A FURTHER CONDITION IN THE EXTENDED MACROSCOPIC APPROACH TO RELATIVISTIC GASES

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Abstract: An exact macroscopic extended model, with many moments, for relativistic gases has been recently proposed in literature. However, a further condition, arising from the exploitation of the entropy principle, has not been imposed, even if its presence is evident in the case of a charged gas and when the electromagnetic field acts as an external force. In the present paper we exploit it and we prove that it amounts in many identities plus some residual conditions which allow to determine the arbitrary single variable functions present in the general theory. The result is that they are polynomials of increasing degree with respect to equilibrium, which coefficients are arbitrary constants. Even in such case the macroscopic model remains more general than the kinetic one.

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1. Introduction

In order to describe the context to which this work applies, let us consider the Vlasov equation (see [5]) multiplied by the rest particle mass \( m \), i.e.,

\[ p^\mu \frac{\partial f}{\partial x^\mu} + eF^{\alpha\mu}p_\mu \frac{\partial f}{\partial p^\alpha} = 0, \]  

(1)

where \( F^{\alpha\mu} \) is the skew-symmetric electromagnetic tensor which can be expressed...
in terms of the electric field \( E_i \) and the magnetic field \( H_i \) as

\[
F^{\alpha\mu} = \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & H_3 & -H_2 \\
-E_2 & -H_3 & 0 & H_1 \\
-E_3 & H_2 & -H_1 & 0
\end{pmatrix}.
\]

\( f \) is the distribution function, \( e \) the particle charge and \( p^\mu \) the four momentum satisfying the relation \( p^\mu p_\mu = -m^2 \) (we assume unitary the light speed). We neglect here the term due to collisions between the atoms, which can be found in [5], [10] or [7], because it does not affect our following considerations. Obviously, if \( F^{\alpha\mu} = 0 \), equation (1) is the Boltzmann equation. Let us now choose two fixed numbers \( M \) and \( N \) such that \( M + N \) is odd, and \( M > N \). If we multiply equation (1) by \( p^{\alpha_1} \cdots p^{\alpha_M} \) and \( p^{\alpha_1} \cdots p^{\alpha_N} \) and integrate it with respect to \( dP = \sqrt{-g} dp^1 dp^2 dp^3 \) (the invariant element of momentum space), we obtain the fields equations

\[
\begin{align*}
\partial_\alpha A^{\alpha_1 \cdots \alpha_M} &= M e F^{(\alpha_1 \cdots \alpha_M)} p_\mu, \\
\partial_\alpha A^{\alpha_1 \cdots \alpha_N} &= N e F^{(\alpha_1 \cdots \alpha_N)} p_\mu,
\end{align*}
\]

where the following definition

\[
A^{\alpha_1 \cdots \alpha_n} = \int fp^{\alpha_1} \cdots p^{\alpha_n} dP,
\]

has been used for the generic moment of the distribution function. The maximum traces of these equations are the conservation laws of mass and momentum-energy.

From equation (3), the trace condition

\[
A^{\alpha_1 \cdots \alpha_n} g_{\alpha_{n-1}\alpha_n} = -m^2 A^{\alpha_1 \cdots \alpha_{n-2}}
\]

follows. Consequently, the moment appearing in the right hand side of equation (2)\(_2\) can be expressed in terms of that differentiated in the left hand side of equation (2)\(_1\), as

\[
A^{\mu_2 \cdots \alpha_N} = (-m^2)^{-\frac{M-N+1}{2}} A^{\mu_2 \cdots \alpha_{M+1}} g_{\alpha_{N+1}\alpha_{N+2}} \cdots g_{\alpha_M\alpha_{M+1}}.
\]

The moment appearing in the right hand side of equation (2)\(_1\) has order \( M \) and it cannot be expressed in terms of that differentiated in the left hand side of equation (2)\(_2\) (which has order \( N + 1 \)) unless \( N = M - 1 \).
This is a restriction on the general case (see [3]), where \( N \) and \( M \) were constrained only by the condition \( M + N \) odd.

Now the entropy principle (see [8], for example) affirms that \( \partial_\alpha h^\alpha = \sigma \geq 0 \) holds for every solution of system (2), where \( h^\alpha \) is the entropy-entropy flux tensor and \( \sigma \) is the entropy production. This is equivalent to assume the existence of the symmetric Lagrange multipliers \( \mu_{\alpha_1\ldots\alpha_{M-1}} \) and \( \lambda_{\alpha_1\ldots\alpha_M} \) such that

\[
A^{\alpha_1\ldots\alpha_{M-1}} = \frac{\partial h'^\alpha}{\partial \mu_{\alpha_1\ldots\alpha_{M-1}}}, \quad A^{\alpha_1\ldots\alpha_M} = \frac{\partial h'^\alpha}{\partial \lambda_{\alpha_1\ldots\alpha_M}},
\]

\[(M-1) e \mu_{\alpha_1\ldots\alpha_{M-1}} F^{\alpha_1}_\mu A^{\alpha_2\ldots\alpha_{M-1}\mu} + M e \lambda_{\alpha_1\ldots\alpha_M} F^{\alpha_1}_\mu A^{\alpha_2\ldots\alpha_M\mu} \geq 0,
\]

where \( h'^\alpha = \mu_{\alpha_1\ldots\alpha_{M-1}} A^{\alpha_1\ldots\alpha_{M-1}} + \lambda_{\alpha_1\ldots\alpha_M} A^{\alpha_1\ldots\alpha_M} - h^\alpha \).

If the electromagnetic field acts as an external force, the residual inequality (5) is linear in \( F^{\alpha}_\mu \) and it would be violated unless

\[(M-1) A^{\alpha_2\ldots\alpha_M}[\mu^{\alpha_1}]_{\alpha_2\ldots\alpha_{M-1}} + MA^{\alpha_2\ldots\alpha_M}[\mu^{\lambda^{\alpha_1}}]_{\alpha_2\ldots\alpha_M} = 0.
\]

Thanks to equations (4), (5), this relation can be written as

\[
-m^2(M-1) \frac{\partial h'^\mu_\alpha}{\partial \mu^\alpha_2\ldots\mu^\alpha_{M-1}} \mu^{\alpha_1} A^{\alpha_2\ldots\alpha_{M-1}} g_{\alpha M \alpha M+1} + M \frac{\partial h'^\mu_\alpha}{\partial \mu^{\lambda^{\alpha_1}}}_2 A^{\alpha_2\ldots\alpha_M} = 0.
\]

In this paper we will impose this further condition for a relativistic gas, as it has already be done for an ultra-relativistic gas (see [6]), with a completely different procedure. We will take into account the case with \( M \) even. The case \( M \) odd will be the argument of a future work.

The left hand sides of equations (5) are symmetric, so the right hand sides must be symmetric too. Thanks to this consideration the problem of solving of system (2) comes down to find the tensor \( h'^\alpha \) that satisfies the following symmetry conditions

\[
\frac{\partial h'^\alpha}{\partial \mu_{\beta_1\ldots\beta_{M-1}}} = 0, \quad \frac{\partial h'^\alpha}{\partial \lambda_{\beta_1\ldots\beta_M}} = 0,
\]

plus equation (6).

The symmetry conditions (7) have been exploited in [3] and [4]. The result is expressed in terms of the Taylor expansion of \( h'^\alpha \) around equilibrium. In order to define equilibrium it is firstly necessary to define the quantities

\[
\lambda = \frac{2^{(M-1)!!}}{(M+2)!!} \lambda_{\alpha_1\alpha_2\ldots\alpha_{M-1}\alpha_M} g^{\alpha_1\alpha_2} \cdots g^{\alpha_{M-1}\alpha_M} (-m^2)^{M/2},
\]
\[ \mu_\alpha = \frac{8(M - 1)!!}{(M + 2)!!} \mu_{\alpha_1 \alpha_2 \cdots \alpha_{M-3} \alpha_{M-2}} g_{\alpha_1 \alpha_2} \cdots g_{\alpha_{M-3} \alpha_{M-2}} (-m^2)^{\frac{M-2}{2}}, \]
\[ \gamma = \sqrt{-\mu_\alpha}, \quad v_\alpha = \frac{1}{\gamma} \mu_\alpha \quad \text{(so that } v_\alpha v^\alpha = -1). \]

After that, equilibrium can be defined as the state described by the independent variables \( \lambda \) and \( \mu_\beta \) such that
\[ \lambda^{eq}_{\alpha_1 \cdots \alpha_M} = \lambda g_{(\alpha_1 \alpha_2 \cdots \alpha_{M-1} \alpha_M)} (-m^2)^{-M/2}, \]
\[ H^{eq}_{\beta_1 \cdots \beta_{M-1}} = H(\beta_1 \beta_2 \cdots \beta_{M-2} \beta_{M-1}) (-m^2)^{(M-2)/2}. \]

The Taylor expansion of \( h^\alpha \) around equilibrium is
\[ h^\alpha = \sum_{h,k=0}^{\infty} \frac{1}{h!k!} H^{\alpha A_1 \cdots A_h B_1 \cdots B_k} \tilde{\lambda}_{A_1} \cdots \tilde{\lambda}_{A_h} \tilde{\mu}_{B_1} \cdots \tilde{\mu}_{B_k}, \]
where \( A_i \) and \( B_j \) are multi index notations standing, respectively for \( \alpha_{i_1} \cdots \alpha_{i_M} \) and \( \beta_{j_1} \cdots \beta_{j_{M-1}} \) and \( \tilde{\lambda}, \tilde{\mu} \) are the deviations of the Lagrange multipliers from equilibrium.

In [3] and [4] we proved that equations (7) amounts in imposing that the tensor \( H^{\alpha A_1 \cdots A_h B_1 \cdots B_k} \), depending only on the variables \( \lambda \) and \( \mu_\beta \), is symmetric together with its partial derivatives with respect to \( \mu_\beta \) and that satisfies the following conditions
\[ H^{\alpha A_1 \cdots A_h \gamma_1 \cdots \gamma_{M-1} \gamma_M B_1 \cdots B_k} g_{\gamma_1 \gamma_2} \cdots g_{\gamma_{M-1} \gamma_M} = \frac{\partial H^{\alpha A_1 \cdots A_h B_1 \cdots B_k}}{\partial \lambda} (-m^2)^{\frac{M}{2}}, \]
\[ H^{\alpha A_1 \cdots A_h B_1 \cdots B_k \beta_1 \cdots \beta_{M-2} \gamma_1 \gamma_2} g_{\gamma_1 \gamma_2} \cdots g_{\gamma_{M-3} \gamma_{M-2}} = \frac{\partial H^{\alpha A_1 \cdots A_h B_1 \cdots B_k}}{\partial \mu_\beta} (-m^2)^{\frac{M-2}{2}}. \]

In paper [4] we already proved that from equations (10) it follows
\[ \frac{\partial h^\alpha}{\partial \lambda_{\alpha_1 \cdots \alpha_M}} = \frac{\partial h^\alpha}{\partial \lambda_{\alpha_1 \cdots \alpha_{M-1}}}, \quad \frac{\partial h^\alpha}{\partial \mu_{\alpha_1 \cdots \alpha_{M-1}}} = \frac{\partial h^\alpha}{\partial \mu_{\alpha_1 \cdots \alpha_{M-1}}}. \]

The equations above affirms that calculating the partial derivatives of \( h^\alpha \) with respect to the Lagrange multipliers, or with respect to their deviations from equilibrium (treating them as independent variables) gives the same result.

In [3] and [4] we found the general solution of conditions (10) and we saw that it depends on a enumerable family of functions depending only on \( \lambda \). In
literature condition (6) has been imposed only in the particular case $M = 2$ (see [9]). In this paper we will impose it for whatever even value of $M$. In order to simplify the calculations we will not use the final results of papers [3] or [4] but we will impose it together with the intermediate result (10). In the next section, we will write equation (6) in terms of the tensors $H_{h,k}^{\alpha_1 \cdots \alpha_h B_1 \cdots B_k}$, obtaining that it is equivalent to

$$0 = \frac{\partial}{\partial \lambda} H_{h,k}^{\gamma_1 \cdots \gamma_{Mh}(M-1)k}[\mu \beta_1] + M h g^{\alpha_1}(\gamma_1 H_{h-1,k+1}^{\gamma_2 \cdots \gamma_{Mh}(M-1)k}[\mu \beta_1] - m^{-2}(M-1)k$$

$$
\cdot g^{\alpha_1}(\gamma_{Mh+1} H_{h+1,k-1}^{\gamma_{Mh+2} \cdots \gamma_{Mh}(M-1)k} \gamma_1 \cdots \gamma_{Mh} \alpha_M^{\alpha_M+1}[\mu \beta_1] g_{\alpha_1} g_{\alpha_M^{\alpha_M+1}}).$$

(12)

In Section 3 we will exploit this condition and we will find that it is equivalent to

$$H_{h,k}^{\gamma_1 \cdots \gamma_{Mh}(M-1)k} = (-m^2)^{-k/2}$$

$$\cdot g^{\alpha_1 \alpha_2 \cdots \alpha_{k-1} \alpha_k} H_{h+k,0}^{\gamma_1 \cdots \gamma_{Mh}(M-1)k} \alpha_1 \alpha_2 \cdots \alpha_{k-1} \alpha_k \mu$$

if $k$ is even,

$$H_{h,k}^{\gamma_1 \cdots \gamma_{Mh}(M-1)k} = (-m^2)^{-(k-1)/2}$$

$$\cdot g^{\alpha_1 \alpha_2 \cdots \alpha_{k-2} \alpha_{k-1}} H_{h+k-1,1}^{\gamma_1 \cdots \gamma_{Mh}(M-1)k} \alpha_1 \alpha_2 \cdots \alpha_{k-2} \alpha_{k-1} \mu$$

if $k$ is odd,

and

$$\frac{\partial}{\partial \lambda} H_{h,0}^{\gamma_1 \cdots \gamma_{Mh} \gamma_{Mh+1} \cdots \gamma_{Mh}} \mu \beta_1] + M h g^{\alpha_1}(\gamma_1 H_{h-1,1}^{\gamma_2 \cdots \gamma_{Mh}}[\mu \beta_1] g_{\alpha_1} = 0,$$

(14)

$$\frac{\partial}{\partial \lambda} H_{h,1}^{\gamma_1 \cdots \gamma_{Mh}(h+1)-1}[\mu \beta_1] - m^{-2}[M(h+1) - 1]$$

$$\cdot g^{\alpha_1}(\gamma_1 H_{h+1,0}^{\gamma_2 \cdots \gamma_{Mh}(h+1)-1} \alpha_2 \alpha_3[\mu \beta_1] g_{\alpha_1} g_{\alpha_2} g_{\alpha_3} = 0.$$

This result is very interesting. In fact, equation (13) determines all the tensors $H_{h,k}$ in terms of $H_{h,0}$ and $H_{h,1}$, which are restricted by equation (14).

In Section 4 we will impose equation (10) and we will find that all the conditions on $H_{h,0}$ and $H_{h,1}$ are equivalent to the following ones

$$H_{h+1,0}^{\alpha_1 \cdots \alpha_{M(h+1)}} g_{\alpha_{Mh+1} \alpha_{Mh+2} \cdots \alpha_{Mh+1} \alpha_{M(h+1)}}$$

$$\frac{\partial}{\partial \lambda} H_{h,0}^{\alpha_1 \cdots \alpha_{Mh}} (-m^2)^{\frac{M}{2}},$$

$$H_{h,1}^{\alpha_1 \cdots \alpha_{M(h+1)-1}} g_{\alpha_{Mh+2} \alpha_{Mh+3} \cdots \alpha_{Mh+2} \alpha_{M(h+1)-1}}$$

(15)
As consequence of the conditions above we will find also that

\[ g_{\beta\delta} \frac{\partial^2}{\partial \mu_\beta \partial \mu_\delta} H_{0,0}^\alpha = -m^2 \frac{\partial^2}{\partial \lambda^2} H_{0,0}^\alpha. \]  

(18)

The equation above concerns \( H_{0,0}^\alpha \), which, as we can see from equation (9), is equal to \( h^\alpha \) at equilibrium. Thanks to the Representation Theorems we have that

\[ H_{0,0}^\alpha = H_0^\alpha(\lambda, \gamma) \mu^\alpha. \]  

(19)

The definition above converts equation (18) into

\[ \frac{\partial^2}{\partial \gamma^2} H_{0,0}^0 - m^2 \frac{\partial^2}{\partial \lambda^2} H_{0,0}^0 + \frac{5}{\gamma} \frac{\partial}{\partial \gamma} H_{0,0}^0 = 0, \]  

(20)

which is a second order hyperbolic partial differential equation. Its general solution can be written in integral form, in terms of two arbitrary single variable functions \( F \) and \( G \) and of an arbitrary constant \( \lambda \); it reads

\[ H_{0,0}^0 = -4\pi \frac{m^3}{\gamma} \cdot \int_0^{\infty} [F(\lambda - \bar{\lambda} + m\gamma \cosh \rho) + G(-\lambda + \bar{\lambda} + m\gamma \cosh \rho)] \sinh^2 \rho \cosh \rho \, d\rho. \]  

(21)

By substituting it into equation (20) and integrating by parts twice, we prove that (21) is a solution of (20) (see Section 4 for the details). To prove uniqueness, let us recall that the general solution of equation (20) depends on two arbitrary single variable functions (for example, the values of \( H_{0,0}^0 \) and of \( \frac{\partial}{\partial \lambda} H_{0,0}^0 \) calculated in \( \lambda = \bar{\lambda} \) and this is realized by the solution (21)), by simply relating \( F \) and \( G \) to these values.
From equation (19) and (21) it follows
\[ H_{0,0}^\alpha = \int [F(\lambda - \mu \gamma p^\gamma) + G(-\lambda + \mu \gamma p^\gamma)] p^\alpha dP \]
which clearly satisfies equation (18). After that, we see that
\[ H_{\gamma_1 \cdots \gamma_M h + (M-1)k+1}^\gamma = \int [F^{(h+k)}(\lambda - \mu \gamma p^\gamma) + (-1)^h G^{(h+k)}(-\lambda + \mu \gamma p^\gamma)] p^{\gamma_1 \cdots \gamma_M h + (M-1)k+1} dP \]
satisfies all conditions (15), (16), (17) and (13). In such way we have obtained a particular solution of our closure. By substituting equation (22) in equation (9), we find the corresponding expression for \( \dot{h}_1^\alpha \), i.e.,
\[ \dot{h}_1^\alpha = \int [F(\lambda_{A_1 \cdots A_M} p^{\alpha_1} \cdots p^{\alpha_M} - \lambda + \mu_{A_1 \cdots A_{M-1}} p^{\alpha_1} \cdots p^{\alpha_{M-1}}) \]
\[ + G(-\lambda_{A_1 \cdots A_M} p^{\alpha_1} \cdots p^{\alpha_M} + \lambda + \mu_{A_1 \cdots A_{M-1}} p^{\alpha_1} \cdots p^{\alpha_{M-1}})] p^\alpha dP , \]
and we appreciate that, in the particular case \( G = 0 \), it becomes the solution of the kinetic approach. It is the most general solution at equilibrium. Consequently, we can define \( \Delta \dot{h}_1^\alpha = \dot{h}_1^\alpha - \dot{h}_1^\alpha \) and expand it around equilibrium, finding the counterpart of equation (9), i.e.,
\[ \Delta \dot{h}_1^\alpha = \sum_{h,k=0}^{\infty} \frac{1}{h!k!} C_{h,k}^{\alpha A_1 \cdots A_k B_1 \cdots B_k}(\lambda, \gamma) \bar{\lambda}_{A_1} \cdots \bar{\lambda}_{A_h} \bar{\mu}_{B_1} \cdots \bar{\mu}_{B_k} ; \]
after that, we see that \( \Delta \dot{h}_1^\alpha \) must satisfy the counterparts of equations (15), (16), (17) and (13) and the condition \( \left( \Delta \dot{h}_1^\alpha \right)_{eq.} = 0 \).

In Section 5 we will find that the general solution of these conditions is
\[ C_{h,k}^{\gamma_1 \cdots \gamma_M h + (M-1)k+1}^\gamma \alpha = \left[ \sum_{s=0}^{\frac{Mh+(M-1)k+1}{2}} C_{h,k}^s(\lambda, \gamma) g^\gamma g^{\gamma_2} \cdots g^{\gamma_2 s - 1} g^{\gamma_2 s} \mu^{\gamma_{2s+1}} \cdots \mu^{\gamma_{Mh+(M-1)k}} \mu^\alpha , \right. \]
where the scalar coefficients \( C_{h,k}^s \) are determined in the following way. First of all, let us introduce the family of polynomial in the variable \( \lambda \) defined recursively...
by

\[
\begin{cases}
  f'_0 = 0, & f_0 \text{ is a constant;} \\
  f''_q = -\frac{2q+4}{m^2} f_{q-1}, & \text{if } q \text{ is not a multiple of } \frac{M}{2}; \\
  f'_{Mk} = -\frac{Mk+4}{m^2} f_{Mk-1}, & \text{if } q \text{ is a multiple of } \frac{M}{2}, \text{ i.e. } q = \frac{M}{2} k.
\end{cases}
\] (26)

Note that, by integrating the second of these relations, two new arbitrary constants arise; while, only one new arbitrary constant arises from integration of the third of the above relations.

After that, let us define the following quantities

\[
C_{h,q} = \begin{cases}
  f_q^{(h-1)}, & \text{for } q = 0, \ldots, \frac{M-2}{2}; \\
  f_q^{(h-2)}, & \text{for } q = \frac{M}{2}, \ldots, \frac{2M-2}{2}; \\
  f_q^{(h-3)}, & \text{for } q = M, \ldots, \frac{3M-2}{2}; \\
  \cdots, & \ldots \\
  f_q^{(h-k)}, & \text{for } q = \frac{M}{2} (k-1), \ldots, k \frac{M}{2} - 1;
\end{cases}
\] (27)

and for \( k = 1, \ldots, h \).

This definition can be synthesized in

\[
C_{h,q} = \frac{\partial^{h-1}[-2q/M]}{\partial \lambda^{h-1}[-2q/M]} f_q.
\]

Finally, the scalar coefficients \( C_{h,k}^s \) in equation (25) are

- when \( k \) is even

\[
C_{h,k}^s = \frac{[Mh + (M - 1)k + 1]!}{(2s)!!} \frac{1}{[Mh + (M - 1)k + 1 - 2s]!!} \cdot \gamma^{-6} \sum_{q=0}^{Mh+(M-1)k-1} \left( \frac{1}{\gamma^2} \right)^{Mh+(M-1)k+q-s} \cdot (\frac{Mh + (M - 1)k - 2s + 2q + 4}{{(2q + 4)}!!}) \cdot (-m^2)^{-\frac{Mh+(M-1)k}{2}} \cdot \frac{1}{\gamma^6} \cdot \frac{1}{[Mh + (M - 1)k - 2q - 2]!!}.
\]
• when \( k \) is odd

\[
C_{h,k}^* = \frac{[Mh + (M - 1)k + 1]!}{(2s)!!} \frac{1}{[Mh + (M - 1)k + 1 - 2s]!} \cdot \gamma^{-6} \sum_{q=0}^{\frac{Mh + (M - 1)k + 1}{2}} \left( \frac{1}{\gamma^2} \right)^{\frac{Mh + (M - 1)k + 1 + q - s}{2}} \\
\cdot [Mh + (M - 1)k - 2s + 2q + 5]!! \cdot \frac{(2q + 6)!!}{(2s)!!} \cdot \gamma^{-6} \sum_{q=0}^{\frac{Mh + (M - 1)k + 1}{2}} \left( \frac{1}{\gamma^2} \right)^{\frac{Mh + (M - 1)k + 1 + q - s}{2}} \\
\cdot [Mh + (M - 1)k - 2q - 3]!!.
\]

After that,

\[
h'_{\alpha} = h'_{1\alpha} + \Delta h'_{\alpha}
\] (28)

is the general solution. It is interesting notice that our results enclose those of [9] for the particular case \( M = 2 \). In [9] only a new arbitrary constant arises from each integration, while equation (26) produces two of these constants, but when \( M = 2 \) every \( q \) is a multiple of \( M/2 \) so the case in (26) is not present. We consider these results very satisfactory.

### 2. Condition (6) up to Whatever Order

In this section we will formulate condition (6) in terms of the tensor \( H_{h,k}^{\alpha A_1 \cdots A_k B_1 \cdots B_k} \).

Thanks to equations (10) and (11), equation (6) at equilibrium is

\[
-m^2(M - 1)H_{1,0}^{\alpha_2 \cdots \alpha_{M+1}}[\mu^1_{\beta_1}] \cdot g_{\alpha_1 \alpha_2 \cdots \alpha_M - 2 \alpha_{M-1}}(\alpha_1 \gamma_{\beta_1}, \alpha_2 \alpha_3 \cdots \alpha_M)(-m^2) - \frac{M^2}{2} = 0,
\]

that, using the identities

\[
\mu_{(\alpha_1 \alpha_2 \cdots \alpha_M - 2 \alpha_{M-1})} = \frac{1}{M - 1} \left[ \mu_{\alpha_1 \alpha_2 \cdots \alpha_M - 2 \alpha_{M-1}} (\alpha_1 \gamma_{\beta_1}, \alpha_2 \alpha_3 \cdots \alpha_M) \right] + (M - 2) \mu_{(\alpha_2 \alpha_3 \cdots \alpha_{M-1})}.
\]
and \( g(\alpha_1 \alpha_2 \cdots \alpha_{M-1} \alpha_M) = g_{\alpha_1}(\alpha_2 \cdots g_{\alpha_{M-1}} \alpha_M) \), converts into
\[
-m^{-2} \cdot (-m^2)^{-\frac{M-2}{2}} \left[ H_{1,0}^{\alpha_2 \cdots \alpha_{M-1} \alpha_M | \mu \beta_1} g_{\alpha_2 \alpha_3} \cdots g_{\alpha_{M-2} \alpha_{M-1} \alpha_M} g_{\alpha_M \alpha_{M+1}} \right] \\
+(M - 2) H_{1,0}^{\alpha_2 \cdots \alpha_{M-1} \alpha_M | \mu \beta_1} g_{\alpha_2 \alpha_3} \cdots g_{\alpha_{M-2} \alpha_{M-1} \alpha_M} g_{\alpha_M \alpha_{M+1}} \\
+M \lambda H_{1,0}^{\alpha_2 \cdots \alpha_{M} | \mu \beta_1} g_{\alpha_1 \alpha_2} \cdots g_{\alpha_{M-1} \alpha_M} \left( -m^2 \right)^{-\frac{M}{2}} = 0 .
\]

The first terms, because of (11), is equal to \( \frac{\partial}{\partial x} H_{0,0}^{\mu \beta_1} \). The second term is zero because \( H_{1,0}^{\alpha_2 \cdots \alpha_{M-1} \alpha_M | \mu \beta_1} g_{\alpha_M \alpha_{M+1}} = H_{0,1}^{\alpha_2 \cdots \alpha_{M-2} \alpha_M | \beta_1 | \mu} = 0 \). The third term is zero because \( H_{0,1}^{\alpha_2 \cdots \alpha_{M} | \mu \beta_1} g_{\alpha_1 \alpha_2} = H_{0,0}^{\alpha_3 \cdots \alpha_{M} | \mu \beta_1} = 0 \). Hence the equation above reduces to \( \frac{\partial}{\partial x} H_{0,0}^{\mu \beta_1} = 0 \), which is an identity. In fact, for the representation theorems, we have \( H_{1,0}^{\mu_0} = H_{0,0}^{0,0} \mu^\mu \) with \( H_{0,0}^{0,0} \) a suitable scalar function.

Equation (6) at order zero with respect to equilibrium is an identity. Let us impose it at order \( h \) with respect to \( \lambda_A \), and at order \( k \) with respect to \( \tilde{\mu}_B \). To this end let us firstly calculate the derivative \( \frac{\partial^{h+k}}{\partial \lambda_A \cdots \partial \lambda_A \cdots \partial \mu_B \cdots \partial \mu_B} \) of equation (6) and calculate the result at equilibrium. We have that the derivative \( \frac{\partial^{k}}{\partial \mu_B \cdots \partial \mu_B} \) of equation (6) is
\[
-m^{-2} (M - 1) \frac{\partial^{k+1}}{\partial \mu_B \cdots \partial \mu_B} \lambda_{\alpha_2 \cdots \alpha_{M+1}}^{\alpha_1} g_{\alpha_{M+1}} \\
+M \frac{\partial^{k+1}}{\partial \mu_B \cdots \partial \mu_B} \lambda_{\alpha_2 \cdots \alpha_{M}}^{\beta_1} g_{\alpha_{M+1}} - m^{-2} (M - 1) (-m^2)^{-\frac{M-2}{2}} \\
+ \left[ \frac{\partial^{k}}{\partial \mu_B \cdots \partial \mu_B} \lambda_{\alpha_2 \cdots \alpha_{M+1}}^{\mu \beta_1} g_{\alpha_{M+1}} \\
+ \frac{\partial^{k}}{\partial \mu_B \cdots \partial \mu_B} \lambda_{\alpha_2 \cdots \alpha_{M}}^{\mu \beta_1} g_{\alpha_{M+1}} \\
+ \frac{\partial^{k}}{\partial \mu_B \cdots \partial \mu_B} \lambda_{\alpha_2 \cdots \alpha_{M+1}}^{\mu \beta_1} g_{\alpha_{M+1}} \right] = 0 ,
\]
as it can be easily seen with the iterative procedure, taking into account that \( \frac{\partial}{\partial \mu_B} \lambda_{\alpha_2 \cdots \alpha_{M-1}}^{\beta_1} \) is constant.

After that, we take the derivative \( \frac{\partial^{h}}{\partial \lambda_A \cdots \partial \lambda_A} \) of the previous equation. It is
\[
\psi_1^{A_1 \cdots A_k B_1 \cdots B_k \mu \beta_1} + \psi_2^{A_1 \cdots A_k B_1 \cdots B_k \mu \beta_1} + \psi_3^{A_1 \cdots A_k B_1 \cdots B_k \mu \beta_1} = 0 ,
\]
with

\[ \psi^A_1\cdots A_k B_1\cdots B_k \mu \beta_1 = -m^{-2}(M - 1) \]

\[ \partial \lambda_{A_1} \cdots \partial \lambda_{A_h} \partial \mu_{B_1} \cdots \partial \mu_{B_k} \partial \lambda_{\alpha_2\cdots \alpha_{M+1}} \mu_{\beta_1} \alpha_2\cdots \alpha_{M-1} g_{\alpha M \alpha_{M+1}} \]

\[ \mu_{\beta_1} \alpha_2\cdots \alpha_{M-1} g_{\alpha M \alpha_{M+1}} \]

\[ \lambda \beta_1 |_{\alpha_2\cdots \alpha_M} \]

\[ \psi^2_{A_1\cdots A_h B_1\cdots B_k \mu} = M \left[ \partial \lambda_{A_2} \cdots \partial \lambda_{A_h} \partial \mu_{B_1} \cdots \partial \mu_{B_k} \partial \mu_{\alpha_2\cdots \alpha_M} \right] ; \]

\[ \psi^3_{A_1\cdots A_h B_1\cdots B_k \mu} = -m^{-2}(M - 1) \]

\[ \partial \lambda_{A_1} \cdots \partial \lambda_{A_h} \partial \mu_{B_1} \cdots \partial \mu_{B_k} \partial \lambda_{\alpha_2\cdots \alpha_{M+1}} \mu_{\beta_1} \alpha_2\cdots \alpha_{M-1} g_{\alpha M \alpha_{M+1}} \]

\[ \partial \lambda_{A_1} \cdots \partial \lambda_{A_h} \partial \mu_{B_1} \cdots \partial \mu_{B_k} \partial \lambda_{\alpha_2\cdots \alpha_{M+1}} \mu_{\beta_1} \alpha_2\cdots \alpha_{M-1} g_{\alpha M \alpha_{M+1}} \]

as it can be easily seen with the iterative procedure, taking into account that \( \frac{\partial}{\partial \lambda_{A_h}} \lambda \beta_1 |_{\alpha_2\cdots \alpha_M} \) is constant.

Let us now evaluate the tensors \( \psi^A_1\cdots A_h B_1\cdots B_k \mu \beta_1 \) at equilibrium, taking into account that \( \frac{\partial}{\partial \lambda_{A_h}} \mu_{\alpha_1\cdots \alpha_M} = g^A_{\alpha_1\cdots \alpha_M} \), \( \frac{\partial}{\partial \mu_{B_k}} \mu_{\alpha_1\cdots \alpha_{M-1}} = g^B_{\alpha_1\cdots \alpha_{M-1}} \) with obvious meaning of the notation.
2.1. Evaluation of the Tensor $\psi_{1}^{A_{1}\cdots A_{h} B_{1}\cdots B_{k} \mu \beta_{1}}$ at Equilibrium

By using equation (9), (11) and (8), we obtain

$$
\psi_{1, eq.}^{A_{1}\cdots A_{h} B_{1}\cdots B_{k} \mu \beta_{1}} = -m^{-2}(M-1)H_{h+1,k}^{A_{1}\cdots A_{h} B_{1}\cdots B_{k} \alpha_{2}\cdots \alpha_{M+1} [\mu \beta_{1} \alpha_{1}}$$

where we have written explicitly the symmetrization over the index $\alpha_{1}$. We have omitted the symmetrization over the remaining indexes because of the contraction with the symmetric tensor $H_{\cdots}^{\alpha_{1}}$

$$
\text{From equations (9) we have}
\psi = \partial_{\lambda} H_{h,k}^{A_{1}\cdots A_{h} B_{1}\cdots B_{k} [\mu \beta_{1}]}.
$$

2.2. Evaluation of the Tensor $\psi_{2}^{A_{1}\cdots A_{h} B_{1}\cdots B_{k} \mu \beta_{1}}$ at Equilibrium

In order to evaluate $\psi_{2, eq.}^{A_{1}\cdots A_{h} B_{1}\cdots B_{k} \mu \beta_{1}}$, we need to prove the following useful property. From equations (9) we have

$$
\frac{\partial h_{\mu}^{A_{1}\cdots A_{h} B_{1}\cdots B_{k} \mu \beta_{1}}}{\partial \mu_{\alpha_{2}\cdots \alpha_{M}}} = \sum_{h=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{h!(k-1)!} H_{h,k}^{\mu_{A_{1}\cdots A_{h} B_{1}\cdots B_{k-1} \alpha_{2}\cdots \alpha_{M}}}
$$

$$
\tilde{\lambda}_{A_{1}} \cdots \tilde{\lambda}_{A_{h}} \tilde{\mu}_{B_{1}} \cdots \tilde{\mu}_{B_{k-1} \mu_{\alpha_{2}\cdots \alpha_{M}}} g_{\alpha_{2}\cdots \alpha_{M}} g_{\alpha_{1}}
$$

$$
+ \sum_{h=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{h!(k-1)!} H_{h,k}^{\mu_{A_{1}\cdots A_{h} B_{1}\cdots B_{k-1} (\delta_{1}\cdots \delta_{M-1} \delta_{M})} g_{\alpha_{1}} \tilde{\lambda}_{A_{1}} \cdots \tilde{\lambda}_{A_{h}}
$$

$$
\tilde{\mu}_{B_{1}} \cdots \tilde{\mu}_{B_{k-1} g_{\beta_{1} \alpha_{1}}}
$$

$$
= \sum_{h=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{h!(k-1)!} H_{h,k}^{\mu_{A_{1}\cdots A_{h} B_{1}\cdots B_{k-1} (\delta_{1}\cdots \delta_{M-1} \delta_{M}) \beta_{1}}
$$

$$
\tilde{\lambda}_{A_{1}} \cdots \tilde{\lambda}_{A_{h} \tilde{\mu}_{B_{1}} \cdots \tilde{\mu}_{B_{k-1}}}
$$
A FURTHER CONDITION IN THE EXTENDED...

\[ \frac{1}{M} \sum_{h=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{h!(k-1)!} \left[ H_{h,k}^{\mu A_1 \cdots A_h B_1 \cdots B_{k-1} \delta_1 \cdots \delta_{M-1} g_{M-1} \beta_1} + \cdots + H_{h,k}^{\mu A_1 \cdots A_h B_1 \cdots B_{k-1} \delta_2 \cdots \delta_{M-2} \delta M g_{M-2} \beta_1} + \cdots + H_{h,k}^{\mu A_1 \cdots A_h B_1 \cdots B_{k-1} \delta_1 \cdots \delta_{M-1} \beta_1} \right] \tilde{\lambda}_{A_1} \cdots \tilde{\lambda}_{A_h} \tilde{\mu}_{B_1} \cdots \tilde{\mu}_{B_{k-1}}. \] (31)

Let us take now the derivative of the relation above with respect to \( \lambda_{\varepsilon_1 \varepsilon M} \) and then let us add to the result the same expression but with \( \lambda_{\delta_1 \delta M} \) and \( \lambda_{\varepsilon_1 \varepsilon M} \) interchanged. We obtain

\[
\frac{\partial^2 h_{\mu}}{\partial \lambda_{\varepsilon_1 \varepsilon M} \partial \mu_{\alpha_2 \cdots \alpha_M}} \frac{\partial \lambda_{\varepsilon_2 \alpha_M}}{\partial \lambda_{\delta_1 \delta M}} + \frac{\partial^2 h_{\mu}}{\partial \lambda_{\delta_1 \delta M} \partial \mu_{\alpha_2 \cdots \alpha_M}} \frac{\partial \lambda_{\varepsilon_2 \alpha_M}}{\partial \lambda_{\varepsilon_1 \varepsilon M}} = \frac{1}{M} \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(h-1)!(k-1)!} \left[ H_{h,k}^{\mu A_1 \cdots A_h-1 B_1 \cdots B_{k-1} \varepsilon_1 \cdots \varepsilon M \delta_1 \cdots \delta_{M-1} g_{M-1} \beta_1} + \cdots + H_{h,k}^{\mu A_1 \cdots A_h-1 B_1 \cdots B_{k-1} \varepsilon_1 \cdots \varepsilon M \delta_2 \cdots \delta_{M-2} \delta M g_{M-2} \beta_1} + \cdots + H_{h,k}^{\mu A_1 \cdots A_h-1 B_1 \cdots B_{k-1} \delta_1 \cdots \delta_{M-1} \varepsilon_1 \varepsilon M g_{M-1} \beta_1} \right].
\]

We note that the tensors enclosed between the square brackets are all symmetric with respect to the indexes \( \varepsilon_1 \) and \( \delta_1 \), except the third and the last one, that, however, interchange one another under a corresponding change between the indexes \( \varepsilon_1 \) and \( \delta_1 \). So the whole expression is symmetric with respect to \( \varepsilon_1 \) and \( \delta_1 \) and not only with respect to the sets \( \varepsilon_1 \cdots \varepsilon_M \) and \( \delta_1 \cdots \delta_M \). We can now apply this result to the evaluation of \( \psi_{2,eq}^{A_1 \cdots A_h B_1 \cdots B_k, \mu, \beta_1} \). We can use one index from \( A_i \) and the other index from another set \( A_j \). All the \( h \) terms of \( \psi_{2,eq}^{A_1 \cdots A_h B_1 \cdots B_k, \mu, \beta_1} \) become equal one another. This fact, jointly with equation (31), allows us to obtain

\[
\psi_{2,eq}^{A_1 \cdots A_h B_1 \cdots B_k, \mu, \beta_1} = M h g_{\alpha_1}^{\gamma_1 H_{h-1, k+1}^{\gamma_2 \cdots \gamma M_h}} g_{\alpha_1}^{\gamma M_h+(M-1)k} \left[ \mu, \beta_1 \right], \quad (32)
\]

where \( \gamma_1 \cdots \gamma_{M_h} \) are the indexes of \( A_1 A_2 \cdots A_h \) and \( \gamma_{M_h+1} \cdots \gamma_{M_h+(M-1)k} \) are those of \( B_1 B_2 \cdots B_k \).
2.3. Evaluation of the Tensor $\psi^{A_1\ldots A_h B_1\ldots B_k \mu \beta_1}_3$ at Equilibrium

Also before evaluating $\psi^{A_1\ldots A_h B_1\ldots B_k \mu \beta_1}_3$, we need to prove a useful property. From equations (9) we have

$$
\frac{\partial h'^\mu_1}{\partial \lambda_{\alpha_2\ldots\alpha_{M+1}}} \frac{\partial}{\partial \mu_{\delta_1\ldots\delta_{M-1}}} \mu_{\beta_1}^{\alpha_2\ldots\alpha_{M-1} g_{\alpha_M \alpha_{M+1}}}
= \sum_{h=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(h-1)!(k)!} H^\mu_{A_1\ldots A_{h-1} B_1\ldots B_{k} \alpha_2\ldots\alpha_{M+1}}_{h,k}
\cdot \tilde{\lambda}_{A_1} \cdots \tilde{\lambda}_{A_{h-1}} \tilde{\mu}_{B_1} \cdots \tilde{\mu}_{B_k} g_{\beta_1}^{\alpha_1} g_{\delta_1}^{\alpha_1 \cdots \delta_{M-1}} g_{\alpha_M\alpha_{M+1}}
= \sum_{h=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(h-1)!(k)!} H^\mu_{A_1\ldots A_{h-1} B_1\ldots B_{k} \alpha_M\alpha_{M+1}}_{h,k}(\delta_1\ldots\delta_{M-2} g_{\delta_{M-1}}^\beta_1)
\cdot g_{\alpha_M\alpha_{M+1}}
= \frac{1}{M-1} \sum_{h=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(h-1)!(k)!} \left[ H^\mu_{A_1\ldots A_{h-1} B_1\ldots B_{k} \alpha_M\alpha_{M+1}}_{h,k}(\delta_1\ldots\delta_{M-2} g_{\delta_{M-1}}^\beta_1)ight] g_{\alpha_M\alpha_{M+1}}
\cdot \tilde{\lambda}_{A_1} \cdots \tilde{\lambda}_{A_{h-1}} \tilde{\mu}_{B_1} \cdots \tilde{\mu}_{B_k}.
$$

Let us take now the derivative of the relation above with respect to $\mu_{\xi_1\ldots\xi_{M-1}}$ and then let us add to the result the same expression but with $\mu_{\delta_1\ldots\delta_{M-1}}$ and $\mu_{\xi_1\ldots\xi_{M-1}}$ interchanged. We obtain

$$
\frac{\partial^2 h'^\mu_1}{\partial \mu_{\xi_1\ldots\xi_{M-1}} \partial \lambda_{\alpha_2\ldots\alpha_{M+1}}} \frac{\partial^2 h'^\mu_1}{\partial \mu_{\delta_1\ldots\delta_{M-1}} \partial \lambda_{\alpha_2\ldots\alpha_{M+1}}} \frac{\partial^2 h'^\mu_1}{\partial \mu_{\xi_1\ldots\xi_{M-1}} \partial \lambda_{\alpha_2\ldots\alpha_{M+1}}}
= \frac{1}{M-1} \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(h-1)!(k-1)!} \left[ H^\mu_{A_1\ldots A_{h-1} B_1\ldots B_{k} \alpha_M\alpha_{M+1} \xi_1\ldots\xi_{M-1} \delta_1\ldots\delta_{M-2} g_{\delta_{M-1}}^\beta_1}_{h,k}
+ H^\mu_{A_1\ldots A_{h-1} B_1\ldots B_{k} \alpha_M\alpha_{M+1} \xi_1\ldots\xi_{M-1} \delta_2\ldots\delta_{M-1} g_{\delta_{M-2}}^\beta_1}_{h,k} + \ldots
+ H^\mu_{A_1\ldots A_{h-1} B_1\ldots B_{k} \alpha_M\alpha_{M+1} \xi_1\ldots\xi_{M-1} \delta_1\ldots\delta_{M-3} g_{\delta_{M-3}}^\beta_1}_{h,k}
+ H^\mu_{A_1\ldots A_{h-1} B_1\ldots B_{k} \alpha_M\alpha_{M+1} \xi_1\ldots\xi_{M-1} \delta_{M-1}}_{h,k} + \ldots
+ H^\mu_{A_1\ldots A_{h-1} B_1\ldots B_{k} \alpha_M\alpha_{M+1} \xi_1\ldots\xi_{M-1} \xi_{M-2}}_{h,k} + \ldots
+ H^\mu_{A_1\ldots A_{h-1} B_1\ldots B_{k} \alpha_M\alpha_{M+1} \xi_1\ldots\xi_{M-1} \xi_{M-1}}_{h,k} + \ldots
+ H^\mu_{A_1\ldots A_{h-1} B_1\ldots B_{k} \alpha_M\alpha_{M+1} \xi_1\ldots\xi_{M-1} \xi_{M-2}}_{h,k}
\right].
$$
The tensors enclosed between the square brackets are all symmetric with respect to the indexes $\varepsilon_1$ and $\delta_1$, except the third and the last one that interchange one another under a corresponding change between the indexes $\varepsilon_1$ and $\delta_1$. So the whole expression is symmetric with respect to $\varepsilon_1$ and $\delta_1$ and not only with respect to the sets $\varepsilon_1 \cdots \varepsilon_M$ and $\delta_1 \cdots \delta_M$. We can now apply this result to evaluate $\psi_{3, eq.}^{A_1 \cdots A_h B_1 \cdots B_k \mu \beta_1}$. It means that this tensor is symmetric with respect to every couple of indexes, even if one index is taken from one set $B_i$ and the other index from another set $B_j$ and that all the $k$ terms of $\psi_{3, eq.}^{A_1 \cdots A_h B_1 \cdots B_k \mu \beta_1}$ become equal one another. This fact, jointly with equation (33), allows us to obtain

$$\psi_{3, eq.}^{A_1 \cdots A_h B_1 \cdots B_k \mu \beta_1} = -m^{-2}(M - 1)k$$

$$\cdot g^{\alpha_1(\gamma_{Mh+1})} H_{h-1,k+1}^{\gamma_{Mh+2} \cdots \gamma_{Mh+(M-1)k}\gamma_1 \cdots \gamma_{Mh}\alpha M\alpha M+1}[\mu \beta_1] g_{\alpha_1} g_{\alpha M\alpha M+1}. \quad (34)$$

Blending together the partial results (30), (32) and (34) we prove that equation (29), calculated at equilibrium, converts into equation (12).

3. Exploitation of Condition (12)

In this section we will exploit equation (12) together with equations (10). Let us consider firstly the case with $h \geq 1$ and $k \geq 1$. By taking the skew-symmetric part of equation (12), with respect to the indexes $\gamma_1$ and $\gamma_{Mh+1}$ we have that:

- The first term of equation (12) gives zero contribute, because it is symmetric.

- Let us take into account the identity

$$M h g^{\alpha_1(\gamma_1)} H_{h-1,k+1}^{\gamma_{2} \cdots \gamma_{Mh}} H_{h-1,k+1}^{\gamma_{Mh+1} \cdots \gamma_{Mh+(M-1)k}\mu \beta_1}$$

$$= g^{\alpha_1(\gamma_1)} H_{h-1,k+1}^{\gamma_{2} \cdots \gamma_{Mh}} H_{h-1,k+1}^{\gamma_{Mh+1} \cdots \gamma_{Mh+(M-1)k}\mu \beta_1}$$

$$+ (Mh - 1)g^{\alpha_1(\gamma_2)} H_{h-1,k+1}^{\gamma_{3} \cdots \gamma_{Mh}} H_{h-1,k+1}^{\gamma_{Mh+1} \cdots \gamma_{Mh+(M-1)k}\mu \beta_1},$$

where we made plain the symmetrization with respect to the index $\gamma_1$. Thanks to this identity, the skew-symmetric part of the second term of (12), with respect to the indexes $\gamma_1$ and $\gamma_{Mh+1}$, is

$$g^{\alpha_1(\gamma_1)} H_{h-1,k+1}^{\gamma_{Mh+1}} g_{\alpha_1}.$$
Let us consider the identity

\[(M - 1) k g^{(\gamma M + 1) H}_{h+1,k-1} = g^{(\gamma M h + 1) H}_{h+1,k-1} + [(M - 1)k - 1] \cdot g^{(\gamma M h + 2) H}_{h+1,k-1} + \cdots + [(M - 1)k - 1] \cdot g^{(\gamma M h + (M - 1)k) H}_{h+1,k-1} + \sum_{\mu} g^{(\gamma M h + \mu) H}_{h+1,k-1},\]

where we made plain the symmetrization with respect to \(\gamma M h + 1\). The identity above converts the skew-symmetric part of the third term of equation (12), with respect to the indexes \(\gamma_1\) and \(\gamma M h + 1\), in

\[\frac{-1}{m^2} g^{(\gamma M h + 1) H}_{h+1,k-1} g^{(\gamma M h + 1) H}_{h+1,k-1} = 0,\]

Consequently, the skew-symmetric part of equation (12), with respect to the indexes \(\gamma_1\) and \(\gamma M h + 1\) is

\[g^{(\gamma M h + 1) H}_{h+1,k-1} = 0,\]

with

\[\psi^{(\gamma M h + 1) H + \mu} = H^{(\gamma M h + 1) H + \mu}_{h+1,k-1} + g^{(\gamma M h + 1) H}_{h+1,k-1} g^{(\gamma M h + 1) H}_{h+1,k-1}.\]

Equation (35) can be written also as

\[\frac{1}{4} \left( g^{(\gamma M h + 1) H + \mu}_{h+1,k-1} g^{(\gamma M h + 1) H + \mu}_{h+1,k-1} - g^{(\gamma M h + 1) H + \mu}_{h+1,k-1} g^{(\gamma M h + 1) H + \mu}_{h+1,k-1} \right) = 0,\]

that, multiplied by \(g_{\beta_1} \gamma_1\), is

\[\frac{1}{4} \left( 2\psi^{(\gamma M h + 1) H + \mu}_{h+1,k-1} + g^{(\gamma M h + 1) H + \mu}_{h+1,k-1} \psi^{(\gamma M h + 1) H + \mu}_{h+1,k-1} \right) = 0.\]

If we multiply the equation above by \(g^{(\gamma M h + 1) H + \mu}_{h+1,k-1}\) we obtain \(\psi^{(\gamma M h + 1) H + \mu}_{h+1,k-1} = 0\). This result converts equation (37) into \(\psi^{(\gamma M h + 1) H + \mu}_{h+1,k-1} = 0\), so that equation (36) becomes

\[H^{(\gamma M h + 1) H + \mu}_{h+1,k-1} + m^2 g^{(\gamma M h + 1) H + \mu}_{h+1,k-1} = 0.\]

Equation (38) is very interesting, but it does not exhaust the consequences of (12).
Until now we have only imposed that the left hand side of equation (12) is symmetric with respect to the indexes $\gamma_1$ and $\gamma_{M+h+1}$. It is already symmetric with respect to the indexes $\gamma_i$ and $\gamma_j$ with $i \leq Mh$ and $j \leq Mh$ or with $Mh+1 \leq i \leq Mh+(M-1)k$ and $Mh+1 \leq j \leq Mh+(M-1)k$. Consequently, we can affirm that it is symmetric for whatever $i$ and $j$ and this fact allows us to take the symmetrization of equation (12) with respect to the indexes $\gamma_1 \cdots \gamma_{M+h+(M-1)k}$ leaving unchanged the result. The case $k = 0$ is equation (14)$_1$, while when $k \geq 1$ we have

$$
\frac{\partial}{\partial \lambda} H_{h,k}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k} \left[ \mu \beta_1 \right]} - m^{-2} \left[ Mh + (M-1)k \right] \left[ Mh + (M-1)k \right] = 0,
$$

where equation (38) has been used. It is not necessary to impose equation (39) for every values of $k$ but only for $k = 1$. To prove this let us consider separately the cases $k \geq 3$ and $k = 2$.

- Case $k \geq 3$. From equation (38) we obtain $H_{h,k}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k}}$ in terms of $H_{h+2,k-2}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k}}$ and $H_{h+3,k-3}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k}}$ and substitute them in equation (39). The result is equation (39) with $h + 2$ instead of $h$ and $k - 2$ instead of $k$, contracted with $-m^{-2} g_{\gamma_{M+h+(M-1)k+1} \gamma_{M+h+(M-1)k+2}}$. This fact can be easily seen by taking into account the following identity

$$
[Mh + (M - 1)k + 2]g_{\alpha_1}^{\alpha_1} (\gamma_1 H_{h+3,k-3}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k+2}} \alpha M \alpha M+1 [\mu \beta_1] g_{\alpha_1}^{\alpha_1}
$$

\begin{align*}
&= g_{\alpha_1}^{\alpha_1} (\gamma_1 H_{h+3,k-3}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k+2}} \alpha M \alpha M+1 [\mu \beta_1] g_{\alpha_1}^{\alpha_1} \\
&= -g_{\alpha_1}^{\alpha_1} (\gamma_1 H_{h+3,k-3}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k+2}} \alpha M \alpha M+1 [\mu \beta_1] g_{\alpha_1}^{\alpha_1}) \\
&= 2H_{h+3,k-3}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k+1} \alpha M \alpha M+1 [\beta_1 \mu]}
$$

\begin{align*}
&= [Mh + (M - 1)k] + [Mh + (M - 1)k] \\
&= g_{\alpha_1}^{\alpha_1} (\gamma_1 H_{h+3,k-3}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k+2}} \gamma M \gamma M+1 [\mu \beta_1] g_{\alpha_1}^{\alpha_1} \\
&= [Mh + (M - 1)k] + [Mh + (M - 1)k] \\
&= g_{\alpha_1}^{\alpha_1} (\gamma_1 H_{h+3,k-3}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k+2}} \gamma M \gamma M+1 [\mu \beta_1] g_{\alpha_1}^{\alpha_1}) \\
&= [Mh + (M - 1)k].
\end{align*}

$$
\frac{\partial}{\partial \lambda} H_{h,k}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k} \left[ \mu \beta_1 \right]} - m^{-2} \left[ Mh + (M-1)k \right] \left[ Mh + (M-1)k \right] = 0,
$$

\begin{align*}
&= g_{\alpha_1}^{\alpha_1} (\gamma_1 H_{h+3,k-3}^{\gamma_1 \cdots \gamma_{M+h+(M-1)k+2}} \gamma M \gamma M+1 [\mu \beta_1] g_{\alpha_1}^{\alpha_1}) \\
&= [Mh + (M - 1)k].
\end{align*}
\[ g_{\gamma Mh + (M-1)k + 1} \gamma Mh + (M-1)k + 2 \]

where we have developed the symmetrization with respect to the indexes \( \gamma Mh + (M-1)k + 1 \gamma Mh + (M-1)k + 2 \).

- Case \( k = 2 \). From equation (38) we obtain \( H_{h,2} \) in terms of \( H_{h,2,0} \) and let us substitute it in equation (39). The result is equation (14) with \( h + 2 \) instead of \( h \), contracted with \( -m^{-2} g_{\gamma Mh + 2M - 1} \gamma Mh + 2M \). This fact can be easily seen by taking into account the following identity

\[
[Mh + 2M] g^{\alpha_1 (\gamma_1 H^2_{h+1,0}^2) [\mu \beta_1] g_{\gamma Mh + 2M - 1} \gamma Mh + 2M} + [Mh + 2M - 2] [\mu \beta_1] g_{\gamma Mh + 2M - 1} \gamma Mh + 2M} \]

where we have developed the symmetrization with respect to the indexes \( \gamma Mh + 2M - 1 \gamma Mh + 2M \).

By applying recursively the conclusions of the two cases above we can express whatever tensor \( H_{h,k} \) in terms of \( H_{h,0} \). It means that from equation (39) with \( k = 1 \) follows its validity for all the other values of \( k \).

Knowing that equation (39) is a consequence of (14) we can write it with \( k = 1 \) obtaining the above reported equation (14). We conclude this section noting that equation (38) allows to determine the above reported equation (13), that is very interesting because express all the tensors \( H_{h,k} \) in terms of \( H_{h,0} \) and \( H_{h,1} \), which are restricted by equation (14). In the next section we will impose also equation (10).

### 4. A Particular Solution

Equation (10) restricts the tensors \( H_{h,k} \). These tensors are now determined in terms of \( H_{h,0} \) and \( H_{h,1} \), so equation (10) restricts \( H_{h,0} \) and \( H_{h,1} \). If we write
equation (10) for the particular cases \( k = 0, 1 \), we obtain the above reported equation (15) plus

\[
H_{h+1,1}^{\alpha_1 \ldots \alpha_M (h+2)^{-1}} g_{\alpha_M (h+1) \alpha_M (h+1)+1} \cdots g_{\alpha_M (h+2)-1 \alpha_M (h+2)} = \frac{\partial}{\partial \mu} H_{h+1}^{\alpha_1 \ldots \alpha_M (h+1)-1} \left( -m^2 \right)^{\frac{M}{2}},
\]

\[
H_{h,2}^{\alpha_1 \ldots \alpha_M (h+2)^{-2}} g_{\alpha_M (h+1)+1 \alpha_M (h+1)+2} \cdots g_{\alpha_M (h+2)-3 \alpha_M (h+2)-2} = \frac{\partial}{\partial \mu} H_{h,1}^{\alpha_1 \ldots \alpha_M (h+1)-1} \left( -m^2 \right)^{\frac{M-2}{2}}. \tag{40}
\]

For all the others values of \( k \) equation (10) is a consequence of equations (15) and (40). In fact, thanks to equation (13)\(_1\), equation (10)\(_1\) with \( k \) even is equivalent to equation (15)\(_1\) (with \( h+k \) instead of \( h \)) multiplied by \((-m^2)^{-k/2}\) and then contracted with \( k/2 \) metric tensors. Similarly, equation (10)\(_1\) with \( k \) odd, thanks to equation (13)\(_2\), is equivalent to equation (40)\(_1\) (with \( h+k-1 \) instead of \( h \)) multiplied by \((-m^2)^{-(k-1)/2}\) and contracted with \((k-1)/2\) metric tensors.

Moreover equation (10)\(_2\) with \( k \) even, thanks to equation (13)\(_1\), is equal to equation (15)\(_2\) (with \( h+k \) instead of \( h \)) multiplied by \((-m^2)^{-k/2}\) and contracted with \( k/2 \) metric tensors. To prove this we have to use also the symmetry of \( H_{h,k} \) together with its derivatives with respect to \( \mu \).

In order to prove that equation (10)\(_2\) with \( k \) odd is a consequence of equations (15) and equation (40), we note that, when \( k \) is even, from equation (13)\(_1\) it follows that

\[
H_{h,k}^{\gamma_1 \ldots \gamma_M (h+k-1)\mu} = (-m^2)^{-(k-2)/2} g_{\gamma_1 \alpha_2} \cdots g_{\gamma_{k-3} \alpha_{k-2}} H_{h+k-2,2}^{\gamma_1 \ldots \gamma_M (h+k-1)\mu},
\]

By using this equation for the left hand side of equation (10)\(_2\), with \( k \) odd, and equation (13)\(_2\) for its right hand side, we see that equation (10)\(_2\) with \( k \) odd is equal to equation (40)\(_2\) (with \( h+k-1 \) instead of \( h \)), multiplied by \((-m^2)^{-(k-1)/2}\) contracted with \((k-1)/2\) metric tensors. Consequently we can now avoid to impose equation (10) and we can impose only equations (15) and (40). In the last of these equations the tensor \( H_{h,2} \) appears. By using equation (13)\(_1\) it converts into

\[
H_{h+2,0}^{\alpha_1 \ldots \alpha_M (h+2)} g_{\alpha_M (h+1)+1 \alpha_M (h+1)+2} \cdots g_{\alpha_M (h+2)-1 \alpha_M (h+2)} = \frac{\partial}{\partial \mu} H_{h,1}^{\alpha_1 \ldots \alpha_M (h+1)-1} \left( -m^2 \right)^{\frac{M}{2}}.
\]
If we subtract equation (15) with \( h + 1 \) instead of \( h \) to the relation above it becomes the above reported equation (16). Similarly, if we subtract equation (15) with \( h + 1 \) instead of \( h \), from equation (40) and contract the result with a metric tensor, then equation (40) becomes the above reported equation (17).

For the determinations of the unknowns tensors \( H_{h,0} \) and \( H_{h,1} \) we have to impose equations (14), (15), (16) and (17); after that, equation (13) will give all the others \( H_{h,k} \).

Thanks to the following theorem we will prove that we can avoid to impose equations (14).

**Theorem 1.** If \( \psi_{\gamma_1 \cdots \gamma_n} \) is a symmetric tensor depending only on \( \lambda \) and \( \mu_\beta \), then the following identity holds

\[
\frac{\partial \psi_{\gamma_1 \cdots \gamma_n}}{\partial \mu_\alpha} \mu^\beta + n g^{\alpha_1 (\gamma_1 \psi_{\gamma_2 \cdots \gamma_n})} g^{\beta \alpha} = 0. \tag{41}
\]

**Proof.** From Representation Theorems we know that a symmetric tensor like \( \psi_{\gamma_1 \cdots \gamma_n} \) can be represented as

\[
\psi_{\gamma_1 \cdots \gamma_n} = \sum_{s=0}^{[n/2]} \phi_s^\alpha (\lambda, \gamma) g^{(\gamma_1 \gamma_2 \cdots \gamma_s-1 \gamma_s \mu_\gamma^{s+1} \cdots \mu_\gamma^m)}. \tag{42}
\]

It follows that

\[
\frac{\partial \psi_{\gamma_1 \cdots \gamma_n}}{\partial \mu_\alpha} \mu^\beta + n g^{\beta (\gamma_1 \psi_{\gamma_2 \cdots \gamma_n})} = 0.
\]

\[
\frac{\partial \psi_{\gamma_1 \cdots \gamma_n}}{\partial \mu_\alpha} \mu^\beta = (n + 1) g^{\beta (\gamma_1 \psi_{\gamma_2 \cdots \gamma_n})} - g^{\beta \alpha} \psi_{\gamma_1 \cdots \gamma_n}.
\]

\[
\sum_{s=0}^{[n/2]} \left[ \frac{\partial \phi_s^\alpha}{\partial \gamma} g^{(\gamma_1 \gamma_2 \cdots \gamma_s-1 \gamma_s \mu_\gamma^{s+1} \cdots \mu_\gamma^m)} \gamma^{-1} \mu^\alpha \mu^\beta
\]

\[
+(n - 2s) \phi_s^\alpha \psi_{\gamma_1 \cdots \gamma_s-1 \gamma_s \mu_\gamma^{s+1} \cdots \mu_\gamma^m=0 \]
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\[ + (n - 2s) \phi_s^\alpha g^{\gamma_1 \gamma_2 \ldots \gamma_{2s-1} \gamma_{2s} \mu_{\gamma_{2s+1} \ldots \gamma_n} \alpha} \mu^\beta 
\]

\[ + g^\beta \phi_s^\alpha g^{\gamma_1 \gamma_2 \ldots \gamma_{2s-1} \gamma_{2s} \mu_{\gamma_{2s+1} \ldots \gamma_n} \alpha} \mu^\beta 
\]

\[ + 2s g^\beta (\gamma_1 \phi_s^\alpha g^{\gamma_2 \gamma_3 \ldots \gamma_{2s-2} \gamma_{2s-1} \gamma_{2s} \mu_{\gamma_{2s+1} \ldots \gamma_n} \alpha} + (n - 2s) 
\]

\[ \cdot \phi_s^\alpha g^{\gamma_1 \gamma_2 \ldots \gamma_{2s-1} \gamma_{2s} \mu_{\gamma_{2s+1} \ldots \gamma_n} \alpha} \mu^\beta 
\]

\[ + \gamma_1 \gamma_2 \ldots \gamma_n \cdot \partial_{\alpha \beta} \partial_{\gamma_1 \gamma_2 \ldots \gamma_n} \alpha \mu^\beta 
\]

\[ + \gamma_1 \gamma_2 \ldots \gamma_n \cdot \partial_{\alpha \beta} \partial_{\gamma_1 \gamma_2 \ldots \gamma_n} \alpha \mu^\beta 
\]

We note that all the tensors appearing in the right hand side are symmetric with respect to \( \beta \alpha \), except for the second and fifth term whose sum is, however, symmetric with respect to \( \beta \alpha \). Consequently, also the left hand side is symmetric with respect to \( \beta \alpha \), so its skew-symmetric part with respect to \( \beta \alpha \) is zero.

Now we want to prove that equation (14) is identically satisfied as consequence of equations (15), (16) and (17). Let us take the expression of \( \frac{\partial}{\partial \alpha} H^\alpha \), with \( h - 1 \) instead of \( h \), from equation (16) and put it into equation (14). In the resulting equation we can exchange the indexes \( \mu \) and \( \alpha_M h \) because \( H^\alpha_{h-1,1} \) is symmetric together with its derivatives with respect to \( \mu \). Thanks to the theorem above the resulting expression is an identity. We can proceed similarly for equation (14). Let us take the expression of \( \frac{\partial}{\partial \alpha} H^\alpha_{h,1} \) from equation (17) and put it into equation (14). In the resulting equation we can exchange the index \( \mu \) and \( \gamma_{M(h+1)+1} \) because \( H^\alpha_{h+1,0} \) is symmetric together with its derivatives with respect to \( \mu \). Thanks to the theorem above the resulting expression is an identity.

It remains to impose equations (15), (16) and (17). If we firstly draw a consequence of these equations a simplification will occur.

If we take the derivative of equation (15) with respect to \( \mu_\beta \) and contract the result with \( g_{\alpha_M h+1 \beta} \), we obtain

\[ g_{\alpha_M h+1 \beta} \frac{\partial^2}{\partial \mu_\beta \partial \mu_{\alpha_M(h+1)}} H^\alpha_{h,0} \alpha_M (\frac{-m^2}{2})^{M+2} 
\]

\[ = g_{\alpha_M h+1 \beta} \frac{\partial}{\partial \mu_\beta} H^\alpha_{h,1} \alpha_M (\frac{-m^2}{2})^{M+1} \]

\[ = g_{\alpha_M h+1 \beta} \frac{\partial}{\partial \lambda} H^\alpha_{h-1,0} \alpha_M (\frac{-m^2}{2})^{M+1} \]

\[ = \frac{\partial^2}{\partial \lambda^2} H^\alpha_{h,0} \alpha_M (\frac{-m^2}{2})^{M+1} 
\]

where, in the second passage we have used the expression of \( \frac{\partial}{\partial \mu_\beta} H^\alpha_{h,1} \) taken from equation (16) and, in the second passage, we have used equation (15). In this
way we have found the condition
\[ g_{\alpha M h+1\beta} \frac{\partial^2}{\partial \mu_\beta \partial \mu_\alpha M (h+1)} H^{\alpha_1 \cdots \alpha_{M h}}_{h,0} = -m^2 \frac{\partial^2}{\partial \lambda^2} H^{\alpha_1 \cdots \alpha_{M h}}_{h,0}. \] (43)

We have not to impose this condition as it is a consequence of equations (15), (16) and (17), but its restriction to the case \( h = 0 \), labeled as equation (18), helps us to obtain a particular solution.

In paper [3] we have characterized the family \( F \) of functions \( \psi_{\gamma_1 \cdots \gamma_n} \) depending only on \( \lambda, \mu_\beta \) which are symmetric with respect to their derivatives with respect to \( \mu_\beta \). There we proved that they have the shape (42) with the scalar coefficients \( \phi_s^n \) satisfying the relation
\[ \frac{\partial^2 \phi^n_s}{\partial \gamma^n} 12s + (n + 2 - 2s)(n + 1 - 2s)\phi^n_{s-1} = 0 \quad \text{for } s = 0, \ldots, [n/2]. \] (44)

This relation gives all the scalar functions \( \phi^n_s \) in terms of that with the highest value of \( s \), i.e. \( \phi^n_{[n/2]} \), that we called leading term of order \( n \). The tensors \( H^{\gamma_1 \cdots \gamma_{M h} +(M-1)\mu} \) belongs to the family \( F \) and so it can be written as
\[ H^{\gamma_1 \cdots \gamma_{M h} +(M-1)\mu}_{h,k} = \sum_{s=0}^{[Mh+(M-1)k+1]} H^s_{h,k}(\lambda, \gamma) g^{\gamma_1 \gamma_2 \cdots \gamma_{2s-1} \gamma_{2s} \mu^{2s+1} \cdots \mu^{M h +(M-1) k \mu}}, \] (45)
where the scalar coefficients satisfy equation (44).

Equation (43) for the particular case \( h = 0 \) is labelled as equation (20) and is reported in Section 1. It is a condition on the expression of \( h'^\alpha \) at equilibrium (which is \( H^{\alpha}_{0,0} \) for (9)), which was until now arbitrary. We said also that (21) is a solution of (20), without proving that. So let us do it now. To this end, let us firstly calculate the following integrals
\[ \int_0^\infty (F'' + G'') \sinh^4 \rho \cosh \rho d \rho \]
\[ = \frac{1}{m\gamma} \int_0^\infty \left[ \frac{d}{d \rho} (F' + G') \right] \sinh^3 \rho \cosh \rho d \rho \]
\[ = \frac{1}{m\gamma} \left| (F' + G') \sinh^3 \rho \cosh \rho \right|_0^\infty \]
\[ - \frac{1}{m\gamma} \int_0^\infty (F' + G')(3 \sinh^2 \rho \cosh^2 \rho + \sinh^4 \rho) d \rho \]
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where we have integrated by parts and used the condition that the integrals of $F'$ and $G'$ times an arbitrary polynomial in the variables $\sinh \rho$, $\cosh \rho$ must be finite (which is the integrability condition for equation (22)), so it must be infinitesimal before integration. Another useful integral is the following

$$
\int_{0}^{\infty} (F' + G') \sinh^4 \rho \, d\rho = \frac{1}{m\gamma} \int_{0}^{\infty} \left[ \frac{d}{d\rho} (F + G) \right] \sinh^3 \rho \, d\rho
= \frac{1}{m\gamma} \left[ (F + G) \sinh^3 \rho \right]_{0}^{\infty} - \frac{3}{m\gamma} \int_{0}^{\infty} (F + G) \sinh^2 \rho \cosh \rho \, d\rho
= -\frac{3}{m\gamma} \int_{0}^{\infty} (F + G) \sinh^2 \rho \cosh \rho \, d\rho.
$$

(47)

After that, by using (21), we see that

$$
\frac{\partial^2 H_{0,0}}{\partial \gamma^2} - m^2 \frac{\partial^2 H_{0,0}}{\partial \lambda^2} + \frac{5}{\gamma} \frac{\partial H_{0,0}}{\partial \gamma}
= 12\pi \frac{m^3}{\gamma^3} \int_{0}^{\infty} (F + G) \sinh^2 \rho \cosh \rho \, d\rho
-12\pi \frac{m^4}{\gamma^2} \int_{0}^{\infty} (F' + G') \sinh^2 \rho \cosh^2 \rho \, d\rho
-4\pi \frac{m^5}{\gamma} \int_{0}^{\infty} (F'' + G'') \sinh^4 \rho \cosh \rho \, d\rho
= 12\pi \frac{m^3}{\gamma^3} \int_{0}^{\infty} (F + G) \sinh^2 \rho \cosh \rho \, d\rho
+4\pi \frac{m^4}{\gamma^2} \int_{0}^{\infty} (F' + G') \sinh^4 \rho \, d\rho = 0,
$$

where in the second passage we have used (46) and, in the third passage, we have used (47). So we have proved that (21) is a solution of (20). Uniqueness has been discussed in Section 1 and we have also found the particular solution $h'_{\alpha} = h'_{1,\alpha}$ of our problem, whose expression truncated at equilibrium is the most general one. In the next section we will find the general solution also outside equilibrium, by adding to $h'_{\alpha}$ a suitable function $\Delta h'_{\alpha}$.

5. The General Solution

If we put the expression of $h'_{\alpha}$ taken from equation (28) into equations (6) and (12) we see that they are satisfied also with $\Delta h'_{\alpha}$ instead of $h'_{\alpha}$. Everything we
have said for $h'\alpha$ can be now affirmed also for $\Delta h'\alpha$, so that it and the quantities $C_{h,k}^{\alpha_1\cdots\alpha_M(h+1)}$, appearing in its expansion (24), must satisfy the counterparts of equations (15), (16), (17) and (13), which written explicitly are respectively

$$C_{h+1,0}^{\alpha_1\cdots\alpha_M(h+1)}g^{\alpha_{Mh+1}\alpha_{Mh+2}\cdots\alpha_{M(h+1)-1}\alpha_{M(h+1)}} = \frac{\partial}{\partial \lambda}C_{h,0}^{\alpha_1\cdots\alpha_M(h+1)}(-m^2)^{\frac{M}{2}},$$

$$C_{h,1}^{\alpha_1\cdots\alpha_M(h+1)-1}g^{\alpha_{Mh+2}\alpha_{Mh+3}\cdots\alpha_{M(h+1)-2}\alpha_{M(h+1)-1}} = \frac{\partial}{\partial \mu_{\alpha_M(h+1)}}C_{h,0}^{\alpha_1\cdots\alpha_M(h+1)}(-m^2)^{\frac{M-2}{2}}.$$  

$$\frac{\partial}{\partial \mu_{\alpha_M(h+1)}}C_{h,1}^{\alpha_1\cdots\alpha_M(h+1)-1} = \frac{\partial}{\partial \lambda}C_{h+1,0}^{\alpha_1\cdots\alpha_M(h+1)}.$$  

Moreover, we have the condition

$$\left(\Delta h'\alpha\right)_{eq.} = 0,$$

or, equivalently, $c_{0,0}^{\alpha} = 0.$

It is necessary to impose equation (43) because we have already seen that it is a consequence of the others equations.

Let us start to impose these conditions from (48). Both sides of these equations are elements of the family $\mathcal{F}$ so it suffices to impose its restrictions on their leading terms. If $C_h$ is the leading term of $C_{h,0}^{\alpha}$ it is possible to prove that

$$C_h = \gamma^{-6} \sum_{q=0}^{Mh-1} (-m^2)^{hM/2}c_{h,q}^{(\lambda)}\left(\frac{1}{\gamma^2}\right)^q \frac{[Mh + 1]!!}{[Mh - 2q - 2]!!},$$

(53)
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via an iterative procedure. It is true for $h = 0$ because $C_{h,0}^\alpha = 0$. We assume that it holds up to $h$ and we prove that it holds also with $h + 1$ instead of $h$. By applying equation (51) of [3], with $m = Mh + 1$, $r = M/2$ we obtain equation (53) with $h + 1$ instead of $h$ and

$$c_{h+1,q} = c_{h,q}' \text{ for } q = 0, \ldots, \frac{M}{2}h - 1. \quad (54)$$

For $q = \frac{M}{2}h, \ldots, \frac{M}{2}(h + 1) - 1$ $c_{h,q}$ are new arbitrary functions of $\lambda$.

Starting with $h = 0$ we obtain a family of arbitrary single variable functions $c_{h,q}(\lambda)$.

Let us now impose condition (49). Both sides of this equation are elements of the family $\mathcal{F}$ so it suffices to impose the restriction on their leading terms. If $C_{h,1}$ is the leading term of $C_{\cdot,1}$, from equation (40b) of paper [3] we have that the leading term of the left hand side of (49) is $- \frac{M(h+1)+1}{\gamma} \frac{\partial}{\partial \gamma} C_{h,1}$. By using equation (53) for the right hand side of equation (49) it becomes

$$- \frac{M(h + 1) + 1}{\gamma} \frac{\partial}{\partial \gamma} C_{h,1} = \gamma^{-6} \sum_{q=0}^{\frac{M(h+1)-1}{2}} (-m^2)^{(h+1)M/2} c_{h+1,q}'(\lambda) \cdot \left( \frac{1}{\gamma^2} \right)^q \frac{[M(h + 1) + 1]!!}{[M(h + 1) - 2q - 2]!!} \quad (55)$$

which can be integrated and gives

$$C_{h,1} = \gamma^{-4} \sum_{q=0}^{\frac{M(h+1)-1}{2}} (-m^2)^{(h+1)M/2} c_{h+1,q}'(\lambda) \left( \frac{1}{\gamma^2} \right)^q \frac{[M(h + 1) - 1]!!}{[M(h + 1) - 2q - 2]!!} \frac{1}{2q + 4} + d_{h,1}(\lambda), \quad (56)$$

with $d_{h,1}(\lambda)$ a new arbitrary function which arise from the integration with respect to $\gamma$.

Let us impose now condition (50). From equation (40b) of [3] we have that the leading term of $\frac{\partial}{\partial \mu_\alpha M(h+1)+1} C_{h+1,0}^{\alpha \alpha_1 \cdots \alpha M(h+1)}$ is the same of $C_{h+1,0}^{\alpha \alpha_1 \cdots \alpha M(h+1)}$: $C_{h+1}$. From equation (49) of [3] we have that the leading term of

$$\frac{\partial}{\partial \mu_\alpha M(h+1)+1} C_{h+1,0}^{\alpha \alpha_1 \cdots \alpha M(h+1)} g_{\alpha M(h+1) \alpha M(h+1)+1}$$
is \( \frac{1}{M(h+1)+1}[(Mh + M + 4)C_{h+1} + \gamma \frac{\partial}{\partial \gamma} C_{h+1}] \). Consequently, equation (50) becomes

\[
\frac{\partial}{\partial \lambda} C_{h,1} = - \frac{1}{m^2} \frac{1}{M(h + 1) + 1} [(Mh + M + 4)C_{h+1} + \gamma \frac{\partial}{\partial \gamma} C_{h+1}].
\]

By using equations (53) and (56) it turns into

\[
\frac{1}{2q + 4} + d'_{h,1} = - \frac{1}{m^2} \frac{(Mh + M + 4)}{M(h + 1) + 1} \sum_{q=0}^{\frac{M(h+1)}{2} - 1} (-m^2)^{(h+1)M/2} c''_{h+1,q}(\lambda) \gamma^{-2q-4} \frac{[M(h + 1) + 1]!!}{[M(h + 1) - 2q - 2]!!} + \frac{1}{m^2} \frac{1}{M(h + 1) + 1} \sum_{q=0}^{\frac{M(h+1)}{2} - 1} (-m^2)^{(h+1)M/2} c''_{h+1,q}(\lambda) \gamma^{-2q-6} \frac{[M(h + 1) + 1]!!}{[M(h + 1) - 2q - 2]!!}.
\]

\[
(2q + 6) = - \frac{1}{m^2} \sum_{q=0}^{\frac{M(h+1)}{2} - 1} (-m^2)^{(h+1)M/2} c''_{h+1,q}(\lambda) \gamma^{-2q-6} \frac{[M(h + 1) + 1]!!}{[M(h + 1) - 2q - 2]!!} [M(h + 1) - 2q - 2],
\]

from which it follows

\[
d'_{h,1} = 0, \quad c''_{h+1,0} = 0, \quad c''_{h+1,q+1} = - \frac{2q + 6}{m^2} c''_{h+1,q}, \quad \text{for } q = 0, \ldots, \frac{M(h+1)}{2} - 2. \tag{57}
\]

The relation above is expressed in terms of a polynomial in the variable \( \gamma^{-2} \). The term of zero degree is \( d'_{h,1} \), while the term with degree 2 comes from the first sum, with \( q = 0 \) and is \( c''_{h+1,0} \). After that, in the first sum \( q \) goes from 1 to \( \frac{M(h+1)}{2} - 1 \) and we can change index according to \( q = Q + 1 \). In the last sum, the value \( q = \frac{M(h+1)}{2} - 1 \) gives zero contribution because of the factor \( [M(h + 1) - 2q - 2] \), so \( q \) can go from 0 to \( \frac{M(h+1)}{2} - 2 \) and the coefficient of \( \gamma^{-2q-6} \) is (57)3. From (57)1, we see that \( d_{h,1} \) is a constant.
Let us now impose condition (48). From equation (40b) of [3] we have that the leading term of \( \frac{\partial}{\partial p_{\alpha \lambda (h+1)}} C^{\alpha \lambda \ldots \alpha M h}_{h,0} \) is the same of \( C^{\alpha \lambda \ldots \alpha M h}_{h,0} : C_h \). By applying equation (51) of [3], with \( m = M h + 2 \), \( r = (M/2) - 1 \) we obtain

\[
C_{h,1} = \gamma^{-6} \sum_{q=0}^{M h - 1} [M(h+1) - 1]!! [M(h+1) - 2q - 4]!! (-m^2)^{M(h+1)-2} c_{h,q}(\lambda)
\]

\[
\cdot \left( \frac{1}{\gamma^2} \right)^q + \sum_{i=0}^{M - 2} f_i,_{M h - 2 - 1}(\lambda) \gamma^{-(Mh+6+2i)},
\]

with \( f_i,_{M h - 2 - 1}(\lambda) \) arbitrary functions. By comparing this expression with (56), we find

\[
d_{h,1} = 0, \quad c_{h+1,0}^\prime = 0, \quad c_{h+1,q+1}^\prime = -2q + 6 m^2 c_{h,q}, \text{ for } q = 0, \ldots, \frac{M h}{2} - 1.
\]

The first term is of zero degree in \( \left( \frac{1}{\gamma^2} \right)^2 \), the second is the coefficient of \( \left( \frac{1}{\gamma^2} \right)^2 \) and comes from the summation in (56) with \( q = 0 \). For the other values we can change index in this sum according to \( q = Q + 1 \). After that, the coefficient of \( \left( \frac{1}{\gamma^2} \right)^{3+q} \) is reported in (58) for \( q = 0, \ldots, \frac{M h}{2} - 1 \), while for \( q = \frac{M h}{2}, \ldots, \frac{M(h+1)}{2} - 2 \) it gives simply the functions \( f_i,_{M h - 2 - 1}(\lambda) \).

We have now to impose the conditions (54), (57) and (58) on the single variable functions \( C_{h,q} \). From equation (58) we see that the function \( d_{h,1} \), introduced in equation (56), is not only a constant, it is zero. Moreover, from (58)2, we see that the function \( c_{h,1} \) is a constant. Condition (57)2 follows as a consequence of this fact.

Regarding equations (57)3 and (58)3, we see that they can be substituted by the following ones

\[
c_{h+1,\frac{M h}{2}}^\prime = -\frac{M h + 4}{m^2} c_{h,\frac{M h}{2} - 1}^\prime,
\]

\[
c_{h+1,q+1}^\prime = -\frac{2q + 6}{m^2} c_{h+1,q}, \text{ for } q = \frac{M h}{2}, \ldots, \frac{M(h+1)}{2} - 2.
\]

Equation (59)1 is (58)3 when \( q = \frac{M h}{2} - 1 \) and (59)2 is a subset of equations (57)3. Vice versa, if equations (59) hold, the derivative of (59)1 is \( c_{h+1,\frac{M h}{2}}^\prime = -\frac{M h + 4}{m^2} c_{h,\frac{M h}{2} - 1}^\prime \) where in the last passage, equation (54)
has been used. From the result we see that equation (59)$_2$ holds now for $q = \frac{Mh}{2} - 1, \ldots, \frac{M(h+1)}{2} - 2$.

For $0 \leq q \leq \frac{Mh}{2} - 2$ and for every integer $k < h$ we have that (59)$_2$ with $k$ instead of $h$ is

$$c''_{k+1,q+1} = -\frac{2q + 6}{m^2} c_{k+1,q}, \text{ for } q = \frac{Mk}{2} - 1, \ldots, \frac{M(k+1)}{2} - 2,$$

whose $(h - k)^{th}$ derivative is

$$c^{(h-k+2)}_{k+1,q+1} = -\frac{2q + 6}{m^2} c^{(h-k)}_{k+1,q}, \text{ for } q = \frac{Mk}{2} - 1, \ldots, \frac{M(k+1)}{2} - 2. \hspace{1cm} (61)$$

From equation (54) we have $c^{(r)}_{h,q} = c_{h,r,q}$ for $q = 0, \ldots, \frac{M}{2}h - 1$. This equation with $h = k + 1$ and $r = h - k$ is

$$c^{(h-k)}_{k+1,q} = c_{h+1,q}, \text{ for } q = 0, \ldots, \frac{M(k+1)}{2} - 1,$$

that with $q + 1$ instead of $q$ reads

$$c^{(h-k)}_{k+1,q+1} = c_{h+1,q+1}, \text{ for } q = 0, \ldots, \frac{M(k+1)}{2} - 2.$$

By substituting $c^{(h-k)}_{k+1,q+1}$ and $c^{(h-k)}_{k+1,q}$ from the two equations above into equation (61) we obtain

$$c''_{h+1,q+1} = -\frac{2q + 6}{m^2} c_{h+1,q}, \text{ for } q = \frac{Mk}{2} - 1, \ldots, \frac{M(k+1)}{2} - 2.$$

If $I_k$ is the set of integers belonging to the interval $[\frac{Mk}{2} - 1, \frac{M(k+1)}{2} - 2]$, we have that $\bigcup_{k=0}^{h-1} I_k$ is the set of integers belonging to $[0, \frac{M}{2}h - 2]$ so that equation (57)$_3$ is proven.

Finally, equation (58)$_3$ for $q = \frac{Mh}{2} - 1$ comes from (59)$_1$. It means that we have to prove (58)$_3$ only for $q = 0, \ldots, \frac{M}{2}h - 2$. To this end, let us consider (57)$_3$ with $h$ instead of $h + 1$ and, in the resulting expression, let us substitute $c'_{h,q+1}$ from equation (54). In this way we obtain (58)$_3$ for the remaining values of $q$.

Let us define $f_q$ as the set of functions $c_{h,q}$, appearing in equations (53) and (54), as new arbitrary functions when increasing $h$ to $h + 1$. In other words, we
have

\[
f_q = \begin{cases} 
  c_{1,q}, & \text{for } q = 0, \ldots, \frac{M-2}{2}; \\
  c_{2,q}, & \text{for } q = \frac{M}{2}, \ldots, \frac{2M-2}{2}; \\
  c_{3,q}, & \text{for } q = M, \ldots, \frac{3M-2}{2}; \\
  \ldots, & \\
  \ldots, & \\
  c_{h+1,q}, & \text{for } q = \frac{M}{2}(h), \ldots, (h+1)\frac{M}{2} - 1; \\
\end{cases}
\]

(62)

From these relations and from equation (54) it follows the above reported equation (27). Thanks to equation (27), conditions (58) and (59) transform into equations (26).

Let us now impose conditions (51) distinguishing two cases.

- If \( k \) is even, equation (51)1, thanks to equation (50) of [3] with \( n = M(h+k) + 1, r = k/2 \), gives the leading term of \( C_{h,k}^{\gamma_1 \cdots \gamma_{Mh+(M-1)}k} \), that is

\[
C_{h,k} = (-m^2)^{-\frac{k-1}{2}} \frac{[Mh+(M-1)k+1]!!}{[M(h+k)+1]!!} \gamma^{-6} 
\]

(63)

\[
\cdot \sum_{q=0}^{\frac{M(h+k)}{2}-1} (-m^2)^{\frac{M(h+k)}{2}} C_{h+k,q} \left( \frac{1}{\gamma^2} \right)^q \frac{[M(h+k)+1]!!}{[M(h+k)-2q-2]!!} \\
\cdot \eta[Mh+(M-1)k-2q, M(h+k)-2-2q] \\
= (-m^2)^{\frac{Mh+(M-1)k}{2}} \gamma^{-6} \sum_{q=0}^{\frac{Mh+(M-1)k}{2}-1} C_{h+k,q} \left( \frac{1}{\gamma^2} \right)^q \\
\cdot \frac{[Mh+(M-1)k+1]!!}{[Mh+(M-1)k-2q-2]!!},
\]

where \( \eta(a,b) \) denotes the product of all even numbers between \( a \) and \( b \), if \( a < b \), otherwise it is equal 1. In the second passage, we have eliminated all the terms with \( q = \frac{Mh+(M-1)k}{2}, \ldots, \frac{M}{2}(h+k) - 1 \) because, for this values of \( q \), at least one of the factors in \( \eta(\cdots) \) is zero.

- If \( k \) is odd, equation (51)2, thanks to equation (50) of [3] with \( n = M(h+k), r = (k-1)/2 \), gives the leading term of \( C_{h,k}^{\gamma_1 \cdots \gamma_{Mh+(M-1)}k} \), that is

\[
C_{h,k} = (-m^2)^{-\frac{k+1}{2}} \frac{[Mh+(M-1)k]!!}{[M(h+k)-1]!!} \gamma^{-6} 
\]

(64)
We conclude by proving equations (28) and (28) via an iterative procedure start with the case \( k \) even. We have that equation (28) holds for \( s = \frac{Mh+(M-1)k}{2} \), thanks to equation (63). Let us assume that it holds also for values of \( s < \frac{Mh+(M-1)k}{2} \) up to the index \( s \). Thanks to equation (37) of [3], we see that it holds also when \( s - 1 \) replaces \( s \).

In the case with \( k \) odd, we have that equation (28) holds for \( s = \frac{Mh+(M-1)k+1}{2} \), thanks to equation (64). Let us assume that it holds also for values of \( s < \frac{Mh+(M-1)k+1}{2} \), up to the index \( s \). Thanks to equation (37) of [3], we see that it holds also when \( s - 1 \) replaces \( s \).

### 6. Conclusions

We consider very interesting the results of the present paper, firstly because they fill a gap in the literature on this subject. Only particular cases \( (M = 2) \) have been studied previously and for this case the present results coincides with those already known. Here we have considered the macroscopic model and it includes the kinetic one (see [1]) as the particular case with \( F = \exp^{-X/k}, G = 0 \), where \( K \) is the Boltzmann constant. This fact gives a further confirmation to the present results, because it is well known from other contexts that the kinetic approach is more restrictive than the macroscopic one which we have here considered. It is also interesting that the particular solution (23) is still more general than that of kinetic approach.
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References


