ON THE SCHOUTEN TENSOR OF SOME SPECIAL RIEMANNIAN MANIFOLDS

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Abstract: The purpose of this paper is to study some properties of the Schouten tensor arising from the considerations of conformal geometry.

In the special cases when the Schouten tensor is recurrent, generalized recurrent and generalized 2-recurrent, it is found the conditions over the manifold. Additionally, in a Riemannian manifold with the constant curvature, it is proved that the Schouten tensor is an Einstein tensor.

In the last section of this paper, an example is given for the existence of the Schouten tensor for a 4-dimensional Riemannian manifold.

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1. Introduction

Let \((M_n, g)\) be an \(n\)-dimensional smooth Riemannian manifold \((n \geq 3)\), and let the Ricci tensor and the scalar curvature be denoted by \(S(X, Y)\) and \(r\), respectively. The Riemannian curvature tensor decomposes into a conformally invariant part, the Weyl tensor and a non-conformally invariant part, the Schouten tensor. The Schouten tensor is defined by the following form

\[
P(X, Y) = \frac{1}{n-2} (S(X, Y) - \frac{r}{2(n-1)} g(X, Y)).
\] (1.1)

There is a decomposition formula

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where $W$ is the Weyl tensor of $g$, and $\bigodot$ denotes the Kulkarni-Nomizu product, $R$ is the Riemannian tensor and $P$ is the Schouten tensor, see [4]. Since the Weyl tensor is conformally invariant, to study the deformation of the conformal metric, we only need to understand the Schouten tensor, see [2-4].

A non-flat $n$-dimensional Riemannian manifold $(M_n, g)$ is called recurrent (see [15]) if its curvature tensor satisfies

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W,$$  \hspace{1cm} (1.3)

for a 1-form $A(X)$. In 1979, R.S.D. Dubey introduced the generalized recurrent Riemannian manifold, [8] as if its curvature tensor satisfies the condition

$$\nabla R = A \otimes R + B \otimes G,$$  \hspace{1cm} (1.4)

where $A$ and $B$ are 1-forms of which $B$ is non-zero, $\otimes$ is the tensor product, $\nabla$ denotes the Levi-Civita connection, and $G$ is a tensor of type $(0, 4)$ given by

$$G(X, Y, Z, U) = g(X, U)g(Y, Z) - g(X, Z)g(Y, U)$$  \hspace{1cm} (1.5)

for all $X, Y, Z, U \in \chi(M_n)$, $\chi(M_n)$ being the Lie algebra of smooth vector fields on $M_n$. This manifold is denoted by $GK_n$. Especially, if $B = 0$, the manifold reduces to a recurrent manifold, denoted by $K_n$, see [4]. These manifolds have been studied by some authors before, see [6-10].

A non-flat Riemannian space $M_n$ ($n > 3$) is called a generalized 2-recurrent space (see [11]), if its curvature tensor satisfies

$$(\nabla_X \nabla_Y R)(Z, U, V, W)$$

$$= \lambda(X)(\nabla_Y R)(Z, U, V, W) + a(X, Y)R(Z, U, V, W)$$  \hspace{1cm} (1.6)

where $a(X, Y)$ is non-zero. $\lambda(X)$ and $a(X, Y)$ are called its vector and tensor of recurrence. Such a space has been denoted by $G(2K_n)$.

If the Ricci tensor of a Riemannian manifold satisfies the following relation

$$(\nabla_X \nabla_Y S)(U, V) = \lambda(X)(\nabla_Y S)(U, V) + a(X, Y)S(U, V),$$  \hspace{1cm} (1.7)

then this space is called a generalized Ricci 2-recurrent space and it is denoted by $G(2R_n)$, [14]. If in particular, $\lambda(X) = 0$, then the space reduces to a Ricci 2-recurrent space, see [5].
2. The Schouten Tensor of Some Special Riemannian Manifolds

Now, let us consider the Schouten tensor of $M_n$ in the form (1.1). Thus, contracting on $X$ and $Y$, we get

$$P = \frac{r}{2(n-1)}, \quad r = 2(n-1)P. \quad (2.1)$$

**Theorem 2.1.** A necessary and sufficient condition the Schouten tensor of a Riemannian manifold $M_n$ to be recurrent is that $M_n$ is a Ricci-recurrent.

**Proof.** By using (1.1) and (2.1), we find

$$P(Y, Z) + \frac{1}{n-2}Pg(Y, Z) = \frac{1}{n-2}S(Y, Z). \quad (2.2)$$

By taking the covariant derivative of (2.2), we get

$$(\nabla_X P)(Y, Z) + \frac{1}{n-2}(\nabla_X P)g(Y, Z) = \frac{1}{n-2}(\nabla_X S)(Y, Z). \quad (2.3)$$

If we assume that the Schouten tensor of $M_n$ is recurrent then we have from (1.3)

$$(\nabla_X P)(Y, Z) = \lambda(X)P(Y, Z) \quad (2.4)$$

and

$$\nabla_X P = \lambda(X)P. \quad (2.5)$$

In this case, by putting (2.4) and (2.5) in (2.3), we obtain

$$\lambda(X)(P(Y, Z) + \frac{1}{n-2}Pg(Y, Z)) = \frac{1}{n-2}(\lambda(X)S(Y, Z)). \quad (2.6)$$

Thus, from (2.6), we can get

$$(\nabla_X S)(Y, Z) = \lambda(X)S(Y, Z).$$

This means that $M_n$ is Ricci-recurrent. Conversely, by taking the covariant derivative of (1.1) and assuming that $M_n$ is Ricci-recurrent manifold, from (1.3), we get

$$(\nabla_X P)(Y, Z) = \lambda(X)P(Y, Z).$$

In this case, we can say that $P(X, Y)$ is recurrent. The proof is completed. \(\square\)

**Theorem 2.2.** A necessary and sufficient condition the Schouten tensor of a Riemannian manifold $M_n$ to be a generalized recurrent is that $M_n$ is a generalized Ricci-recurrent.
Proof. Let us consider that the Schouten tensor is generalized recurrent. Thus, from (1.1) and (1.4), we find

\[(\nabla_X P)(Y, Z) = \alpha(X)P(Y, Z) + \beta(X)g(Y, Z). \quad (2.7)\]

If we substitute (2.7) into (2.3), we can be obtained that

\[
\alpha(X)P(Y, Z) + \beta(X)g(Y, Z) + \frac{1}{n-2} (\alpha(X)P + n\beta(X)g(Y, Z)) = \frac{1}{n-2}(\nabla_X S)(Y, Z). \quad (2.8)
\]

If we use (1.1) and (2.11), (2.8) reduces to

\[(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + 2(n-1)\beta(X)g(Y, Z). \quad (2.9)\]

Thus, we get

\[(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \gamma(X)g(Y, Z) \quad (2.10)\]

provided that \(\gamma(X) = 2(n-1)\beta(X)\).

(2.10) shows that \(M_n\) is also generalized Ricci-recurrent manifold. Conversely, assuming that \(M_n\) is a generalized Ricci-recurrent and using (1.4), we have

\[(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(X)g(Y, Z). \quad (2.11)\]

Contracting on \(X\) and \(Y\) in (2.11), we find

\[
\nabla_X r = \alpha(X)r + n\beta(X). \quad (2.12)
\]

With the help of (2.11) and (2.12), (2.3) takes the form

\[(\nabla_X P)(Y, Z) = \frac{\alpha(X)}{n-2}S(Y, Z) + \frac{1}{2(n-1)}(\beta(X) - \frac{1}{n-2}(n-2)\lambda(X)r)g(Y, Z). \quad (2.13)\]

From (1.1), we find

\[
S(Y, Z) = (n-2)P(Y, Z) + \frac{r}{2(n-1)}g(Y, Z). \quad (2.14)
\]

Thus, by using (2.14), (2.13) reduces to

\[(\nabla_X P)(Y, Z) = \alpha(X)P(Y, Z) + \frac{1}{2(n-1)}\beta(X)g(Y, Z). \quad (2.15)\]
Finally, we get
\[ (\nabla_X P)(Y, Z) = \alpha(X)P(Y, Z) + \gamma(X)g(Y, Z), \] (2.16)
where \(\gamma(X) = \frac{1}{2(n-1)}\beta(X)\). In this case, from (2.16), we can say that the Schouten tensor is also generalized recurrent. Thus the proof is completed. \(\square\)

**Definition 1.** A Riemannian manifold \(M_n\) of quasi-constant curvature was given by Chen and Yano (1972) as a conformally flat space with the curvature tensor, [3]
\[ R^h_{ijk} = a(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + b((\delta^h_k v_j - \delta^h_j v_k)v_i - (v_k g_{ij} - v_j g_{ik})v^h), \] (2.17)
where \(a, b\) are differentiable functions and \(v^i\) is a unit vector field. The vector field \(v^i\) is called the generator vector of the manifold.

**Theorem 2.3.** If \(M_n\) is of quasi-constant curvature then the Schouten tensor of \(M_n\) is a quasi-Einstein tensor.

**Proof.** We assume that \(M_n\) is of quasi-constant curvature. By the aid of (2.17), the Ricci tensor of \(M_n\) is in the form
\[ S(Y, Z) = ((n-1)a + b)g(Y, Z) + (n-2)bv(Y)v(Z). \] (2.18)
Thus, from (1.1) and (2.18), we find
\[ P(Y, Z) = \frac{a}{2}g(Y, Z) + bv(Y)v(Z). \]
The proof is completed. \(\square\)

**Theorem 2.4.** Let \(M_n\) be of quasi-constant curvature. If the generator vector field of this manifold is recurrent then the Schouten tensor of \(M_n\) is generalized recurrent.

**Proof.** We consider that \(M_n\) is of quasi-constant curvature. Thus, from Theorem 2.3, we have
\[ P(Y, Z) = \frac{a}{2}g(Y, Z) + bv(Y)v(Z). \] (2.19)
By taking the covariant derivative of (2.19), we find
\[ (\nabla_X P)(Y, Z) = b((\nabla_X v)(Y)v(Z) + (\nabla_X v)(Z)v(Y)) \]
\[ + \frac{1}{2}(\nabla_X a)g(Y, Z) + (\nabla_X b)v(Y)v(Z). \] (2.20)
If \( v \) is recurrent vector field then we get

\[
(\nabla_X P)(Y, Z) = 2b\lambda(X)v(Y)v(Z) + \frac{1}{2}(\nabla_X a)g(Y, Z) + (\nabla_X b)v(Y)v(Z). \tag{2.21}
\]

From (2.19), (2.21) takes the form

\[
(\nabla_X P)(Y, Z) = (2\lambda(X) + \frac{\nabla_X b}{b})P(Y, Z)
+ \left(\frac{\nabla_X a}{2} - \frac{a}{2} \frac{\nabla_X b}{b} - a\lambda(X)\right)g(Y, Z). \tag{2.22}
\]

Thus, we have

\[
(\nabla_X P)(Y, Z) = \alpha(X)P(Y, Z) + \beta(X)g(Y, Z), \tag{2.23}
\]

where \( \alpha(X) = 2\lambda(X) + \frac{\nabla_X b}{b} \) and \( \beta(X) = \frac{\nabla_X a}{2} - \frac{a}{2} \frac{\nabla_X b}{b} - a\lambda(X) \). From (2.23), we can say that the Schouten tensor of \( M_n \) is generalized recurrent.

**Theorem 2.5.** If \( M_n \) is of constant curvature then the Schouten tensor is an Einstein tensor.

**Proof.** Let \( M_n \) be of constant curvature. When \( b = 0 \) in (2.18), \( M_n \) is of constant curvature. Thus, we get from (2.19)

\[
P(Y, Z) = \frac{a}{2} g(Y, Z).
\]

In this case, it can be said that \( P(Y, Z) \) is Einstein tensor.

Now, we can give the following theorems:

**Theorem 2.6.** If \( M_n \) is a \( G(2R_n) \), then the Schouten tensor of \( M_n \) is also generalized 2-recurrent.

**Proof.** By taking the covariant derivatives of (1.1) twice, we obtain that

\[
(\nabla_X \nabla_Y P)(Z, W) = \frac{1}{n-2}((\nabla_X \nabla_Y S)(Z, W) - \frac{\nabla_X \nabla_Y r}{2(n-1)})g(Z, W) \tag{2.24}
\]

If \( M_n \) is a \( G(2R_n) \), then from (1.1) and (1.7), (2.24) reduces to

\[
(\nabla_X \nabla_Y P)(Z, W) = \gamma(X)(\nabla_Y P)(Z, W) + b(X, Y)P(Z, W)
\]

where \( \gamma(X) = \frac{\lambda(X)}{n-2} \) and \( b(X, Y) = \frac{a(X, Y)}{n-2} \). This shows that the Schouten tensor of \( G(2R_n) \) is generalized 2-recurrent.
Theorem. A conformally symmetric $G(2R_n)$ is a $G(2K_n)$ and when the tensor of recurrence is symmetric then the length of Ricci tensor is $\frac{nr^2}{4(n-1)}$, i.e., [14]

$$R_{ij}R^{ij} = \frac{nr^2}{4(n-1)}. \quad (2.25)$$

The following theorem follows directly from the theorem above.

**Theorem 2.7.** For a conformally symmetric $G(2R_n)$, if the tensor of recurrence is symmetric then the lengths of the Ricci tensor and the Schouten tensor are proportional, i.e.,

$$\frac{L}{\overline{L}} = n(n-1),$$

where $L$ and $\overline{L}$ be the lengths of the Ricci tensor and the Schouten tensor, respectively.

**Proof.** In local coordinates, if we consider the equation (1.1), we can find

$$P_{ij}P^{ij} = \frac{1}{(n-1)^2}(R_{ij} - \frac{r}{2(n-1)}g_{ij})(R^{ij} - \frac{r}{2(n-1)}g^{ij}). \quad (2.26)$$

By using the equation (2.25), (2.26) can be seen that

$$P_{ij}P^{ij} = \frac{r^2}{4(n-1)^2}. \quad (2.27)$$

Thus, considering (2.25) and (2.27), it is finally obtained that

$$\frac{L}{\overline{L}} = n(n-1),$$

where $L$ and $\overline{L}$ be the lengths of the Ricci tensor and the Schouten tensor, respectively. \qed

3. Geometrical Symmetries of the Schouten Tensor of a Riemannian Manifold

The geometrical symmetries of a Riemannian manifold are expressed through the equation

$$\mathcal{L}_\xi A - 2\Omega A = 0, \quad (3.1)$$
where $A$ represents a geometrical/physical quantity $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the vector field $\xi$ and $\Omega$ is a scalar, see [12].

One of the most simple and widely used example is the metric inheritance symmetry for which $A = g$ in (3.1); and for this case, $\xi$ is the Killing vector field if $\Omega$ is zero, i.e.

$$ (\mathcal{L}_\xi g)(X, Y) = 2\Omega g(X, Y). \quad (3.2) $$

A Riemannian manifold $M_n$ is said to admit a symmetry called a curvature collineation (CC) provided there exists a vector field $\xi$ such that

$$ (\mathcal{L}_\xi R)(X, Y)Z = 0, \quad (3.3) $$

where $R(X, Y)Z$ is the Riemannian curvature tensor, see [9].

Now, we shall investigate the role of such symmetry inheritance for the Schouten tensor of a Riemannian manifold.

**Theorem 3.1.** If $M_n$ admitting a Killing vector $\xi$ is a (CC) then the Lie derivative of the Schouten tensor of $M_n$ vanishes.

**Proof.** Let $\xi$ be a Killing vector of $M_n$. If we take the Lie derivative of $P(X, Y)$ from (2.1) and using (3.2), we get

$$ (\mathcal{L}_\xi P)(X, Y) = \frac{1}{n-2}((\mathcal{L}_\xi S)(X, Y) - \frac{(\mathcal{L}_\xi r)}{2(n-1)}g(X, Y)). \quad (3.4) $$

If $M_n$ is CC, from (3.3), we then get

$$ (\mathcal{L}_\xi P)(X, Y) = 0. \quad \Box $$

**Theorem 3.2.** The Schouten tensor is a conformal Killing tensor if the curvature tensor of $M_n$ admitting a conformal Killing vector $\xi$ is a conformal Killing tensor.

**Proof.** From (3.4), if we consider the curvature tensor of $M_n$ is conformal Killing tensor and if we use the equation (2.1), we find

$$ (\mathcal{L}_\xi P)(X, Y) = 2\Omega P(X, Y) - \frac{\Omega}{(n-1)(n-2)}rg(X, Y). \quad (3.5) $$

Thus, we can say that if the Schouten tensor is a conformal Killing tensor then it must be $R = 0$. The converse is also true.

**Definition 2.** Let $(M, g)$ be a manifold with Levi-Civita connection $\nabla$. A quadratic conformal Killing tensor is an analogous generalization of a conformal
Killing vector and is defined as a second order symmetric tensor $A$ satisfying the condition, see [19] and [18]

$$(\nabla_X A)(Y, Z) + (\nabla_Y A)(Z, X) + (\nabla_Z A)(X, Y)$$

$$= k(X)g(Y, Z) + k(Y)g(Z, X) + k(Z)g(X, Y). \quad (3.6)$$

Now we have the following theorems:

**Theorem 3.3.** If $M_n$ is of constant curvature then the Schouten tensor of $M_n$ is a quadratic Killing tensor.

**Proof.** Let $M_n$ be of constant curvature. Thus, from Theorem 2.5, we can say that the Schouten tensor is an Einstein tensor and then the covariant derivative of the Schouten tensor is

$$(\nabla_X P)(Y, Z) = \frac{1}{2}(\nabla_X a)g(Y, Z). \quad (3.7)$$

If we consider Definition 2, by putting (3.7) in (3.6), then we have $P(Y, Z)$ is a quadratic conformal Killing tensor where $k(X) = \frac{1}{2}(\nabla_X a)$. Thus, the proof is completed.

**Theorem 3.4.** A necessary and sufficient condition the Schouten tensor of $M_n$ be a quadratic conformal Killing tensor is that the Ricci tensor of $M_n$ be a quadratic conformal Killing tensor.

**Proof.** If we take the covariant derivative of the equation (2.1) then we find

$$(\nabla_X P)(Y, Z) = \frac{1}{(n-2)}(\nabla_X S)(X, Y) - \frac{(\nabla_X r)}{2(n-1)}g(X, Y). \quad (3.8)$$

We assume that the Schouten tensor is a quadratic conformal Killing tensor. Then, from (3.7), we obtain

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y)$$

$$= (k(X) + \frac{1}{2(n-1)}(\nabla_X r))g(Y, Z) + (k(Y) + \frac{1}{2(n-1)}(\nabla_Y r))g(Z, X)$$

$$+ (k(Z) + \frac{1}{2(n-1)}(\nabla_Z r))g(X, Y). \quad (3.9)$$

If we take $\lambda(X) = (k(X) + \frac{1}{2(n-1)}(\nabla_X r))g(Y, Z)$ then, from (3.7), we say that the Ricci tensor is a quadratic conformal Killing tensor.

If we consider that the Ricci tensor is a quadratic conformal Killing tensor, from (3.7) and (3.8), it is clear that the Schouten tensor is a quadratic conformal Killing tensor.
4. An Example for the Schouten Tensor

We define a Riemannian metric $g$ on the 4-dimensional real number space $\mathbb{R}^4$ by the formula

$$\mathrm{d}s^2 = g_{ij} \mathrm{d}x^i \mathrm{d}x^j = t^\frac{4}{3} \left((\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2\right) + (\mathrm{d}t)^2,$$

(4.1)

where $0 < t < \infty$, $x, y, z, t$ are the standard coordinates of $\mathbb{R}^4$. Then the only non-vanishing components of the Christoffel symbols (see [7]), and the curvature tensor are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3} t^{-1},$$

$$\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3} t^\frac{1}{3},$$

$$R_{1441} = R_{2442} = R_{4334} = -\frac{2}{9} t^{-\frac{4}{3}}, \quad R_{2112} = R_{3113} = R_{2332} = \frac{4}{9} t^\frac{4}{3}. \quad (4.2)$$

By using the equations (4.1) and (4.2), we can find the non-vanishing components of the Ricci tensor, the scalar curvature and the only non-vanishing components of Schouten tensor as follows:

$$S_{11} = S_{22} = S_{33} = \frac{2}{3} t^{-\frac{2}{3}}, \quad S_{44} = -\frac{2}{3} t^{-2}, \quad r = \frac{4}{3} t^{-2}, \quad (4.3)$$

$$P_{11} = P_{22} = P_{33} = \frac{2}{9} t^{-\frac{4}{3}}, \quad P_{44} = -\frac{4}{9} t^{-2}, \quad P = \frac{2}{9} t^{-2}. \quad (4.4)$$

Thus, from (4.3) and (4.4) we show that the relation (2.1) holds, i.e. $r = 2(n-1)P$. It shows that the Schouten tensor in $\mathbb{R}^4$ endowed with the metric given by (4.1) can be defined as in (1.1).

References


