ON THE ADJACENT STRONG EQUITABLE
EDGE COLORING OF $C_n \vee C_n$

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Abstract: In this paper, we discuss the adjacent strong equitable edge coloring of join-graphs about $C_n \vee C_n$.

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1. Introduction

The coloring problem of graphs is widely applied in practice. In [9], some conditional coloring problems as introduced. Some network problem can be converted to the strong edge coloring (see [6], [4], [1], [3]) and adjacent strong edge coloring, see [11].

Definition 1. (see [6], [4], [1], [3]) For a graph $G(V, E)$, if a proper coloring $f$ is satisfied with $C(u) \neq C(v)$ for $\forall u, v \in V(G)(u \neq v)$, then $f$ is called $k$-strong edge coloring of $G$, abbreviated $k$-SEC, and

$$\chi'_s(G) = \min\{k|k$-SEC of $G\}$$

is called the strong edge chromatic number of $G$. And for $\forall uv \in E(G)$, $C(u) \neq \ldots$
$C(v)$, $f$ is called $k$-adjacent strong edge coloring of $G$, abbreviated $k$-ASEC, and

$$
\chi_{as}(G) = \min\{k|k$-ASEC of $G\}
$$

is called the adjacent strong edge chromatic number of $G$, see [11]. Here

$$
C(u) = \{f(uv)|uv \in E(G)\}.
$$

**Definition 2.** Let $f$ be a $k$-ASEC of $G$ and satisfy

$$
||E_i| - |E_j|| \leq 1, \quad i, j = 1, 2, \cdots, k,
$$

$f$ is called the adjacent strong equitable edge coloring of $G$, and is denoted by $k$-ASEEC of $G$, and

$$
\chi_{ase}(G) = \min\{k|k$-ASEEC of $G\}
$$

is called the adjacent strong equitable edge chromatic number of $G$. Here

$$
E_i = \{e|f(e) = i\}, \quad i = 1, 2, \cdots, k.
$$

**Conjecture.** (see [11]) For a connected graph with order $p \geq 3$, and $G \neq C_5$ (5-cycle),

$$\chi_{as}(G) \leq \Delta(G) + 2.
$$

Here $p = |V(G)|$, $\Delta(G)$ is maximal degree of $G$.

There are many proofs of this conjecture, for example [10], [2], for $\Delta(G) \leq 3$. For a connected graph with $|V(G)| \geq 3$:

1. If $G$ is a bipartite graph with no isolate edges, then

$$
\chi_{as}(G) \leq \Delta(G) + 2.
$$

2. If $G$ is a $k$-chromatic graph with no isolate edges, then

$$
\chi_{as}(G) \leq \Delta(G) + O(\log k).
$$

**Definition 3.** (see [5]) For graph $G$ and graph $H$, $V(G) \cap V(H) = E(G) \cap E(H) = \emptyset$, and

$$
\begin{align*}
\{ & V(G \vee H) = V(G) \cup V(H), \\
& E(G \vee H) = E(G) \cup E(H) \cup \{uv|u \in V(G), v \in V(H)\},
\end{align*}
$$

then $G \vee H$ is called join-graph of $G$ and $H$. 
Lemma 1. (see [11]) If $G$ is a connected graph with $|V(G)| \geq 3$, and $uv \in E(G)$, $d(u) = d(v) = \Delta(G)$. Then
\[
\chi'_as(G) \geq \Delta(G) + 1.
\]

Lemma 2. (see [5]) If $k \geq \chi'(G)$, then $k$-PEC of $G$ exists
\[
||E_i| - |E_j|| \leq 1; \quad i, j = 1, 2, \ldots, k,
\]
where $e \in E_i, f(e) = i(i = 1, 2, \ldots, k), \chi'(G)$ is the chromatic number of $G$.

Lemma 3. For $n \geq 3$,
\[
|E(C_n \lor C_n)| = n^2 + 2n.
\]

For $m > n \geq 1$, there are many adjacent strong chromatic numbers of $C_m \lor C_n$. In this paper we have the adjacent strong equitable chromatic number of $C_n \lor C_n$, for the others terminologies refer to [5], [7], [8].

2. Adjacent Strong Edge Coloring of $C_n \lor C_n$

Theorem 1. For $n \geq 4$. Then
\[
\chi'_{ase}(C_n \lor C_n) = \begin{cases} 
  n + 3, n = 3 \text{ or } n \equiv 0 \mod 2, \\
  n + 4, n \geq 5 \text{ or } n \equiv 1 \mod 2.
\end{cases}
\]

Proof. Supposing the two cycles are $u_1u_2 \cdots u_nu_1$ and $v_1v_2 \cdots v_nv_1$ with separately.

When $n = 3$, $C_3 \lor C_3 = K_6$ (complete graph with order 6), can be seen in appendix.

Case 1. $n \equiv 0 \mod 2$. By Lemmas 1, 2, and 3, $C_n \lor C_n$ there exist three perfect matching $M_1, M_2, M_3$ and $M_1 \cap M_2 = M_2 \cap M_3 = M_3 \cap M_1 = \phi$.

When $n = 4$, let $f$ be as follows:
\[
\begin{align*}
  f(u_1u_4) &= f(u_2u_3) = f(v_1v_4) = f(v_2v_3) = 7, \\
  f(u_1v_4) &= f(u_2v_1) = f(u_3v_3) = f(u_4v_2) = 6, \\
  f(u_1v_2) &= f(u_2v_4) = f(u_3v_1) = f(u_4v_3) = 5,
\end{align*}
\]
\[ f(u_1v_1) = f(u_2v_3) = f(u_3u_4) = 4, \]
\[ f(u_1u_2) = f(u_3v_2) = f(u_4v_4) = 3, \]
\[ f(u_1v_3) = f(u_3v_4) = f(v_1v_2) = 2, \]
\[ f(u_2v_2) = f(u_4v_1) = f(v_3v_4) = 1. \]

So \( f \) is a 7-ASEEC of \( C_4 \lor C_4 \).

When \( n = 6 \):
\[
\begin{align*}
(C_6 \lor C_6) \setminus \{u_1, u_3\} & \cup_{i=1}^{3} M_i, \text{ there exists a perfect matching } M_4, \\
(C_6 \lor C_6) \setminus \{u_2, u_5\} & \cup_{i=1}^{4} M_i, \text{ there exists a perfect matching } M_5, \\
(C_6 \lor C_6) \setminus \{u_4, u_6\} & \cup_{i=1}^{5} M_i, \text{ there exists a perfect matching } M_6, \\
(C_6 \lor C_6) \setminus \{v_1, v_3\} & \cup_{i=1}^{6} M_i, \text{ there exists a perfect matching } M_7, \\
(C_6 \lor C_6) \setminus \{v_2, v_5\} & \cup_{i=1}^{7} M_i, \text{ there exists a perfect matching } M_8, \\
(C_6 \lor C_6) \setminus \{v_4, v_6\} & \cup_{i=1}^{8} M_i, \text{ there exists a perfect matching } M_9.
\end{align*}
\]

Let
\[
\forall e \in M_i : \quad f(e) = i.
\]

So, \( f \) is a 9-ASEEC of \( C_6 \lor C_6 \).

Similarly we can prove that \( C_n \lor C_n \) exist \((n + 3)\)-ASEEC, when \( n \equiv 0 \pmod{2} \) and \( n \geq 8 \).

By Lemma 3, \( C_n \lor C_n \) there exist three perfect matching \( M_1, M_2, M_3 \) and \( M_1 \cap M_2 = M_2 \cap M_3 = M_3 \cap M_1 = \emptyset \). Suppose \( n = 2k, k \geq 2 \).

\[
(C_n \lor C_n) \setminus \{u_1, u_{k+1}\} \cup_{i=1}^{3} M_i \text{ there exists a perfect matching } M_4,
\]
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\[(C_n \lor C_n) \setminus \{u_2, u_{k+2}\} \setminus \bigcup_{i=1}^{4} M_i \text{ there exists a perfect matching } M_5, \]

\[\vdots\]

\[(C_n \lor C_n) \setminus \{u_k, u_n\} \setminus \bigcup_{i=1}^{k+3} M_i \text{ there exists a perfect matching } M_{k+4},\]

\[(C_n \lor C_n) \setminus \{v_1, v_k\} \setminus \bigcup_{i=1}^{k+4} M_i \text{ there exists a perfect matching } M_{k+5},\]

\[\vdots\]

\[(C_n \lor C_n) \setminus \{v_k, v_n\} \setminus \bigcup_{i=1}^{n+2} M_i \text{ there exists a perfect matching } M_{n+3}.\]

let \[\forall e \in M_i : \quad f(e) = i, \quad i = 1, 2, \ldots, n + 3.\]

For \(f\) we have

\[\overline{C}(u_i) = \overline{C}(u_{k+i}) = \{i\}, \quad i = 1, 2, \ldots, k,\]

\[\overline{C}(v_i) = \overline{C}(v_{k+i}) = \{k + i + 1\}, \quad i = 1, 2, \ldots, k,\]

and

\[|E_i| = \begin{cases} n, & i = 1, 2, 3; \\ n - 1, & i = 4, 5, \ldots, n + 3. \end{cases}\]

So, \(f\) is a \((n + 3)\)-ASEEC of \(C_n \lor C_n\).

Case 2. \(n \equiv 1 \pmod{2}\) and \(n \geq 5\). We prove \(\chi_{as}'(C_n \lor C_n) \geq n + 4\).

If \(\chi_{as}'(C_n \lor C_n) = n + 3\), there exist at least 3 colors represented at every vertex. For \(|E(C_n \lor C_n)| = n^2 + 2n\), and each vertex lack just one color and the vertices lack the same color in a same cycle and not adjacent, and \(n\) is an odd, so it is not possible that odd number vertices lack same color. For the edge number of \(C_n \lor C_n\), the number of vertices lack same color just an even. So there must have two vertices in two different cycle lack same color. It is contradictory. Thus \(\chi_{as}'(C_n \lor C_n) \geq n + 4\) when \(n \equiv 1 \pmod{2}\).

Let \(f\) be as follows: the edges of \(u_1u_2, u_2u_3, \ldots, u_{n-1}u_n\) are coloring colors with \(n + 3, n + 4\) rotate; the edges of \(v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\) are coloring colors with \(n+4,n+3\) rotate;

\[f(u_1u_n) = f(v_1v_n) = n + 1,\]
\[f(u_1 v_j) = j, \quad j = 1, 2 \cdots, n;\]

\[f(u_i v_j) = i + j \quad \text{(when } i + j > n + 2, \text{ then } \mod n + 2),\]

\[i = 2, 3, \cdots, n - 1; \quad j = 1, 2, \cdots, n,\]

\[f(u_n v_1) = n + 2; \quad f(u_n v_j) = j - 1, j = 2, 3, \cdots, n.\]

For the \(f\), we have

\[
\overline{C}(u_1) = \{n + 2, n + 4\}; \quad \overline{C}(u_i) = \{i + 1, n + i\} \text{ (when } i + j > n + 2, \text{ take } \mod n + 2,\]

\[
\overline{C}(u_n) = \{n, n + 3\} \quad (n \equiv 1 \mod 2),\]

\[
\overline{C}(v_1) = \{2, n + 3\}; \quad \overline{C}(v_i) = \{i, n + i\} \quad \text{ (when } i + j > n + 2, \text{ take } \mod n + 2,\]

\[
\overline{C}(v_n) = \{n, n + 4\} \quad (n \equiv 1 \mod 2).\]

So, \(f\) is a \((n + 4)\)-ASEEC of \(C_n \vee C_n\). Theorem 3 is true.

Using the arguments above, it is easy to verify Theorem 1. \(\square\)

Acknowledgments

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References


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