

THE MATHEMATICS OF MINOR KEYS

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Abstract: Why is it that scales beginning with an A note feel sad? Seven-note scales are composed of five whole tones and two semitones. The position of the semitones determines whether the key is major or minor. Not all music in a minor scale necessarily sounds sad, but the arrangement of notes in major and minor scales plays a large role in determining the mood of music.

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1. Why is Starting from an A Note Sad?

Musical scales most commonly start from a C note, but one can also begin from two notes below, an A note. Doing so gives the scale a sad mood. I first learned this in music class in elementary school, where they taught me that this was why some Japanese children's songs like *Akai Kutsu* ("Red Shoes") and *Tsuki no Sabaku* ("Moonlight Desert") sound so sad. Starting from A does indeed change the mood of music, despite the use of the same seven notes to create the scale, just shifting it over two places. I have wondered this for over 50 years, and so would like to do some mathematical reasoning on the topic.

Representing a C-D-E-F-G-A-B major key and an A-B-C-D-E-F-G minor key as whole tones (W) and semitones (S), we have the following:

Major: WWSWWWS
 Minor: WSWWSWW

Both are the same in that they have seven tones – five whole tones and two semitones – but the semitones are in the third and seventh position in the major key and in the second and fifth position for the minor key. It is strange that just changing the positions of the semitones can affect the mood of music so much. Starting the scale on a C key of a piano keyboard produces a cheery sound, but from an A or E key it sounds sad. Examples include Beethoven’s *Für Elise* and Mozart’s *Turkish March*.

2. The Major and Minor Thirds

Despite an extensive search for a clear answer to my question, I could not find a complete explanation and so decided that I would have to come up with one on my own. Yasushi Akutagawa’s *Ongaku no Kiso* (“Fundamentals of Music”) (p. 67) shows a diagram of major and minor scales in a staircase pattern and explains:

The definitive difference between minor and major scales lies in that the distance between the tonic and the third is shorter in the final. In other words, it has a narrower interval (a major third in the major scale, and a minor third in the minor scale). This is where the naming of major and minor scales comes from, see [1].

Hideyuki Kojima’s *Onkai Nyumon* (“An Introduction to Scales”) (p. 87-88) presents an even more detailed illustration than does Akutagawa’s, and this is shown in Figure 1, see [4]. In the illustration, on the left is a major scale (major key) with C as its fundamental note (its tonic), and so the progression is C-D-E-F-G-A-B. I will give more details about the naming scheme later, but the C note is called perfect unison, the D note is the major second, E is the major third, F is the perfect fourth, G is the perfect fifth, A is the major sixth, B is the major seventh, and the higher C is called the perfect octave. The right column shows a minor scale (minor key) with A as the tonic, followed by the B through G notes. Here, A is the perfect unison, B is the major second, C is the minor third, D is the perfect fourth, E is the perfect fifth, F is the minor

8	Do		Ra	8
Major 7th	Ti		Sol	Minor 7th
Major 6th	La		Fa	Minor 6th
5	Sol		Mi	5
4	Fa		Re	4
Major 3rd	Mi		Do	Minor 3rd
Major 2nd	Re		Ti	Major 2nd
1	Do		Ra	1
		Major scales	Minor scales	

Figure 1: Scales in major and minor keys (from [4])

sixth, G is the minor seventh, and the higher A is the perfect octave. Kojima explains:

A major scale is one in which each note other than the perfect fourth and the perfect fifth forms a major interval, and a minor scale is one in which all except the second form a short interval. It is often said that music in a minor scale has a darker mood than does music in a major scale, but there are many differing opinions on this matter.

Comparing the right and left columns of Figure 1, we can see that the first, fourth, fifth, and eighth notes are the same, and these are called the perfect fourth, the perfect fifth, and so on. The numbers indicating other intervals are prefixed with “major” on the left side, and all but one “minor” on the right. Could this be related to the different moods between music written in major and minor keys? Akutagawa emphasizes the difference between the major third and the minor third.

3. What is an Interval?

The separation between two pitches is generally indicated using numbers. Notes with the same pitch are said to be in perfect unison. Note that the separation between two notes with the same number can differ. In a standard diatonic interval, we have the following:

	Minor	Major	Perfect
S1	Minor second		
W1		Major second	
S1+W1	Minor third		
W2		Major third	
S1+W2			Perfect fourth
W3			Augmented fourth
S3+W2			Diminished fourth
S1+W3			Perfect fifth
S2+W3	Minor sixth		
S1+W4		Major sixth	
S2+W4	Minor seventh		
S1+W5		Major seventh	

Table 1: Diatonic intervals

- The minor second (1 semitone) and the major second (1 whole tone).
- The minor third (1 semitones and 1 whole tone) and the major third (2 whole tones).
- The perfect fourth (1 semitone and 2 whole tones) and the augmented fourth (3 whole tones).
- The diminished fifth (2 semitones and 2 whole tones) and the perfect fifth (1 semitone and 3 whole tones).
- The minor sixth (2 semitones and 3 whole tones) and the major sixth (1 semitone and 4 whole tones).
- The minor seventh (2 semitones and 4 whole tones) and the major seventh (1 semitone and 5 whole tones).

To clean this up a bit, I have ordered the note separations from smallest to largest in Table 1. Since an octave has twelve semitones it shows eleven steps from the first. In that table, S1 means a single semitone, and W1 means a single whole tone. S1 + W1 means 1 semitone and 1 whole tone. Writing it this way shows that there are two 2-step tones, the minor second and the major second. Similarly, there are two 3-step tones, the minor third and the major third. There are two exceptions to the “perfect,” “major,” and “minor” classification, the “augmented fourth” with its three whole tones, and the “diminished fifth” with its two semitones and two whole tones. Both have the same separation, however, which is six semitones.

There are no “majors” or “minors” for the fourth and fifth intervals, the perfect fourth (1 semitone and 2 whole tones) and the perfect fifth (1 semitone and 3 whole tones) existing in both the minor and major keys. This alone is not enough to distinguish between major and minor keys. The difference when it comes to scales is that the perfect fourth comes between the C and F notes in the major scale and the A and D notes in a minor scale, and the perfect fifth comes between the C and G notes in the major scale and between the A and E notes in the minor scale.

There are four intervals that take the “perfect” prefix, the first (“unison”), fourth, fifth, and eighth. The perfect unison is a sound with the same pitch, and the eighth the note is one octave above, or twice the pitch, and is made with a string of inverse, or $1/2$, length. So what do we mean by the perfect fourth and the perfect fifth?

Pythagoras performed his sound experiments in ancient Greece using an instrument called a monochord. This instrument has a single string above a sound box, and the player changes its pitch by sliding a movable bridge. The string is divided into twelve parts, and using combinations of 12, 9, 8, and 6 allow it to be used with various ratios. For example, making the string length $2/3$ of the full length produces the interval between C and G, and making it $3/4$ of the full length produces the interval from C to F. The interval from C to G is $3/2$, and is called the perfect fifth. The interval from C to F is $4/3$, and is called the perfect fourth. The perfect fifth is particularly important in that by beginning at an interval such as C and G and continuing by a fifth twelve times either up or down one can produce a twelve-note scale (including semitones). A similar scale was also used in ancient China, where it was called *sanfen sunyi*.

A musical scale is a progression of sounds covering a single octave, each step is a fixed distance and is ordered according to pitch. Considering the tonalities of the twelve semitones in an octave, there would be 24 possibilities, twelve in each of the major and minor keys, and both the C-based tonality and the A-based one are fine examples. In ancient Greece there were seven practical scales, but the best developed ones were the Ionian mode which began in the C major scale and the Aeolian mode which began in the A minor scale.

Figure 1 shows the Aeolian mode beginning with A as its minor key, but there are other minor scales as well, such as modal minor scales and harmonic minor scales. The sixth and seventh intervals with a sharp (\sharp) are examples, with invariant notes without the sharp called natural minor scales to differentiate. The natural minor scale begins with A and is unable to express the final G and A notes as minor, so this is likely a workaround for that:

A-B-C-D-E-F \sharp -G \sharp -A	(the modal minor scale)
A-B-C-D-E-F-G \sharp -A	(the harmonic minor scale)
A-B-C-D-E-F-G-A	(the natural minor scale)

Sound is the transmission of the vibration of some object through air, and the sounds that humans can hear fall within the range of 20-20,000 Hz (the “zone of audibility”). Humans produce sounds by vibrating the vocal cords found in the upper part of our throat, and the resulting vibrations in the air in turn vibrate the eardrum, which in turn move tiny bones in our ear that excite our auditory nerves. We are able to continuously produce sound, and furthermore to continuously distinguish between sounds. There are no scales in the sound of wind or the murmuring of a brook—scales are an artificial thing. Associating the sounds of nature with cheer or gloom is purely a subjective matter for the listener.

4. Major and Minor Chords

So, are all measures that make up music written in C major cheerful, while all those written in A minor are gloomy? Things are not so simple. A piece of music can be broken down into melodies, and melodies into individual measures, just as writing can be broken down into paragraphs, sentences, and clauses. What we must look at is not entire pieces of music, but its component parts (melodies and measures).

While there are individual differences, the vocal range of most humans is about one octave and a third, see [5]. The range does not depend on whether one is singing in a major or a minor key, and if one uses each of the possible diatonic scales then there is of course no differentiation between major and minor scales. The problem lies in the higher and lower notes in smaller ranges of music. In this case, the concept of “chords” becomes important.

A chord is the superposition of a third and a fifth onto their tonic note. When a major third interval (two whole tones) and a minor third interval (one whole tone and one semitone) are placed above their tonic, the result is a major cord when the major third is on top and a minor cord when a minor third is on top. Here again, the role of the major and minor thirds is key. For example, playing the C, E, and G notes simultaneously results in a chord called C major, its name coming from the tonic (key) note. In the case of A, C, and E the tonic is A, and combining it with its minor and major thirds gives an A minor chord.

Music is commonly made up of combinations of chords, and such combinations are called chord progressions. There are a limitless number of possible

progressions, but let us look at the basic patterns of guitar chords.

- | | |
|--------------|---------------|
| (1) C major | (2) A minor |
| C-F-G | Am-Dm-Em |
| C-G7-C | Am-Em-Am |
| C-F-G7-C | Am-Dm-Em-Am |
| C-Dm-G7-C | Am-Bm-Em-Am |
| C-Em-Dm-G7-C | Am-C-Bm-Em-Am |
| C-Am-F-G7-C | Am-F-Dm-Em-Am |

Chord name	Chord	
C	C-E-G	Major third + Minor third
Dm	D-F-A	Minor third + Major third
Em	E-G-B	Minor third + Major third
F	F-A-C	Major third + Minor third
G	G-B-D	Major third + Minor third
Am	A-C-E	Minor third + Major third
Bm	B-D-F \sharp	Minor third + Major third

Table 2. The basic chords of the C key

It is a matter of fact that some of these chords are easier to play on a guitar than others. When the popularity of folk music was at its peak in the 1970s, it seems like most students had a guitar in their dorm rooms. The F and G7 chords are among the difficult ones, while Em and Dm are relatively easy, so it is probably easier to play songs written in minor keys.

One popular song in Japan, *Tabibito Yo* (“Oh, Traveler,” lyrics by Tokiko Iwatani, music by Kosaku Dan), is written in D minor, but it switches between minor and major chords. The following are the strummed chords along with the lyrics:

The green meadow / shudders in the wind	Dm-Dm-Dm-Dm
Oh, traveler / how your eyes shine	F-Gm-A7-Dm

Converting the lyrics to notes, we can see their connection to the chords:

AFFFEDA / FEDEFGG
 AAAAAGFGG / GFDEFGFED

Many folk songs use a similar technique, so it is not necessarily true that singing a song in a minor key makes it sad. Songs are made up of combinations of major and minor chords, and it is the type which dominates that establishes the mood of the piece. Mozart's Symphony No. 40 was written in G minor, but changes to a major key in parts. It is a piece in a minor key, but with a minor-major-minor pattern.

There are many other exceptions to the rule that music in a major key is cheerful and that in a minor key is gloomy. In August 2008 I was working in a laboratory in Cambridge, England. One day after dinner I overheard a familiar tune coming from the PC of a graduate student from North Vietnam who was watching YouTube videos. Looking at his screen, I saw that it was the song *Mukashi no Namae de Deteimasu* ("You'll find me under my old name," lyrics by Tetsuro Hoshino, music by Gendai Kano). The song is known as *Hoa Ann Dao* in Vietnam, which I understand means "the cherry blossoms are in full bloom." One tends to consider lyrics and music as a single entity, but I found this an interesting case in that only the music spread overseas, and as a result the pathos of the original song was lost, transforming the music into something that others can enjoy as a completely different, even cheerful melody.

5. The Pentatonic Scale and Traditional Japanese Scales

We have focused on heptatonic (7-note) scales so far, but I would also like to mention the pentatonic (5-note) scales used in Japan. Many of the scales used in Japanese music, especially the court music of ancient Japan, are a modification of the Chinese chromatic scale of twelve pitches. The scales of traditional Japanese music, however, are based on four tetrachords, the *minyo*, *ritsu*, *miyako-bushi*, and *ryukyu*, each with corresponding scales. The *minyo* and *ritsu* scales use the *yo-sen* scale (a scale without semitones) and the *miyako-bushi* scale uses the *in-sen* (a scale with semitones), see [3]:

· <i>Minyo</i> scale:	DFG	ACD
· <i>Ritsu</i> scale:	DEG	ABD
· <i>Miyako-bushi</i> scale:	DE \flat G	AB \flat D
· <i>Ryukyu</i> scale:	DF \sharp G	AC \sharp D

Another scale is the *yona-nuki* ("fourth and seventh omitted") scale, an amalgam of Japanese and Western styles, which was introduced in the Meiji Era. The *yona-nuki* scales uses the Western C-D-E-F-G-A-B-C scale, but with the fourth and seventh notes dropped. The major scales of *yona-nuki* do not

use F as the fourth note and B as the seventh, making a C-D-E-G-A pentatonic scale; the minor scales of *yona-nuki* do not use D as the fourth note and G as the seventh from A-B-C-D-E-F-G-A, making a A-B-C-E-F-A pentatonic scale, see [3]. This scale is often used in Japanese *enka* music and in folk songs such as *Kitaguni no Haru* (“Spring in the North Country”) and *Subaru* (“The Pleiades”).

Pentatonic scales use five notes of the octave, with a variety of methods used to determine the combination. Pentatonic scales are most common in Japan and China, and moreover have a strong presence in southeast and southwest Asia. The gamelan music of Java uses scales called *slendro* and *pelog*, both pentatonic scales. Some Indian music uses pentatonic scales as well. The use of pentatonic scales is indeed widespread, reaching beyond Asia as far as Africa and Europe (Ireland and Hungary, at least), see [3].

The use of five notes (an odd number) in a pentatonic scale can likely be connected to the five fingers on a human hand. Heptatonic scales instead divide the octave up into twelve semitones (an even number), an extremely useful value because it is so easy to divide and multiply with. The twelve semitones of the heptatonic scale are arranged as five whole tones and two non-combined semitones to create a seven note scale (an odd number). It seems as though the pentatonic, heptatonic, and twelve note scales use even and odd numbers in culturally specific ways.

Folk music was very popular from the late 1960s to the 1970s, and I remember many of the songs from that time being in minor keys. To confirm, I used one of the referenced works [2] to list the 572 most popular songs in a spreadsheet and look for trends. When looking at the ratio of minor to major chords used in those songs, the minor chords were indeed more numerous. Perhaps minor chords did a better job of reflecting the mood of the people at the time?

There were also many songs in minor keys that were popular in Japan before the 1970s. Many of the Russian folk songs (collectively known as *utagoe*) that became popular in Japan after World War II were in minor keys, as were many prewar songs, post-1970s “group sound” pop songs, and traditional ballads and *enka* pieces. Overseas, minor keys are often seen in Latin music, while major keys are more common in American folksongs. Curious whether there were more songs in major or minor keys, I discovered on Wikipedia that there were approximately an equal number of representative songs in each key. It seems as if liking music in major or minor keys is largely a matter of personal preference. Given that, I now have a newfound respect for Pythagorean tuning and its ability to convert the same pentatonic scale into major or minor forms just by changing the position of the semitones.

References

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