inequality:*

$$|w_k(p)| < 1 \leq \lim_{|z|\to 1^-} \inf |w_k(z)|$$

for $k = 1, 2$. Then $(M_{w_2} C_\varphi M_{w_1} C_\varphi)^*$ is supercyclic on the Banach space X.

Proof. Put

$$\Phi = \alpha_p \circ \varphi \circ \alpha_p$$

and

$$W_k = w_k \circ \alpha_p$$

for $k = 1, 2$, where

$$\alpha_p(z) = \frac{p - z}{1 - \overline{p}z}.$$

Since Φ is an automorphism with $\Phi(0) = 0$, thus Φ is a rotation $z \to e^{i\theta} z$ for some $\theta \in [0, 2\pi]$ and every $z \in \mathbb{D}$. Since $|\alpha_p(z)| \to 1^-$ when $|z| \to 1^-$, so

$$\lim_{|z|\to 1^-} \inf |w_k(z)| \leq \lim_{|z|\to 1^-} \inf |w_k \circ \alpha_p(z)|$$

for $k = 1, 2$. Thus

$$|W_k(0)| < 1 \leq \lim_{|z|\to 1^-} \inf |W_k(z)|$$

for $k = 1, 2$. Therefore there exists a constant λ and a positive number $\delta < 1$ such that $|W_k(z)| < \lambda < 1$ when $|z| < \delta$, and $|W_k(z)| \geq 1$ when $|z| > 1 - \delta$ for $k = 1, 2$. Set

$$F_1 = \{z : |z| < \delta\}$$

and

$$F_{-1} = \{z : |z| > 1 - \delta\}.$$

So if $z \in F_1$, then for each positive integer n,

$$|W_k(\Phi_n(z))| < \lambda < 1,$$

and if $z \in F_{-1}$, then

$$|W_k((\Phi_n)(z))| \geq 1.$$

Hence for $m = -1, 1$, F_m is a subset of E_m where E_m is defined as in Theorem 2.1 depending on W_k instead of w_k. This says that the sets E_m have limit points in $\mathbb{D}$ for $m = -1, 1$ and so by Theorem 2.1, the operator $S =$

$(M_{W_2}C_{\Phi}M_{W_1}C_{\Phi})^*$ is supercyclic. But $(M_{w_2}C_{\varphi}M_{w_1}C_{\varphi})C_{\alpha_p} = C_{\alpha_p}S^*$, thus by similarity $(M_{w_2}C_{\varphi}M_{w_1}C_{\varphi})^*$ is supercyclic. This completes the proof. □

Corollary 2.3. *Under the conditions of Theorem 2.2, the operator* $T \oplus T$ *is supercyclic on the Banach space* $X \oplus X$ *where* $T = (M_{w_2}C_{\varphi}M_{w_1}C_{\varphi})^*$.

Proof. It is well known that $T \oplus T$ is supercyclic $X \oplus X$ if and only if T satisfies the supercyclicity criterion on X. This completes the proof. □

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