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QUASILINEAR ELLIPTIC EQUATIONS WITH A DAMPING TERM VIA PICONE-TYPE IDENTITIES: OSCILLATORY SOLUTIONS AND UNIQUENESS

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Abstract: We establish a uniqueness theorem and some oscillation criterion for the equation

$$Pu := \nabla\Big\{a(x)\Phi(\nabla u)\Big\} + f(x,u,\nabla u) + c(x)\phi(u) = 0 \quad \text{in } \mathbb{R}^n.$$

We use solely some selected Picone's type formulas and the known such results for the related half linear equation. The work is an extension of the earlier one on semilinear equations (see [7]). This work underlines the fact that such results can follow mainly from some properties of the main coefficients $a(x)$ and $c(x)$ of the equation. The usual methods based on the Riccati techniques (see [4], [8]) make our methods quite simple. Equations with $p-$Laplacian displayed above and associated half-linear equations (when $f \equiv 0$) have been widely investigated lately because of the interest in their applications (e.g. in physical and biological problems, see, e.g. [1], [4] and referrences therein).

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1. Introduction

This work deals with equations of the types

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$$Pu := \nabla\Big\{a(x)\Phi(\nabla u)\Big\} + f(x,u,\nabla u) + c(x)\phi(u) = 0 \quad \text{in } \mathbb{R}^n \tag{1.1}$$

$$\text{and} \qquad Qv := \nabla\Big\{a(x)\Phi(\nabla v)\Big\} + c(x)\phi(v) = 0 \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

For the boundary conditions, we set

$$u|_{\partial G} = 0 \quad \text{ and } \; u \neq 0 \quad \text{in } G, \tag{1.3}$$

where $G \subset \mathbb{R}^n$ will always denote any regular (i.e. $\partial G \in C^1$) bounded domain in $\mathbb{R}^n$ in which (1.3) holds. When the term $f(x,u,\nabla u)$ contains explicitly ∇u, it denotes the damping term for (1.1). The problem where $f(x,u,\nabla u)$ does not contain ∇u has been largely investigated lately, using mainly some Picone inequalities (see [2, 5, 6] and references therein). To our knowledge, a part from [7], cases with damping terms in multidimensional spaces have been approached via some Riccati techniques (see [4, 8]) for $p-$Laplacian cases where f has the form $f(x,u,\nabla u) = ||\nabla u||^{p-2}\sum_{i=1}^n B_i \frac{\partial u}{\partial x_i}$. As in our earlier results, by extending the Picone's method beyong the half-linear operators developed mainly in [2, 3] to the perturbation cases (non zero $f(x,u,.)$) we show some very potent applications of the Picone's identities and inequalities. We recall that most of the methods for obtaining oscillation results are based on known ones obtained for one dimensional cases. The autors of [2, 3] have largely contributed to that end.

For the notations, define for some $\alpha > 0$, $\forall t,s \in \mathbb{R}$ and $\zeta \in \mathbb{R}^n$

$$\phi(t) = |t|^{\alpha-1}t \quad \text{and} \quad \Phi(\zeta) = |\zeta|^{\alpha-1}\zeta; \text{ which satisfy } \; \phi(s)\phi(t) = \phi(st),$$
$$t\phi(t) = |t|^{\alpha+1};\; \zeta\Phi(\zeta) = |\zeta|^{\alpha+1};\; t\phi'(t) = \alpha\phi(t) \quad \text{and} \quad \phi(t)\Phi(\zeta) = \Phi(t\zeta).$$

The main hypotheses are:

(H1) The functions $a \in C^1(\mathbb{R}^n\,;\,(0,\,\infty))$ and $c \in C(\mathbb{R}^n\,;\,(0,\,\infty))$.

(H2) $f \in C(\,\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n\,;\,\mathbb{R})\,)$.

(H3) For any function $w \in C^1(\mathbb{R}^n)$ and any regular bounded domain G, if $w|_{\partial G} = 0$ and $w \neq 0$ inside G then:

(H3+) either $wf(x,w,\nabla w) > 0$ inside G;

(H3-) or $wf(x,w,\nabla w) < 0$ inside G.

(H4) $\forall w \in C^1(\mathbb{R}^n)$, $f(x,w,\nabla w)$ has the sign of w.

The (classical) solutions for (1.1) or (1.2) with (1.3) will be those of solutions which belong to

(S) $w \in C^1(\mathbb{R}^n;\ \mathbb{R})$ such that $w \in C^1(\overline{\Omega};\ \mathbb{R})$ and $a\Phi(\nabla w) \in C^1(\Omega;\ \mathbb{R}) \bigcap C(\overline{\Omega};\ \mathbb{R})$ whenever $w \neq 0$ in Ω and $w|_{\partial\Omega} = 0$.

Here a solution w will be said to be positive in G if $w > 0$ inside G.

Definition 1.1. (see [3, 5, 6]) Here a function $w \in C^1(\mathbb{R}^n; \mathbb{R})$ will be said to be (weakly) oscillatory (in $\mathbb{R}^n$) if $\forall R > 0, \exists x \in \mathbb{R}^n;\quad |x| > R$ and $w(x) = 0$ or strongly oscillatory or nodal if $\forall R > 0, \exists G \subset \{x \in \mathbb{R}^n; |x| > R\}$, a nodal component say, such that $w|_{\partial G} = 0$ and $w \neq 0$ inside G.

Here, since the function $c \in C(\mathbb{R}^n)$, we will be dealing with strongly oscillatory solutions (see [4], Chapter 1, §2).

In a large sense a solution of (1.1),(1.3) will be said to be oscillatory if its extension in the whole $\mathbb{R}^n$ is strongly oscillatory; G being any of its nodal components.

2. Preliminaries

2.1. Picone Types Identities

From the properties of the functions Φ and ϕ, if $w \in C^1(\mathbb{R}^n)$ then

$\nabla\phi(w) = \phi'(w)\nabla w = \dfrac{\alpha\phi(w)\nabla w}{w}$ and $\nabla[\dfrac{1}{\phi(w)}] = -\dfrac{\phi'(w)\nabla w}{\phi(w)^2} = -\dfrac{\alpha\nabla w}{w\phi(w)}$.

Let u be a solution of (1.1), (1.3) and v be that of (1.2) such that $v \neq 0$ in G. Then easy verifications show that

$$\nabla\left\{ua\Phi(\nabla u)\right\} = a|\nabla u|^{\alpha+1} - c|u|^{\alpha+1} - uf(x,u,\nabla u); \tag{2.1}$$

$$\nabla\left\{va\Phi(\nabla v)\right\} = a|\nabla v|^{\alpha+1} - c|v|^{\alpha+1}; \tag{2.2}$$

$$\begin{aligned}\nabla\left\{u\frac{\phi(u)}{\phi(v)}a\Phi(\nabla v)\right\} &= \phi(\frac{u}{v})a\Phi(\nabla v).\nabla u + \frac{u}{\phi(v)}[\ \nabla\phi(u)\ ]a\Phi(\nabla v)\\ &+ u\phi(u)[\ \nabla(\frac{1}{\phi(v)})\ ]a\Phi(\nabla v) + u\phi\left(\frac{u}{v}\right)\nabla(a\Phi(\nabla v));\end{aligned} \tag{2.3}$$

$$\begin{aligned}\nabla\left\{v\frac{\phi(v)}{\phi(u)}a\Phi(\nabla u)\right\} &=\phi(\frac{v}{u})a\Phi(\nabla u).\nabla v + \frac{v}{\phi(u)}[\ \nabla\phi(v)\ ]a\Phi(\nabla u)\\ &+ v\phi(v)[\ \nabla(\frac{1}{\phi(u)})\ ]a\Phi(\nabla u) + v\phi\left(\frac{v}{u}\right)\nabla(a\Phi(\nabla u)).\end{aligned} \tag{2.4}$$

Thus, with the two-form on $C^1(\mathbb{R}^n)\times C^1(\mathbb{R}^n)$ defined by

$$\begin{aligned} Z(w,h) :=&|\nabla w|^{\alpha+1}+\alpha|\frac{w}{h}\nabla h|^{\alpha+1}-(\alpha+1)|\frac{w}{h}\nabla h|^{\alpha-1}\ \frac{w}{h}\nabla h.\nabla w \\ &\text{which is strictly positive if } \nabla w\neq\nabla h \\ &\text{and zero only if } \ \exists k\in\mathbb{R},\ w=kh, \end{aligned} \tag{2.5}$$

we get for classical solutions u of (1.1), (1.3) and v of (1.2), (1.3) with $v\neq 0$ in G

$$\begin{aligned} \nabla\Big\{ua\Phi(\nabla u)-u\phi(\frac{u}{v})a\Phi(\nabla v)\Big\} &= aZ(u,v)-uf(x,u,\nabla u) \\ &=a\Big\{|\nabla u|^{\alpha+1}+\alpha|\frac{u}{v}\nabla v|^{\alpha+1} \\ &-(\alpha+1)|\frac{u}{v}\nabla v|^{\alpha-1}\ \frac{u}{v}\nabla v.\nabla u\Big\}-uf(x,u,\nabla u). \end{aligned} \tag{2.6}$$

Also if $u\neq 0$ in G then similarly

$$\begin{aligned} \nabla\Big\{va\Phi(\nabla v)-v\phi(\frac{v}{u})a\Phi(\nabla u)\Big\} =&aZ(v,u)+v\phi(\frac{v}{u})f(x,u,\nabla u) \\ =&a\Big\{|\nabla v|^{\alpha+1}+\alpha|\frac{v}{u}\nabla u|^{\alpha+1}+ \\ -(\alpha+1)|\frac{v}{u}\nabla u|^{\alpha-1}\ \frac{v}{u}\nabla u.\nabla uv\Big\}&+v\phi(\frac{v}{u})f(x,u,\nabla u). \end{aligned} \tag{2.7}$$

Lemma 2.1. *Let u be a positive solution of (1.1), (1.3) in G and $v\in C^1(G)$ with $v|_{\partial G}=0$. Then*

$$\int_G a\Big\{|\nabla v|^{\alpha+1}-Z(v,u)\Big\}dx=\int_G\Big\{|\frac{v}{u}|^{\alpha+1}\ uf(x,u,\nabla u)+c|v|^{\alpha+1}\Big\}dx. \tag{2.8}$$

Therefore if (H3+) holds,

$$\int_G\Big\{aZ(v,u)+c|v|^{\alpha+1}\Big\}dx<\int_G a|\nabla v|^{\alpha+1}dx \tag{2.9}$$

and if (H3-) holds $\displaystyle\int_G\Big\{aZ(v,u)+c|v|^{\alpha+1}\Big\}dx>\int_G a|\nabla v|^{\alpha+1}\Big\}dx. \quad (2.10)$

Proof. From (2.4),

$$\nabla\{\ v\phi(\frac{v}{u})a\Phi(\nabla u)\ \}=a\{(1+\alpha)|\frac{v}{u}\nabla u|^{\alpha-1}\ \frac{v}{u}\nabla u.\nabla v-\alpha|\frac{v}{u}\nabla u|^{\alpha+1}\ \}$$

$$-\frac{v}{u}\phi(\frac{v}{u})uf(x,u,\nabla u) - c|v|^{\alpha+1}$$
$$= -aZ(v,u) + a|\nabla v|^{\alpha+1} - |\frac{v}{u}|^{\alpha+1}uf(x,u,\nabla u) - c|v|^{\alpha+1}.$$

The integration over G of the equation above leads to (2.8). From (2.8) we get (2.9) and (2.10). □

Lemma 2.2. *Let u and v be respectively solutions of* (1.1), (1.3) *and* (1.2), (1.3), *both strictly positive inside G. Then*

$$\int_G aZ(u,v)dx = \int_G \Big\{a|\nabla u|^{\alpha+1} - c|u|^{\alpha+1}\Big\}dx = \int_G uf(x,u,\nabla u)dx; \tag{2.11}$$
$$\int_G \Big\{a|\nabla v|^{\alpha+1} - c|v|^{\alpha+1}\Big\}dx = 0 \quad \textit{and} \tag{2.12}$$
$$\int_G aZ(v,u)dx = -\int_G |\frac{v}{u}|^{\alpha+1}\, uf(x,u,\nabla u)dx\ . \tag{2.13}$$

Proof. The equation in (2.12) follows from the integration over G of (2.2). Also (2.13) follows from (2.7). Similarly the integration over G of (2.1) and (2.6) leads to (2.11). □

Remark 2.3. It is worth noticing that:

1) If the solutions in Lemma 2.2 were both negative inside such a G,(2.11)-(2.13) would hold as well.

2) If G_u and G_v denote a connected component respectively of supp.(u) and supp.(v) then:

(a) If (H3+) holds, $u \neq 0$ in G_v cannot hold by (2.13) and

(b) If (H3-) holds, $v \neq 0$ in G_u would not hold either by (2.11).

Therefore if (H4) holds then supp.$(u)\bigcap$supp.(v) contains no connected component of any of them.

Lemma 2.4. *Let u and v be two positive classical solutions of* (1.1), (1.3). *Then*

$$\begin{aligned}
&\nabla\Big\{ua\Phi(\nabla u) - u\phi(\frac{u}{v})a\Phi(\nabla v)\Big\}\\
&= a|\nabla u|^{\alpha+1} - c|u|^{\alpha+1} - uf(x,u,\nabla u)\\
&+ a\{\ -(\alpha+1)|\frac{u}{v}\nabla v|^{\alpha-1}\ \frac{u}{v}\nabla v.\nabla u + \alpha|\frac{u}{v}\nabla v|^{\alpha+1}\ \}\\
&+ u\phi(\frac{u}{v})\{\ f(x,v,\nabla v) + c\phi(v)\ \}\\
&= aZ(u,v) + |u|^{\alpha+1}\Big\{\frac{f(x,v,\nabla v)}{\phi(v)} - \frac{f(x,u,\nabla u)}{\phi(u)}\Big\}.
\end{aligned} \tag{2.14}$$

Also $\forall \mu \in [0\,,\infty)$ *and* $V := v + \mu$,

$$\begin{aligned}
&\nabla\Big\{ u\phi(\frac{u}{V}) a\Phi(\nabla v) - au\Phi(\nabla u)\Big\}\\
=& -aZ(u,v) + a|\nabla u|^{\alpha+1} - |\frac{u}{V}|^{\alpha+1} V f(x,v,\nabla v) - c|u|^{\alpha+1}\phi(\frac{v}{V})\\
&- a|\nabla u|^{\alpha+1} + uf(x,u,\nabla u) + c|u|^{\alpha+1}\\
=& -aZ(u,v) + c|u|^{\alpha+1} + uf(x,u,\nabla u) - u\phi(\frac{u}{V})\Big\{ c\phi(v) + f(x,v,\nabla v)\Big\}\,.
\end{aligned} \tag{2.15}$$

Proof. Because $\nabla V = \nabla v$, $\nabla\{\ u\phi(\frac{u}{V})a\Phi(\nabla v)\ \} = -aZ(u,v) + a|\nabla u|^{\alpha+1} - |\frac{u}{V}|^{\alpha+1}\ Vf(x,v,\nabla v) - c|u|^{\alpha+1}\ \phi(\frac{v}{V})$ and $-\nabla[\ au\Phi(\nabla u)\ ] = -a|\nabla u|^{\alpha+1} + uf(x,u,\nabla u) + c|u|^{\alpha+1}$.

This leads to (2.15). Similarly (2.14) follows from (2.1) and (2.3). □

2.2. Oscillation Criteria for Half-Linear Equations

In the preceding paragraphs we saw that whenever the solution v of (1.2)-(1.3) is positive in a regular and bounded domain G, under the hypotheses (H1) to (H4), any positive solution of (1.1), (1.3) in G has a zero inside G, implying that with those hypotheses the oscillatory character of the equation (1.2) leads to that of (1.1).

For the half linear equation

$$Qv := \nabla\Big\{ a(x)\Phi(\nabla v)\Big\} + c(x)\phi(v) = 0 \quad \text{in } \mathbb{R}^n,$$

define for $r > 0$ and $r_0 > 0$

$$\begin{aligned}
&A(r) := \max_{|x|=r} a(x)\ ; \quad \gamma(r) := \min_{|x|=r} c(x); \quad q(r) := r^{n-1}\gamma(r)\ ;\\
&p(r) := r^{n-1}A(r)\ ; \quad P(t) := \int_{r_0}^{t} p(s)^{-1/\alpha}\, ds \quad \text{and}\\
&\pi(t) := \int_t^\infty p(s)^{-1/\alpha}\, ds.
\end{aligned} \tag{2.16}$$

We have the following well known result (see [5], [3]):

Theorem 2.5. *The problem* (1.2) *has oscillatory solutions in* $\mathbb{R}^n$ *for any of the following cases:*

(i) $\lim_{r\nearrow\infty} P(r) = \infty$ *and* $\int_{r_0}^{\infty} q(r)dr = \infty$ *or* $\int_{r_0}^{\infty} q(r)dr < \infty$ *and* $\lim_{r\nearrow\infty} \inf \left\{ P(r)^\alpha \int_r^\infty q(s)ds \right\} > \dfrac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$;

(ii) $\lim_{r\nearrow\infty} P(r) < \infty$ *and* $\int_{r_0}^{\infty} \pi(r)^{\alpha+1}\, q(r)dr = \infty$ *or* $\int_{r_0}^{\infty} \pi(r)^{\alpha+1} q(r)dr < \infty$ *and*

$$\lim_{r\nearrow\infty} \inf \left\{ \frac{1}{\pi(r)} \int_r^\infty \pi(t)^{\alpha+1} q(t)dt \right\} > \left[\frac{\alpha}{\alpha+1}\right]^{\alpha+1}.$$

3. Main Results

Theorem 3.1. *Assume that (H1) through (H3) hold. Then:*

a) If u is a positive classical solution of (1.1), (1.3), *any classical solution of* (1.2), (1.3) *has a zero inside G.*

b) Reversely, if v is a classical positive solution of (1.2), (1.3) *then any classical solution of* (1.1), (1.3) *has a zero inside G.*

Proof. Assume that (H3+) holds; the statements follow from Lemma 2.2. In fact for such solutions u and v, the right hand side of (2.13) is strictly negative while the left hand side is positive which is absurd.

And if (H3-) holds, similarly (2.11) leads to the same conclusion. □

Theorem 3.2. *For any regular bounded domain $G \in \mathbb{R}^n$, if (H1) through (H3) hold then*

$$Pu := \nabla\left\{a(x)\Phi(\nabla u)\right\} + f(x,u,\nabla u) + c(x)\phi(u) = 0 \quad \textit{in } \mathbb{R}^n \;;$$
$$u|_{\partial G} = 0 \qquad \textit{and} \;\; u \neq 0 \quad \textit{in } G;$$

has at most one positive classical solution in G.

Proof. Assume that the problem has two such solutions, u and v. Then from (2.15) of Lemma 2.4, for any $\mu > 0$ and $V(x) := v(x) + \mu$,

$$0 = \int_G \left\{ -aZ(u,v) + c|u|^{\alpha+1} + uf(x,u,\nabla u) - u\phi(\frac{u}{V})\left\{ c\phi(v) + f(x,v,\nabla v)\right\}\right\} dx$$

after integrating (2.15) over G. Thus $\forall \mu > 0$

$$\begin{aligned} &\int_G \{\, aZ(u,v) - c|u|^{\alpha+1} - uf(x,u,\nabla u)\,\}dx \\ &\quad = \int_G \{\, -u\phi\left(\frac{u}{v+\mu}\right)\{c\phi(v) + f(x,v,\nabla v)\,\}dx. \end{aligned} \tag{3.1}$$

Because the left hand side of (3.1) is independent of μ and the function ϕ not being constant in μ, (3.1) cannot hold unless each member of (3.1) is zero.

Assume that each member of (3.1) is zero.

(a) If (H3+) holds, as u solves (1.1),

$$u\phi\Big(\frac{u}{v+\mu}\Big)\{c\phi(v)+f(x,v,\nabla v)\}=\frac{u}{v}\phi\Big(\frac{u}{v+\mu}\Big)\{vc\phi(v)+vf(x,v,\nabla v)\}$$

and the right hand side of (3.1) is strictly positive.

(b) If rather (H3-) holds, u and v being solutions of (1.1), the left hand side of (3.1) reads

$$aZ(u,v)-c|u|^{\alpha+1}-uf(x,u,\nabla u)=aZ(u,v)+\nabla\{\ ua\Phi(\nabla u)\ \}-a|\nabla u|^{\alpha+1}$$

whose integration over G gives

$$\int_G a\{\ Z(u,v)-|\nabla u|^{\alpha+1}\ \}dx=0. \tag{3.2}$$

For the right hand side,

$$-u\phi(\frac{u}{V})\Big\{f(x,v,\nabla v)+c\phi(v)\Big\}=\nabla.\Big[u\phi(\frac{u}{V})a\Phi(\nabla v)\Big]-a\Phi(\nabla v)\nabla.\Big(u\phi(\frac{u}{V})\Big)$$
$$=\nabla\Big[u\phi(\frac{u}{V})a\Phi(\nabla v)\Big]-a\{\ (\alpha+1)\Phi(\frac{u}{V}\nabla v)\nabla u-\alpha\Phi(\frac{u}{V}\nabla v)\frac{u}{V}\nabla v\ \}$$
$$=\nabla\Big[u\phi(\frac{u}{V})a\Phi(\nabla v)\Big]+aZ(u,V)-a|\nabla u|^{\alpha+1}.$$

So, the right hand side of (3.1) is then $\int_G a\{\ Z(u,V)-|\nabla u|^{\alpha+1}\ \}dx$.

But by the continuity of $Z(u,v+\mu)$ in all its arguments, we cannot get

$$\int_G a\{\ Z(u,v+\mu)-|\nabla u|^{\alpha+1}\ \}dx=0,\quad \forall\mu>0, \tag{3.3}$$

while (3.2) also holds. Thus having two distinct positive solutions is in conflict with (3.1). □

Theorem 3.3. *The equation* (1.1) *has oscillatory solutions in* $\mathbb{R}^n$ *if the following two conditions hold:*

1) (H1), (H2) and (H4) hold and

2) with the elements defined in (2.16), (i) and (ii) of the Theorem 2.5 hold.

Proof. By Theorem 2.5, under the hypotheses 1) and 2), the half linear equation (1.3) has oscillatory solutions in $\mathbb{R}^n$. It is then enough to show that the conditions 1) and 2) above imply that zeros of (1.1) and those of (1.2) alternate in the sense that between any two consecutive zeros of one lies one zero of the other. This is guaranteed by Lemma 2.2. □

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