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AN AK SOLOW MODEL WITH LOGISTIC LAW TECHNOLOGY

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Abstract: This paper analyzes how a logistic law technology hypothesis affects the dynamics of the standard AK Solow model.

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1. Introduction

The neoclassical growth models, due to Ramsey [15] and Solow [17], provide the basic framework for much of modern macroeconomics. If we assume the production function to be linear within the Solow model, we have the so-called AK Solow model. Unlike the standard Solow model, the AK Solow formulation of Rebelo [16] predicts that growth rates do not exhibit any tendency to convergence, i.e. no transitional dynamics. A standard assumption in these models is that the technological growth rate is constant. In this article, we propose to investigate the dynamic effects of considering a non-constant technological growth rate into the Solow model with AK technology. More precisely, we assume a logistic-type population growth law. Within this framework, we find the model's dynamics to be richer than those of the basic Solow model with AK technology. A final comment. The Solow model has been recently generalized in several different directions (see, e.g., Bucci and Guerrini [1]; Ferrara and Guerrini [2]-[4]; Germanà and Guerrini [5]; Guerrini [6]-[14]). For further research, it would be interesting to use the non-constant technological growth rate hypothesis in those contests.

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2. The Model

We consider a closed economy with a single good which can be used for consumption and investment. The equilibrium condition for the market of the good is $Y_t = C_t + I_t$, where Y_t is the supply of the good, C_t and I_t are the demands for consumption and investment, respectively. The single good is produced according to an aggregate production function exhibiting constant returns to scale, $Y_t = A_t K_t$, where K_t is the stock of capital and A_t is the level of technology. Contrary to standard AK Solow-Swan model, the growth rate of technological change is not constant, but it follows the following logistic law (see [18])

$$\dot{A}_t = (a - bA_t)A_t, \quad a > b > 0. \tag{1}$$

For simplicity, $A_0 = 1$. Notice that equation (1) is a Bernoulli differential equation whose solution is $A_t = ae^{at}/(a-b+be^{at})$. Hence, A_t is monotone increasing from A_0 to $A_\infty = a/b$. The consumption behavior is proportional to income, i.e. $C_t = (1-s)Y_t$, where s denotes the rate of savings. Since the economy is closed and there is no public sector, the change in the capital stock equals gross investment, so that we get $\dot{K}_t = sY_t$. For simplicity, we have assume there is no capital depreciation. Let us define a new variable $k_t = K_t/A_t L_t$, the stock of capital per unit of effective labor $k_t = K_t/A_t L_t$, where L_t represents population and it assumed to grow at the constant rate n. Taking derivatives with respect to time in the definition of k_t yields $\dot{k}_t = d(K_t/A_t L_t)/dt = \dot{K}_t/(A_t L_t) - \left(\dot{A}_t/A_t + \dot{L}_t/L_t\right) k_t$. Consequently, we have

$$\dot{k}_t = \left(sA_t - n - \frac{\dot{A}_t}{A_t}\right) k. \tag{2}$$

Together with the initial condition $k_0 > 0$, this equation completely determine the entire time path of the capital stock.

3. Model's Solution and Long-Run Behavior

Proposition 1. *For all $t > 0$, the time path of capital stock is given by*

$$k_t = k_0 e^{(sA_\infty - n)t} A_t^{-1-\frac{s}{b}}. \tag{3}$$

Proof. Equation (2) is a separable differential equation. Separating variables and integrating both sides gives $\ln(k_t/k_0) = s\int_0^t A_t dt - nt - \int_0^t \dot{A}_t/A_t dt$.

From being $\int_0^t A_t dt = (1/b)[\int_0^t a dt - \int_0^t \dot{A}_t/A_t dt] = (a/b)t - (1/b)\ln A_t$, substitution in the previous equation and then exponentiation yield the result. □

Corollary 1. *$\lim_{t\to\infty} k_t = k_0 A_\infty^{-1-\frac{s}{b}}$ if $sa = nb$, $\lim_{t\to\infty} k_t = +\infty$ if $sa > nb$, and $\lim_{t\to\infty} k_t = 0$ if $sa < nb$.*

Proof. The long-run behavior of capital stock follows from equation (3) since we have $\lim_{t\to\infty} k_t = k_0 A_\infty^{-1-\frac{s}{b}} \lim_{t\to\infty} e^{(sA_\infty - n)t}$. □

Remark 1. The convergence behavior of this modified AK Solow model vanishes if there is zero population growth rate. In fact, if $n = 0$, the previous result yields that an economy that starts from a stock of per capita capital equal to k_0 will perpetually accumulate physical capital and its per capita capital stock will rise toward infinity.

Proposition 2. *Starting from k_0, the capital stock k_t increases monotonically if $a - b < s - n$, and it decreases monotonically if $a/b < n/s$.*

Proof. Let $\gamma_{k_t} \equiv \dot{k}_t/k_t$ denote the per capita capital stock growth rate. Using equation (1), we can rewrite equation (2) as $[1/(s+b)]\gamma_{k_t} = A_t - (a+n)/(s+b)$. Recalling that the function A_t increases from 1 to $A_\infty = a/b$, we can conclude that $\gamma_{k_t} > 0$ if $(a+n)/(s+b) < 1$, while $\gamma_{k_t} < 0$ if $A_\infty < (a+n)/(s+b)$. The statement is now immediate. □

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