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LINE BUNDLES ON CURVES OVER A NON-ALGEBRAICALLY CLOSED FIELD

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Abstract: Let Y be an integral projective curve defined over a field K. Set $g := p_a(Y)$. Here we give conditions on $\sharp(Y_{reg}(K))$ assuring the existence of $L \in \mathrm{Pic}^{g-1}(Y)$ defined over K and such that $h^0(Y, L) = h^1(Y, L) = 0$.

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1. Introduction

Let Y be a geometrically integral projective curve defined over a field K. Set $g := p_a(Y)$. We first wrote this paper taking $K := \mathbb{F}_q$, but the only use of the finiteness of K was in the use of the theorem of Hasse-Weil for smooth curves and Abelian varieties. We just write conditions on $\sharp(Y(\mathbb{F}_q))$ which are always satisifed (even if Y is singular) if we fix g and then take $q \gg g$. Here we study the following problem:

Question 1. Under what assumptions on $Y_{reg}(K)$ (e.g. its cardinality) is there $L \in \mathrm{Pic}^{g-1}(Y)$ defined over K and such that $h^0(Y, L) = h^1(Y, L) = 0$?

Set $m := \sharp(Y_{reg}(K)$ with the convention $m = +\infty$ and $m \geq x$ for all $x \in \mathbb{N}$ if $Y_{reg}(K)$ is infinite, i.e. if $Y(K)$ is infinite. Set $g := p_a(Y)$. Let $\overline{K}$ denote the algebraic closure of K.

Let $u : C \to Y$ be the normalization map. For each $P \in Y_{reg}$ set $P' := u^{-1}(P)$. Set $\eta := u^*(\omega_Y)/\mathrm{Tors}(u^*(\omega_Y))$. Since C is a smooth curve, ω is a line bundle on C. Since ω_Y has no torsion, any non-zero element of it vanishes

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only at finitely many points. We have $\deg(\eta) \le \deg(\omega_Y) = 2g-2$ and equality holds if and only if $\mathrm{Tors}(u^*(\omega_Y)) = \{0\}$, i.e. if and only if Y is Gorenstein. Thus the natural map $H^0(Y, \omega_Y) \to H^0(C, \eta)$ is injective. We call Θ its image and Θ' the associated projective space. Thus $|\Theta|$ is a $(g-1)$-dimensional linear system. A theorem of Rosenlicht (see [1]) says that ω_Y is spanned. Thus the linear system Θ' has no base points. For each $P \in Y_{reg}$ set $P' := u^{-1}(P)$. We say that ω_Y has classical generic gap sequence if the linear system Θ' on C has classical generic gap sequence, i.e. for a general $P \in Y(\overline{K})$ we have $\dim(\Theta \cap H^0(\mathcal{I}_{tP'} \otimes \eta)) = \max\{0, g-t\}$ for all $t \in \mathbb{N}$, i.e. for every $t \in \mathbb{N}$ we have $h^0(Y, \mathcal{I}_{tP} \otimes \omega_Y) = \max\{0, g-t\}$ for a general $P \in Y_{reg}(\overline{K})$. If $\mathrm{char}(K) > \deg(\eta)$ (e.g. if $\mathrm{char}(K) > 2g-2$), then Θ' has classical generic gap sequence (see [2]). If $\mathrm{char}(K)$ is small with respect to g there are examples with non-classical generic gap sequence, even in the case Y smooth (see [2], [3]), and ways to produce curves with classical generic gap sequence. For any $P \in Y_{reg}$ let k_P be the maximal integer t such that $h^0(Y, \mathcal{I}_{tP} \otimes \omega_Y) = g$. Thus $k_P \le g$ for all $P \in Y_{reg}$ and equality holds for a general $P \in Y_{reg}$ if and only if ω_Y has classical generic gap sequence. Set $k_{max} := \max_{P \in Y_{reg}} k_P$ and $k_{max,K} := \max_{P \in Y_{reg}(K)} k_P$. We first list some of our results.

Proposition 1. *Assume $k_{max,K} = g$ and $\sharp(Y_{reg}(K)) \ge 2$. Then there is $L \in Pic^{g-1}(Y)$ defined over K and such that $h^0(Y, L) = h^1(Y, L) = 0$.*

Proposition 2. *Assume the existence of a degree 2 separable morphism $f : Y \to \mathbb{P}^1$, i.e. assume Y Gorenstein and hyperelliptic (see [1]). Assume $\sharp(Y_{reg}(K)) \ge 2$ and the existence of $P \in Y_{reg}(K)$ which is not a ramification point of f. Then there is $L \in Pic^{g-1}(Y)$ defined over K and such that $h^0(Y, L) = h^1(Y, L) = 0$.*

Proposition 3. *Assume Y smooth, Y not hyperelliptic and $\sharp(Y(K)) \ge g$. Then there is $L \in Pic^{g-1}(Y)$ defined over K and such that $h^0(Y, L) = h^1(Y, L) = 0$.*

The proof of Proposition 1 will give verbatim also the following result.

Proposition 4. *Fix an integer y such that $0 \le y \le k_{max,K} - 1$. Assume $\sharp(Y_{reg}(K)) \ge 2$. Then there exists $L \in Pic^y(Y)$ defined over K and such that $h^0(Y, L) = 0$.*

If we avoid any assumption on Y it is easy to get the following result.

Proposition 5. *Assume $m \ge 2g-1$. If $g = 1$, then assume $m \ge 2$. Then there is $L \in Pic^{g-1}(Y)$ defined over K and such that $h^0(Y, L) = h^1(Y, L) = 0$.*

Proof. Riemann-Roch gives $h^0(Y, R) = h^1(Y, R)$ for every $R \in \mathrm{Pic}^{g-1}(Y)$.

Thus it is sufficient to find $L \in \mathrm{Pic}^{g-1}(Y)$ defined over K and such that $h^0(Y, L) = 0$. Since $m \geq 2g-1$ there are $P_1, \dots, P_{2g-1} \in Y_{reg}(K))$ such that $P_i \neq P_j$ for all $i \neq j$. For degree reasons we have $h^1(Y, \mathcal{O}_Y(P_1 + \cdots + P_{2g-1})) = 0$. Thus $h^0(Y, \mathcal{O}_Y(P_1 + \cdots + P_{2g-1})) = g > 1 = h^0(Y, \mathcal{O}_Y)$. Thus there is $i \in \{1, \dots, 2g-1\}$ such that P_i is not in the base-locus of the line bundle $\mathcal{O}_Y(P_1 + \cdots + P_{2g-1})$. Up to a permutation we may assume $i = 2g-1$. Hence $h^0(Y, \mathcal{O}_Y(P_1 + \cdots + P_{2g-2})) = g-1$, i.e. $h^1(Y, \mathcal{O}_Y(P_1 + \cdots + P_{2g-2})) = 0$. If $g-1 > 1$, then we may continue and find $i\{1, \dots, 2g-2\}$ such that P_i is not in the base-locus of the line bundle $\mathcal{O}_Y(P_1 + \cdots + P_{2g-2})$. Up to a permutation we may assume $i = 2g-2$. Then we continue until we find a subset of $\{P_1, \dots, P_{2g-1}\}$ of cardinality g, say $P_1, \dots, P_g$, such that $h^0(Y, \mathcal{O}_Y(P_1 + \cdots + P_g)) = 1$, i.e. $h^1(Y, \mathcal{O}_Y(P_1 + \cdots + P_g)) = 0$. Since $h^0(Y, \mathcal{O}_Y(P_1 + \cdots + P_g)) = 1$, the unique non-zero section of $\mathcal{O}_Y(P_1 + \cdots + P_g)$ has the divisor $P_1 + \cdots + P_g$ as its zero-locus. Thus $h^0(Y, \mathcal{O}_Y(P_1 + \cdots + P_g) - P_{g+1}) = 0$. □

Proof of Proposition 1. By assumption there is $P \in Y_{reg}(K)$ such that $k_P = g$, i.e. $h^0(Y, \mathcal{O}_Y(gP)) = 1$. By assumption there is $Q \in Y_{reg}(K)$ such that $Q \neq P$. Set $L := \mathcal{O}_Y(gP - Q)$. □

Proof of Proposition 2. Brill-Noether theory for singular Gorenstein hyperelliptic curves gives $k_P = g$. Apply Proposition 1. □

Proof of Proposition 3. Since Y is not hyperelliptic, we have $g \geq k_P \geq 3$. Fix any $P \in Y(K)$ such that $k_P = k_{max,K}$. Hence $k_P \leq g$ and $h^0(Y, \mathcal{O}_Y(k_P P)) = 1$, i.e. $h^0(Y, \omega_Y(-k_P P)) = g - k_P$. If $k_P = g$, then we may apply Proposition 1. Thus we may assume $k_P < g$ and in particular $g \geq 4$. Let D_1 be the base locus of $|\omega_Y(-k_P P)|$. Since Y is not hyperelliptic and $1 \leq \deg(\omega_Y(-k_P P)) \leq 2g-3$, the strong form of Clifford's inequality gives $2g-2-k_P-\deg(D_1) \geq 2(g-k_P)-1$, i.e. $\deg(D_1) \leq k_P - 1$. Since $\sharp(Y(K)) \geq k_P$, we may find $O_1 \in Y(K)$ not in the support of D. Thus $h^0(Y, \omega_Y(-k_P P - O_1)) = g - k_P - 1$. Since $\sharp(Y(K)) \geq 3$ we may take $Q_1 \in Y(K)$ such that $Q_1 \neq P$ and $Q_1 \neq O_1$. If $k_P = g-1$, then take $L := \mathcal{O}_Y((g-1)P + O_1 - Q_1)$. Now assume $k_P \leq g-2$. Hence $g \geq 5$. Let D_2 be the base locus of $\omega_Y(-k_P P - O_1)$. The strong form of Clifford's inequality gives $2g - 2 - k_P - 1 - \deg(D_2) \geq 2(g - k_P - 1) - 1$, i.e. $\deg(D_2) \leq k_P$. Since $\sharp(Y(K)) \geq k_P + 1$, there is $O_2 \in Y(K)$ not in the support of D_2. Thus $h^0(Y, \omega_Y(-k_P P - O_1 - O_2)) = g - k_P - 2$. Since $\sharp(Y(K)) \geq 4$ if $k_P \leq g-2$, we may take $Q_2 \in Y(K) \setminus \{P, O_1, O_2\}$. If $k_P = g-2$, then take $L := \mathcal{O}_Y((g-2)P + O_1 + O_2 - Q_2)$. And so on. □

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