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A NOTE ON THE STABILITY OF AN ECONOMIC GROWTH LOGISTIC MODEL WITH MIGRATION

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Abstract: Juchem Neto et al [8] have recently analyzed the Solow growth model with the labor force ruled by the logistic equation added by a constant migration rate. In particular, they have showed the global asymptotic stability of the model's solution. In this paper, we present another proof of this result.

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1. Introduction

In neoclassical growth models, a standard assumption is that population grows at a positive given rate. Recently, some authors (see, e.g., Accinelli and Brida [1], Bucci and Guerrini [3], Ferrara and Guerrini [4], Guerrini [5]-[7]) have investigated the implications of studying these models under a variable population growth law hypothesis. Juchem Neto et al [8] have considered the standard one-sector neoclassical growth model of Solow [9] within a framework where the change over time of the labor force is governed by the logistic growth law added by a constant migration rate and found the model's solution to be globally asymptotically stable. The aim of this paper is to show this statement can be proved via the Bendixson-DuLac and Poincarè-Bendixson Theorems.

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2. The Model

Following Juchem Neto et al [8], we consider an economic growth logistic model with migration whose economic environment is as follows: a Cobb-Douglas production function $Y = AK^{\varphi}L^{1-\varphi}$, $A > 0$, $0 < \varphi < 1$, where Y, A, K and L denote output, the level of technology, capital and labor/population, respectively; the law of evolution of capital $\dot{K} = sY - \delta K$, with s and δ the rates of saving and depreciation, respectively; the law of population growth $\dot{L} = L(\alpha - \beta L) + I$, with I a constant migration rate and α, β positive constants such that $\alpha - \beta L(0) > 0$. By defining $k = K/L$ as capital per unit of labor, we derive that the economy is thus described by the following system of two non-linear differential equations

$$\dot{k} = sAk^{\varphi} - \left(\delta + \frac{\dot{L}}{L}\right)k, \qquad \dot{L} = L(\alpha - \beta L) + I. \tag{1}$$

Under the hypothesis $I \geq -\alpha^2/4\beta \vee L(0)[\beta L(0) - \alpha]$, Juchem Neto et al [8] demonstrate that the dynamical system (1) to has a unique non-trivial steady state equilibrium (k_*, L_*). As well, the model's solution happens to be globally asymptotically stable, i.e. all solutions starting near the steady state remain near the steady state for all the time, and furthermore they tend towards (k_*, L_*) as t grows to infinity.

3. Stability Analysis

In this section, we wish to give an alternative proof of the global asymptotic stability of the model's solution, which is based on the Bendixson-DuLac Theorem and the Poincarè-Bendixson Theorem (see, e.g., Boyce and DiPrima [2]). We recall that the Bendixson-DuLac Theorem states that if there exists a function $\psi(k, L)$ such that $\partial(\psi\dot{k})/\partial k + \partial(\psi\dot{L})/\partial L$ has the same sign $(\neq 0)$ almost everywhere in a simply connected region, then the plane autonomous system (1) has no periodic solutions. The Poincarè-Bendixson Theorem instead says that any orbit staying in a bounded region of the plane autonomous system (1) either approaches a fixed point or a periodic orbit.

Theorem 1. *Let $I \geq -\alpha^2/4\beta \vee L(0)[\beta L(0) - \alpha]$. Any solution of* (1) *has global asymptotic stability.*

Proof. First, note that setting $\psi(k, L) = k^{-1}L^{-1}$ yields

$$\frac{\partial(k^{-1}L^{-1}\dot{k})}{\partial k} + \frac{\partial(k^{-1}L^{-1}\dot{L})}{\partial L} = (\varphi - 1)sAk^{\varphi-2}L^{-1} - \beta k^{-1} - Ik^{-1}L^{-2}$$

$$< 0.$$

Thus, by the Bendixson-Dulac Criterion, there will be no periodic orbits. Second, the Inada conditions of the Cobb-Douglas production function yield the boundedness of any solution to system (1). Consequently, an application of the Poincarè-Bendixson Theorem yields that any solution of (1) converges to the steady state equilibrium (k_*, L_*) as $t \to \infty$. □

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