THE SCHUR COMPLEMENT IN AN ALGORITHM FOR CALCULATION OF FOCAL POINTS OF CONJOINED BASES OF SYMPLECTIC DIFFERENCE SYSTEMS

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Abstract: We present new results connecting the number of focal points of conjoined bases of symplectic difference systems $Y_{i+1} = W_i Y_i$, $W_i^T J W_i = J$ and the negative inertia index of the Schur complement ($\Lambda_i / H_i$), where a $2n \times 2n$ symmetric matrix $\Lambda_i$ is associated with $Y_i$ and $W_i$. We offer an algorithm for computing eigenvalues of $2n$-order discrete Sturm Liouville eigenvalue problems based on discrete oscillation theorems and results of this paper.

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1. Introduction

We consider the discrete symplectic eigenvalue problem (see [3, 7])

\begin{align*}
x_{i+1}(\lambda) &= A_i x_i(\lambda) + B_i u_i(\lambda), \\
u_{i+1}(\lambda) &= C_i x_i(\lambda) + D_i u_i(\lambda) - \lambda W_i x_{i+1}(\lambda), \quad i = 0, \ldots, N, \\
x_0(\lambda) &= x_{N+1}(\lambda) = 0,
\end{align*}

(1.1)

where $\lambda \in \mathbb{R}$, $x_i(\lambda), u_i(\lambda) \in \mathbb{R}^n$, and the real $n \times n$ matrices $W_i, A_i, B_i, C_i, D_i$ satisfy the conditions
\( W_i = W_i^T, \quad W_i \geq 0, \) \hspace{1cm} (1.2)

\[ B_i^T D_i = D_i^T B_i, \quad A_i^T C_i = C_i^T A_i, \quad A_i^T D_i - C_i^T B_i = I. \] \hspace{1cm} (1.3)

Conditions (1.2), (1.3) imply (see [3]) that the matrix of the above difference system rewritten in the form
\[
y_{i+1}(\lambda) = W_i(\lambda)y_i(\lambda), \\
y_i(\lambda) = [x_i(\lambda) \ u_i(\lambda)]^T, \quad W_i(\lambda) = \begin{bmatrix} A_i & B_i \\ C_i(\lambda) & D_i(\lambda) \end{bmatrix} \] \hspace{1cm} (1.4)

is symplectic, i.e. \( W_i(\lambda)^T J W_i(\lambda) = J \) for \( i = 0, \ldots, N, \lambda \in \mathbb{R}, J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, I \) being the \( n \times n \) identity matrix. The important particular case of problem (1.1) is the discrete \( 2n \)-order Sturm-Liouville boundary problem (see [1, 15, 17])
\[
\sum_{\mu=0}^{n} (-\Delta)^{\mu} \left\{ r_k^{(\mu)} \Delta^{\mu} y_{k+1-\mu} \right\} = \lambda y_{k+1}, \quad r_k^{(n)} \neq 0, \\
y_{1-n} = \ldots = y_0 = y_{N+2-n} = \ldots = y_{N+1} = 0, \] \hspace{1cm} (1.5)

where \( \Delta x_k = x_{k+1} - x_k, 0 \leq k \leq N - n, \quad N, \ n \in \mathbb{Z} \) and \( 1 \leq n \leq N. \)

The oscillation and spectral theory for (1.1) based on the concepts of finite eigenvalues and multiplicities of focal points for conjoined bases of (1.1) is developed in the fundamental papers [3, 7, 16, 2]. The so-called Global Oscillation Theorem established in [7, Theorem 2] relates the number of finite eigenvalues of (1.1) less than or equal to a given number \( \lambda_1 \) to the number of focal points (counting multiplicity) of the principal solution of the difference system in (1.4) with \( \lambda = \lambda_1. \)

From the point of view of practical implementations of [7, Theorem 2] it seems to be very important to calculate the number of focal points (counting multiplicity) by a stable way. Our aim is to add a new aspect to the definition of the number of focal points connecting this concept with the concept of the Schur complement (see [19], [14, p. 101]) for a symmetric matrix associated with a conjoined basis of (1.4). Suppose \( H \) is a nonsingular principal submatrix of a symmetric \( 2n \times 2n \) matrix \( \Lambda. \) We define the Schur complement of \( H \) in \( \Lambda, \) denoted by \( \Lambda/H, \) as follows. Let \( \Lambda' \) be the matrix obtained from \( \Lambda \) by a simultaneous permutation of rows and columns which puts \( H \) into the upper left corner of \( \Lambda'. \) Then, for \( \Lambda' = \begin{bmatrix} H & P \\ P & G \end{bmatrix} \) we have \( \Lambda/H = G - PH^{-1}P^T. \)

Define the symmetric operator
\[
\Lambda_i = \begin{bmatrix} X_{i+1}^T U_{i+1} & U_{i+1}^T B_i \\ B_i^T U_{i+1} & B_i^T D_i(\lambda) \end{bmatrix}, \] \hspace{1cm} (1.6)
where $B_i, D_i(\lambda)$ are the $n \times n$ blocks of the symplectic matrix $W_i(\lambda)$ in (1.4) and $Y_i = [X_i^T U_i^T]^T$ is a conjoined basis of (1.4), i.e. a $2n \times n$ matrix solution with the conditions

$$\text{rank} Y_i = n, \ X_i^T U_i = U_i^T X_i. \quad (1.7)$$

The main result of this paper (see Theorem 3.5 in Section 3) connects the number of focal points $m_i(Y)$ for a conjoined basis of (1.4) with the number of negative eigenvalues of the Schur complement $\Lambda_i/H_{i+1}$, where the $k_{i+1} \times k_{i+1}$ principal submatrix $H_{i+1}$ of $X_{i+1}^T U_{i+1}$ in (1.6) obeys the condition $k_{i+1} = \text{rank}(X_{i+1}^T U_{i+1})$.

Note that any numerical algorithm based on [7, Theorem 2] has to overcome two problems. The first one is connected with instability in the computation of a conjoined basis of (1.4) as the product $Y_i = W_{i-1}(\lambda)W_{i-2}(\lambda)\ldots W_0(\lambda)Y_0$. The instability is caused by the fact that the matrices $Y_i$ become very large elementwise and property (1.7) is not satisfied. For example, the problem $\Delta^{(4)} y_{i-1} = \lambda y_{i+1}$ shows already this effect, which can be seen by an explicit calculation (see [17, Remark 7(iii)]).

The problem of the unstable computations of $Y_i$ makes it impossible to compute the number of focal points via Theorem 3.5 and the definition introduced in [16]. Recall that this definition explicitly uses the upper blocks of $Y_i$ and their Moor-Penrose inverses. We overcome these problems by use of important algebraic properties of the number of focal points which in turn follows from the properties of the comparative index proved in [13]. In particular, we apply an approach connected with a symplectic block $LU$ factorizations with pivoting for conjoined bases of (1.4) (see [8, 9, 10]) and formulate an analog of Theorem 3.5 in terms of solutions of a Riccati matrix difference equation associated with the $LU$ factorization of $Y_i$ (see Theorem 3.7 in Section 3). In Section 4 we offer an algorithm for computing eigenvalues of (1.5). This algorithm uses a version of [7, Theorem 2] for (1.5), results of this paper and bisection (see [18], [4]).

Our test experiments (see Section 4) are based on the equivalence between (1.5) and eigenvalue problems for symmetric banded matrices $A \in \mathbb{R}^{(N+1-n) \times (N+1-n)}$ with bandwidth $2n + 1$ (see [15, 17]).

2. Preliminaries

We will use the following notation. If $A$ is a symmetric matrix, i.e. $A^T = A$, the inequality $A > 0 \ (\geq, <, \leq)$ means that $A$ is positive (nonnegative, negative, nonpositive) definite, ind$A$ denotes the index or the number of negative
eigenvalues of a symmetric matrix $A$. By $A^\dagger$ we denote the Moore-Penrose generalized inverse of a matrix $A$, and $A^{-T} = (A^T)^{-1}$.

Recall (see [2]) that the conjoined basis $Y_i^{(0)}$ of (1.4) with the initial condition $Y_0^{(0)} = [0 I]^T$ at $i = 0$ is said to be the principal solutions at 0.

According to [13, Lemmas 3.1 and 3.2] we have

$$m_i(Y) = \mu(Y_{i+1}, W_i [0 I]^T),$$

$$m^*_i(Y) = \mu^*(Y_i, W_i^{-1} [0 I]^T).$$

(2.1)

The comparative index is defined by $\mu(Y, \hat{Y}) = \mu_1(Y, \hat{Y}) + \mu_2(Y, \hat{Y})$, where $\mu_1(Y, \hat{Y}) = \text{rank} \mathcal{M}$ and $\mu_2(Y, \hat{Y}) = \text{ind} \mathcal{D}$. Introduce the dual index $\mu^*(Y, \hat{Y}) = \mu_1(Y, \hat{Y}) + \mu_2^*(Y, \hat{Y})$, where $\mu_2^*(Y, \hat{Y}) = \text{ind}(-\mathcal{D})$.

According to [13, Lemmas 3.1 and 3.2] we have

$$m_i(Y) = \mu(Y_{i+1}, W_i [0 I]^T),$$

$$m^*_i(Y) = \mu^*(Y_i, W_i^{-1} [0 I]^T).$$

(2.1)
where \( m_i(Y), m^*_i(Y) \) are the numbers of focal points of \( Y_i = [X_i^T U_i^T]^T \) in \((i, i + 1), [i, i + 1)\) respectively (see Definitions 2.1 and 2.2). By formula (3.5) in [13, p. 451] we get

\[
m^*_i(Y) - m_i(Y) = \Delta \text{rank}_{X_i}, X_i = [I 0]Y_i. \tag{2.2}
\]

Introduce the notation

\[
l(Y, M, N) = \sum_{i=M}^{N} m_i(Y), l^*(Y, M, N) = \sum_{i=M}^{N} m^*_i(Y)
\]

for the numbers of focal points of \( Y_i \) in \((M, N + 1]\) and \([M, N + 1)\), then, by (2.2)

\[
l^*(Y, M, N) - l(Y, M, N) = \text{rank}_{X_{N+1}} - \text{rank}_{X_M}. \tag{2.3}
\]

The following theorem based on the concept of a finite eigenvalue introduced in [7] plays an important role in this paper.

**Theorem 2.4.** (Global Oscillation Theorem, see [7]) There exists an integer \( p \in \{0, 1, \ldots, nN\} \), such that, for all \( \lambda \in \mathbb{R} \)

\[
l(Y^{(0)}(0), 0, N) = \#\{\lambda \in \sigma | \lambda \leq \lambda_1\} + p,
\]

where \( \sigma \) is the finite spectrum of (1.1), \( \#\{\lambda \in \sigma | \lambda \leq \lambda_1\} = \sum_{\lambda \leq \lambda_1} \theta(\lambda) \), and \( \theta_N(\lambda) = \max_{\mu \in \sigma} \text{rank}_{X_{N+1}}(\mu) - \text{rank}_{X_{N+1}}(\lambda) \) is the multiplicity of \( \lambda \).

By [7], there are always only finitely many of finite eigenvalues of (1.1), hence

\[
p = l(Y^{(0)}(0), 0, N), \quad \text{max}_{\mu \in \sigma} \text{rank}_{X_{N+1}}(\mu) = \text{rank}_{X_{N+1}}(\lambda_0), \tag{2.4}
\]

\[
\lambda_0 < \lambda_{\min}, \lambda_{\min} := \min_{\sigma}.
\]

We will use the following modification of Theorem 2.4 connecting the number of focal points \( l^*(Y^{(0)}(0), 0, N) \) in \((0, N + 1]\) and \( \#\{\lambda \in \sigma | \lambda < \lambda_1\} \). We have

\[
l^*(Y^{(0)}(0), 0, N) = \#\{\lambda \in \sigma | \lambda < \lambda_1\} + p^*, \quad p^* = l^*(Y^{(0)}(0), 0, N), \lambda_0 < \lambda_{\min}, \lambda_{\min} := \min_{\sigma}, \tag{2.5}
\]

The proof of (2.5) is presented in [12, p. 1236].

Now we recall results of [8, 9, 11, 10] connected with a \( n \) by \( n \) block LU factorization with pivoting for conjoined bases of (1.4). The following technical notation is used: \( \langle k \rangle = \{1, 2, \ldots, k\} \) for every positive integer \( k \), the
lower case Greek letters $\alpha, \beta$ are used as index sets. $A(\alpha, \beta)$ is the submatrix of $A$ whose rows and columns are indexed by $\alpha, \beta$ respectively, and we denote $A(\alpha) := A(\alpha, \alpha)$. When a row or column index set is empty, the corresponding submatrix is considered vacuous. For a $k \times k$ matrix $A$ the Schur complement of an invertible principal submatrix $A(\alpha)$ in $A$ is $A/A(\alpha) = A(\alpha^c) - A(\alpha^c, \alpha)(A(\alpha))^{-1}A(\alpha, \alpha^c)$, where $\alpha^c$ is the complement of $\alpha$ with respect to $\langle k \rangle$.

Introduce the $2n \times 2n$ matrices

$$\mathcal{H} = \begin{bmatrix} F & G \\ -G & F \end{bmatrix}, \quad (2.6)$$

with $n \times n$ diagonal blocks $F, G$ which obey the conditions:

$$F^2 + G^2 = I, \quad FG = 0, \quad (2.7)$$
$$F \geq 0, \quad G \geq 0. \quad (2.8)$$

The condition (2.7) defines a group of symplectic orthogonal matrices. It is easy to verify that there exist only $2^n$ symplectic orthogonal matrices $\mathcal{H}_j$ defined by (2.7), (2.8). So we number the transformations $\mathcal{H}$ in such a way that the number $j$ of any $\mathcal{H}_j$ takes the values from the set $\{0, 1, \ldots, 2^n - 1\}$, and the diagonal of $G_j$ is composed of the zeros and ones that constitute the binary representations of $j$. In this case, we have $\mathcal{H}_0 = I, \mathcal{H}_{2^n-1} = J, \mathcal{H}_j = J\mathcal{H}_{2^n-1-j}^T$. We also introduce the index sets $\alpha_j \subseteq \langle n \rangle$ defined by the positions of ones in the diagonal of $G_j$. For example, $\alpha_0 = \emptyset$, $\alpha_{2^n-1} = \langle n \rangle$, and $\alpha_j^c = \alpha_{2^n-1-j}$.

The treatment of the set $\mathcal{H}_j$ is justified by the following theorem (see [8, 10, 9]).

**Theorem 2.5.** For any matrix $Y = [XT UT]^T$ with conditions (1.7) there exists $j \in \{0\} \cup \langle 2^n - 1 \rangle$ such that

$$\det(X_j) \neq 0, \quad X_j = F_j X - G_j U. \quad (2.9)$$

Hence, we have the following block LU factorization of $Y$ with the pivoting matrix $\mathcal{H}_j$

$$\mathcal{H}_j^TY = \begin{bmatrix} I & 0 \\ Q_j & I \end{bmatrix} \begin{bmatrix} X_j \\ 0 \end{bmatrix}, \quad (2.10)$$
$$U_j = G_j X + F_j U, \quad Q_j = Q_j^T, \quad Q_j = U_j X_j^{-1}.$$

In the following lemma we formulate some properties of (2.10).
Lemma 2.6. (i) If (2.9) and
\[
\frac{\det(F_lX - G_lU)}{\det(F_jX - G_jU)} \leq 1, \quad l = 0, 1, \ldots, 2^n - 1, \tag{2.11}
\]
hold, then all principal minors of the symmetric matrix $Q_j$ in (2.10) are less than or equal to one in absolute value, in particular, we have
\[
\|Q_j\|_1 \leq 1 + \sqrt{2(n - 1)}, \quad i = 0, \ldots, N + 1. \tag{2.12}
\]

(ii) If $F_j$ in (2.7), (2.8) obeys the condition
\[
\text{rank } X = \text{rank}(F_jX) = \text{rank } F_j, \tag{2.13}
\]
then $X_j = F_jX - G_jU$ is nonsingular and (2.10) holds with the symmetric matrix $Q_j$ which obeys the condition
\[
G_j Q_j G_j = 0. \tag{2.14}
\]

(iii) If (2.10) holds, then
\[
\text{rank } X = \text{rank } F_j + \text{rank } (G_jQ_jG_j),
\text{rank } (X^TU) = \text{rank } (F_jQ_jF_j) + \text{rank } (G_jQ_jG_j), \tag{2.15}
\]
In particular, $\det(X) \neq 0$ iff $\text{rank}(G_jQ_jG_j) = \text{rank } G_j$.

Proof. The proof of (i) is based on the formula (see [8, 10])
\[
\frac{\det(F_lX - G_lU)}{\det(F_jX - G_jU)} = |\det(Q_j(\alpha_p))| \tag{2.16}
\]
which connects all principal minors of $Q_j = U_jX_j^{-1}$ with the corresponding minors of order $n$ of $Y$. In (2.16) the index set $\alpha_p$ is defined by positions of nonzero entries of the diagonal matrix $G_p = F_lG_j - G_lF_j$. Then, if $j$ obeys (2.11) all principal minors of $Q_j$ less then or equal to one in absolute value and then this matrix obeys (2.12).

The proof of (ii) is presented in [11, Lemma 2.4]. See also [6, Theorem 3.1].

To prove (iii) we note that factorization (2.10) implies
\[
X = (I + G_jQ_jF_j) (F_j + G_jQ_jG_j) X_j,
U = (I - F_jQ_jG_j) (-G_j + F_jQ_jF_j) X_j,
X^TU = X_j^T (-G_jQ_jG_j + F_jQ_jG_j) X_j.
\]
Since
\[
(I + G_jQ_jF_j)(I - F_jQ_jG_j)^T = I \tag{2.17}
\]
and $X_j$ is nonsingular, we have proven (2.15).
Assume that (2.9) holds. Then, according to (2.16) we have $\det(F_lX - G_lU) \neq 0$ if and only if $\det(Q_j(\alpha_p)) \neq 0$ and the last condition is equivalent with
\[
\text{rank } (G_pQ_jG_p) = \text{rank } G_p > 0. \quad (2.18)
\]
If (2.9),(2.18) hold, then for $Y$ there exists (2.10) with the pivoting matrix $\mathfrak{N}_l = \mathfrak{N}_l^T$ and the symmetric matrix $Q_l$, such that
\[
Q_l = (I - F_pQ_jG_p) \left( F_pQ_jF_p - (G_pQ_jG_p)^\dagger \right) (I - G_pQ_jF_p) \quad (2.19)
\]
(see [10, p.127]). In the next sections we will use the following result based on (2.19).

**Lemma 2.7.** Assume that $Y$ obeys (1.7) and (2.10) holds. Introduce the $2n \times 2n$ matrix associated with $F_j$, $G_j$ and $Q_j$

\[
\Lambda[V_j] = \begin{bmatrix}
-G_jQ_jG_j & -G_j(I - Q_jF_j)\hat{X} \\
-(G_j(I - Q_jF_j)\hat{X})^T & \hat{X}^T\hat{U} - \hat{X}^T F_jF_j\hat{X}
\end{bmatrix}, \quad (2.20)
\]
where $\hat{X}, \hat{U}$ are arbitrary $n \times n$ matrices. If $G_jQ_jG_j \neq 0$ and $Q_j(\alpha_p)$ is a nonsingular principal submatrix of $Q_j(\alpha_j)$, then
\[
Q_j(\alpha_j)/Q_j(\alpha_p) = Q_l(\alpha_l), \alpha_l \cup \alpha_p = \alpha_j, \alpha_l \cap \alpha_p = \emptyset, \quad (2.21)
\]
where $Q_l$ is given by (2.19), and
\[
\Lambda[V_j](\alpha_l \cup \langle n \rangle^c)/(\langle Q_j(\alpha_p)\rangle) = \Lambda[V_j](\alpha_l \cup \langle n \rangle^c), \quad (2.22)
\]
where $\langle n \rangle^c = \{n + 1, \ldots, 2n\}$, $\Lambda[V_l]$ is defined by (2.20) for the case $j := l$, and $G_l = G_j - G_p$, $F_l = F_j + G_p$.

**Proof.** Note that by the assumption $\det Q_j(\alpha_p) \neq 0$ there exists factorization (2.10) with $Q_l$ given by (2.19) and $\mathfrak{N}_l$, where by $\alpha_p \subseteq \alpha_j$ we have $G_j - G_p = G_l \geq 0$ and then $G_lG_p = 0$, $G_lF_p = G_l(I - G_p) = G_l$. Hence, by (2.19),
\[
G_lQ_lG_l = G_lQ_jG_l - G_lQ_jG_p(G_pQ_jG_p)^\dagger G_pQ_jG_l
\]
or
\[
Q_l(\alpha_l) = Q_j(\alpha_l) - Q_j(\alpha_l, \alpha_p)(Q_j(\alpha_p))^{-1}Q_j(\alpha_p, \alpha_l).
\]
Then, (2.21) is proven because $\alpha_l$ is the complement of $\alpha_p$ with respect to the set $\alpha_j$. 

It can be verified by direct computations that
\[
\mathcal{R} \Lambda[V_j] \mathcal{R}^T = \begin{bmatrix}
-G_p Q_j G_p - G_l Q_l G_l & -G_l (I - Q_l F_l) \hat{X} \\
-(G_l (I - Q_l F_l) \hat{X})^T & \hat{X}^T \hat{U} - \hat{X}^T F_l Q_l F_l \hat{X}
\end{bmatrix},
\]
where we use that \( Q_j \) and \( Q_l \) are connected by (2.19) and (2.21) holds. Then, by \( G_p G_l = 0, \ G_p + G_l = G_j \) we have proven (2.22) for the index sets \( \alpha_j, \alpha_l, \alpha_p \) associated with \( G_j, G_l, G_p \).

For the case when \( Y := Y_i \) is a conjoined basis of (1.4) the number \( j \) in the previous results is a function of \( i \).

**Definition 2.8.** A function \( j = j(i), \ i = 0, \ldots, N + 1 \) is called an integration path for a conjoined basis of (1.4) if the condition (2.9) holds. In this case, one can consider the symmetric matrix \( Q_j := Q^i_{j(i)} \equiv Q_j(i) \) in (2.10) as the solution of the following Riccati equation

\[
\dot{C}_i - Q_{j(i+1)} \dot{A}_i + \dot{D}_i \dot{Q}_{j(i)} = Q_{j(i+1)} \dot{B}_i \dot{Q}_j(i) = 0,
\]

\[
\dot{W}_i = (\Upsilon_{j(i+1)})^T \dot{W}_i \Upsilon_{j(i)} = \begin{bmatrix}
\dot{A}_i & \dot{B}_i \\
\dot{C}_i & \dot{D}_i
\end{bmatrix}
\]

along the path \( j = j(i), \ i = 0, \ldots, N + 1 \).

The function \( j = j(i), \ i = 0, \ldots, N + 1 \) defined by the condition (2.13) for \( Y := Y_i \) is called a special path for a conjoined basis \( Y_i \).

### 3. Main Results

In this section we present the main connections between the concepts of the number of focal points, the comparative index and the Schur complement.

Introduce (see [13])

\[
\Lambda[V] = V^T \begin{bmatrix}
0 & I \\
0 & 0
\end{bmatrix} V - \begin{bmatrix}
0 & w(Y, \hat{Y}) \\
0 & 0
\end{bmatrix}, \quad w(Y, \hat{Y}) = Y^T J \hat{Y},
\]

(3.1)

where a \( 2n \times 2n \) matrix \( V \) is defined as \( V = [Y \ \hat{Y}] \), and \( Y, \ \hat{Y} \) are \( 2n \times n \)-matrices. If \( Y, \ \hat{Y} \) are separated into \( n \times n \) blocks \( Y = [X^T U^T]^T, \ \hat{Y} = [\hat{X}^T \hat{U}^T]^T \), then \( \Lambda[V] \) takes the form

\[
\Lambda[V] = \begin{bmatrix}
U^T X & U^T \hat{X} \\
\hat{X}^T U & \hat{X}^T \hat{U}
\end{bmatrix}.
\]

(3.2)
If we assume additionally that the conditions \( U^T X = X^T U, \hat{X}^T \hat{U} = \hat{U}^T \hat{X} \) hold, then \( \Lambda[V] \) is symmetric. Note that (3.1) generalizes the case when \( V \) is symplectic, i.e. \( Y, \hat{Y} \) obey (1.7) and \( w(Y, \hat{Y}) = I \). For the case when \( Y, \hat{Y} \) obey (1.7) the following important result is proven in [13, Lemma 4.4].

**Lemma 3.1.** If \( Y, \hat{Y} \) obey conditions (1.7) then

\[
\text{ind} \Lambda[V] = \text{ind}(X^T U) + \mu(Y, \hat{Y}) = \text{ind}(\hat{X}^T \hat{U}) + \mu^*(J\hat{Y}, JY). \tag{3.3}
\]

Recall now that if a symmetric \( q \times q \) matrix

\[
\Lambda = \begin{bmatrix} H & P \\ P^T & G \end{bmatrix}
\]

is separated into \( k \times k \) and \( p \times p \) blocks \( H \) and \( G \), where \( k + p = q \) and \( \det H \neq 0 \), then (see [14, p. 101])

\[
\Lambda = \begin{bmatrix} I & 0 \\ P^T H^{-1} & I \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & (\Lambda/H) \end{bmatrix} \begin{bmatrix} I & H^{-T} P^T \\ 0 & I \end{bmatrix},
\]

where \( (\Lambda/H) \) is the Schur complement of \( H \) in \( \Lambda \). From the last factorization we have

\[
\text{ind} \Lambda = \text{ind} H + \text{ind}(\Lambda/H). \tag{3.4}
\]

Using identities (3.4), (3.3) we prove the following

**Lemma 3.2.** If \( Y, \hat{Y} \) obey (1.7), then for any \( k \times k \) nonsingular principal submatrix \( H \) of \( X^T U \) we have

\[
\mu(Y, \hat{Y}) = \text{ind}(\Lambda[V]/H) - \text{ind}((X^T U)/H), \tag{3.5}
\]

where \( \mu(Y, \hat{Y}) \) is the comparative index and \( \Lambda[V] \) is given by (3.1). In particular, if

\[
\text{rank}(X^T U) = k, \tag{3.6}
\]

then

\[
\mu(Y, \hat{Y}) = \begin{cases} 
\text{ind}(\Lambda[V]/H), & k > 0, \\
\text{ind}(\Lambda[V]), & k = 0.
\end{cases} \tag{3.7}
\]

**Proof.** Without loss of generality we assume that \( H \) defined in Lemma 3.2 is located in the upper left corner of \( \Lambda[V] \). Then, using (3.4) for the symmetric matrix \( X^T U \) we have

\[
\text{ind}(X^T U) = \text{ind} H + \text{ind}((X^T U)/H),
\]
and \((X^T U)/H = 0\) iff (3.6) holds. Next, according to (3.4) we have \(\text{ind}(\Lambda[V]) = \text{ind}(\Lambda[H])\), while by (3.3), \(\text{ind}(\Lambda[V]) = \text{ind}(X^T U) + \mu(Y, \hat{Y}) = \text{ind}(X^T U)/H + \mu(Y, \hat{Y})\). Comparing the last identities we prove (3.5), (3.7).

\[\text{Remark 3.3.} \quad \text{Note that (3.5) can be generalized in the following way. Let} \ H_1, H_2, \ldots, H_{k-1}, H_k \text{ be a sequence of nonsingular symmetric matrices such that} \ H_k \text{ is a submatrix of} \ X^T U, \text{ and} \ H_k \text{ is a submatrix of} \ ((X^T U)/H_1, \ldots)/H_{k-1}. \text{ Then we have}
\]

\[
\mu(Y, \hat{Y}) = \text{ind}(((\Lambda[V]/H_1)/H_2)/\ldots)/H_k) - \text{ind}(((X^T U)/H_1)/H_2)/\ldots)/H_k). \tag{3.8}
\]

The algorithm for computing the comparative index and the numbers of focal points offered in this paper is based on the relevant choice of the sequence \(H_1, H_2, \ldots, H_{k-1}, \ldots, H_k\), coupled with the subsequent evaluation of the Schur complements \(\Lambda[V]/H_1, (\Lambda[V]/H_1)/H_2, \ldots, ((\Lambda[V]/H_1)/H_2)/\ldots)/H_k\).

\[\text{Note that we have} \ \mu(Y, \hat{Y}) = \mu(LYC_1, LY C_2) \text{ for any symplectic block low triangular matrix} \ L \text{ and arbitrary nonsingular} \ n \times n \text{ matrices} \ C_1, C_2 \text{ (see [13, p. 448])}. \text{ Then, the Schur complements in (3.5), (3.7) have the similar property. So we can formulate the following}
\]

\[\text{Lemma 3.4.} \quad \text{If} \ Y, \hat{Y} \text{ obey (1.7),} \hat{Y} = L Y C_1, \hat{V} = L V \text{ diag}(C_1, C_2), \text{ where the symplectic matrix} \ L \text{ is given as} \ L = \begin{bmatrix} M & 0 \\ P & M^{-T} \end{bmatrix}, \text{ and} \ C_1, C_2 \text{ are nonsingular}, \text{ then under the notation of Lemma 3.2 we have}
\]

\[
\mu(Y, \hat{Y}) = \text{ind}(\Lambda[V]/(H)) - \text{ind}((X^T U)/H) = \text{ind}(\Lambda[\hat{V}]/(\hat{H})) - \text{ind}((\hat{X}^T \hat{U})/\hat{H}), \tag{3.9}
\]

where the nonsingular \(k \times \tilde{k}\) matrix \(\hat{H}\) is a principal submatrix of \(\hat{X}^T \hat{U} = C_1^T (X^T U + X^T M^T P X) C_1\).

\[\text{As a corollary of Lemma 3.2 and (2.1) we have}
\]

\[\text{Theorem 3.5.} \quad \text{Let} \ m_i(Y) \text{ be the number of focal points in} \ (i, i + 1] \text{ for a conjoined basis} \ Y_i = [X_i^T U_i]^T. \text{ Then for any} \ k_{i+1} \times k_{i+1} \text{ nonsingular principal submatrix} \ H_{i+1} \text{ of} \ X_i^{T^i_1} U_{i+1} \text{ we have}
\]

\[
m(Y_i) = \text{ind}(\Lambda[V_i]/H_{i+1}) - \text{ind}((X_{i+1}^T U_{i+1})/H_{i+1}). \tag{3.10}
\]
If additionally
\[ \text{rank}(X_i U_i) = k_i, \ i = 0, \ldots, N, \] (3.11)
then
\[ m(Y_i) = \begin{cases} 
\text{ind}(\Lambda[V_i]/H_{i+1}), & k_{i+1} > 0, \\
\text{ind}(\Lambda[V_i]), & k_{i+1} = 0,
\end{cases} \] (3.12)
where \( \Lambda[V_i] \) is given by (1.6).

Similarly, if \( m^*(Y) \) is the number of focal points in \([i, i+1)\) for a conjoined basis \( Y_i \), then for any \( k_i \times k_i \) nonsingular principal submatrix \( H_i \) of \(-X_i^T U_i\) we have
\[ m^*(Y_i) = \text{ind}(\Lambda[V_i]/H_i) - \text{ind}((-X_i^T U_i)/H_i), \] (3.13)
where \( \Lambda[V_i] \) is defined as the following
\[ \Lambda[V_i] = \begin{bmatrix} -X_i^T U_i & U_i^T B_i^T \\
B_i U_i & B_i A_i^T \end{bmatrix}, \quad i = 0, \ldots, N. \] (3.14)

If (3.11) holds, then
\[ m^*(Y_i) = \begin{cases} 
\text{ind}(\Lambda[V_i]/H_i), & k_i > 0, \\
\text{ind}(\Lambda[V_i]), & k_i = 0.
\end{cases} \] (3.15)

**Proof.** The proof of (3.10), (3.12) follows from the representation \( m_i(Y) = \mu(Y_{i+1}, W_i[0 I]^T) \) (see (2.1)) and (3.5), (3.7). Consider the proof of (3.15). According to (2.1)
\[ m_i^*(Y) = \mu(Y_i, W^{-1}[0 I]) = \mu^*(\begin{bmatrix} X_i^T \\
U_i \end{bmatrix}, \begin{bmatrix} -B_i^T \\
A_i^T \end{bmatrix}) = \mu(\begin{bmatrix} -X_i \\
U_i \end{bmatrix}, \begin{bmatrix} B_i^T \\
A_i^T \end{bmatrix}), \]
were we use that \( \mu^*(Y, \hat{Y}) = \mu(\text{diag}(-I, I)Y, \text{diag}(-I, I)\hat{Y}) \) because of the definition of the dual comparative index (see [13]). Applying Lemma 3.2 to \( \mu(\begin{bmatrix} -X_i \\
U_i \end{bmatrix}, \begin{bmatrix} B_i^T \\
A_i^T \end{bmatrix}) \), we derive (3.13), (3.15). \( \Box \)

In the last part of this section we show that it is possible to reduce the dimensions of \( \Lambda[V_i] \) in Theorem 3.5 using Lemma 3.4 and the \( LU \)-factorization with pivoting for a conjoined basis \( Y_i \) (see Section 2). The consideration is based on the following

**Lemma 3.6.** If \( Y, \hat{Y} \) obey (1.7), (2.10) holds for \( Y \), and the symmetric \( 2n \times 2n \) matrix \( \Lambda[V_j] \) is given by (2.20), then for any \( k \times k \) nonsingular principal submatrix \( H \) of \(-G_j Q_j G_j \neq 0\) we have
\[ \mu(Y, \hat{Y}) = \text{ind}(\Lambda[V_j]/H) - \text{ind}(-G_j Q_j G_j/H). \] (3.16)
In particular, for the case \( \text{rank}(G_j Q_j G_j) = k \) we have \( \mu(Y, \hat{Y}) = \text{ind}(\Lambda[V_j]/H), \) \( k > 0, \mu(Y, \hat{Y}) = \text{ind}(\Lambda[V_j]), \) \( k = 0. \)

Proof. From (2.10) we derive

\[
Y X_j^{-1} = \mathcal{R}_j \begin{bmatrix} I \\ Q_j \end{bmatrix} = \tilde{L} \begin{bmatrix} F_j + G_j Q_j G_j \\ -G_j \end{bmatrix},
\]

where the symplectic matrix \( \tilde{L} \) is given by

\[
\tilde{L} = \begin{bmatrix} (I + G_j Q_j F_j) & 0 \\ F_j Q_j F_j & (I - F_j Q_j G_j) \end{bmatrix}
\]

(note that we have (2.17) because of (2.7)). Applying Lemma 3.4 for \( \Lambda[V] \) given by (3.2) for \( C_1 := X_j^{-1}, C_2 := I, L := L^{-1} \) and using Lemma 3.2 we complete the proof. \( \Box \)

The following theorem is based on Lemma 3.6.

**Theorem 3.7.** Let \( j = j(i) \) be an integration path for a conjoined basis \( Y_i \) and let \( Q_{j(i)} \) be the solution of (2.23) along this path. Then for any \( k_i+1 \times k_i+1 \) nonsingular principal submatrix \( H_{i+1} \) of \( -G_{j(i+1)} Q_{j(i+1)} G_{j(i+1)} \) we have \( m_i(Y) = \text{ind}(\Lambda[V_i]/H_{i+1}) - \text{ind}((-G_{j(i+1)} Q_{j(i+1)} G_{j(i+1)})/H_{i+1}) \) with

\[
\Lambda[V_i] = \begin{bmatrix} -G_{j(i+1)} Q_{j(i+1)} G_{j(i+1)} & P_i \\ P_i^T & R_i \end{bmatrix},
\]

\[
P_i = -G_{j(i+1)} (I - Q_{j(i+1)} F_{j(i+1)}) B_i,
\]

\[
R_i = B_i^T D_i - B_i^T F_{j(i+1)} Q_{j(i+1)} F_{j(i+1)} B_i, \quad i = 0, \ldots, N.
\]

If additionally \( k_{i+1} = \text{rank}(G_{j(i+1)} Q_{j(i+1)} G_{j(i+1)}) \), then (3.12) holds for \( \Lambda[V_i] \) given by (3.17).

Similarly, for any \( k_i \times k_i \) nonsingular principal submatrix \( H_i \) of \( G_{j(i)} Q_{j(i)} G_{j(i)} \) we have \( m_i^*(Y) = \text{ind}(\Lambda[V_i]/H_i) - \text{ind}((G_{j(i)} Q_{j(i)} G_{j(i)})/H_i) \) with

\[
\Lambda[V_i] = \begin{bmatrix} G_{j(i)} Q_{j(i)} G_{j(i)} & P_i \\ P_i^T & R_i \end{bmatrix},
\]

\[
P_i = -G_{j(i)} (I - Q_{j(i)} F_{j(i)}) B_i^T,
\]

\[
R_i = B_i A_i^T + B_i F_{j(i)} Q_{j(i)} F_{j(i)} B_i^T.
\]

If \( k_i = \text{rank}(G_{j(i)} Q_{j(i)} G_{j(i)}) \), then (3.15) holds with \( \Lambda[V_i] \) given by (3.18).
Proof. The proof of the both assertions is similar. Applying Lemma 3.6 for the case \( j := j(i + 1), k := k_{i+1}, \hat{X} := B_i, \hat{U} := D_i \) and using Theorem 3.5 we prove the first assertion associated with \( m_i(Y) \) and (3.17). Similarly, applying Lemma 3.6 for the case \( j := j(i), k = k_i, \hat{X} := B_i^T, \hat{U} := A_i^T, Y := [-X_i^T, U_i^T]^T \) by use of Theorem 3.5 we prove the second assertion associated with \( m_i^*(Y) \) and (3.18).

Note that by Lemma 2.6(ii) for the special integration path defined by (2.13) (see Section 2) we have \( k_i \equiv 0 \), \( k_i = \text{rank}(G_{j(i)}Q_{j(i)}G_{j(i)\bar{}}) \) because of (2.14), then we derive the following

**Corollary 3.8.** Let \( j = j(i) \) be the special integration path defined by (2.13) for a conjoined basis \( Y_i \) and let \( Q_{j(i)} \) be the solution of (2.23) along this path. Then we have for the numbers of focal points \( m(Y_i), m^*(Y_i) \) in \((i, i + 1), [i, i + 1)\) that \( m(Y_i) = \text{ind}(\Lambda[Y_i]) \), where \( \Lambda[Y_i] \) is given by (3.17) with \( G_{j(i)}Q_{j(i)}G_{j(i)\bar{}} \equiv 0 \). Similarly, \( m^*(Y_i) = \text{ind}(\Lambda[Y_i]) \), where \( \Lambda[Y_i] \) is given by (3.18) with \( G_{j(i)}Q_{j(i)}G_{j(i)\bar{}} \equiv 0 \).

**Remark 3.9.** Note that by Lemma 2.7 we have in (3.16) that \( \text{ind}(\Lambda[Y_j]/H) = \text{ind}(\Lambda[Y_j]) \), \( \text{ind}(-G_jQ_jG_j/H) = \text{ind}(-G_lQ_lG_l) \), where \( \Lambda[Y_j] \) is defined by (2.20) with \( j := l \). Then, under the assumptions of Lemma 2.7 the choice of a nonsingular principal submatrix \( H \equiv Q_j(\alpha_p) \) of \( Q_j(\alpha_j) \neq 0 \) coupled with the evaluation of the Schur complement \( \Lambda[Y_j]/Q_j(\alpha_p) \) leads to the definition of \( \mu(Y, \hat{Y}) \) in terms of \( \Lambda[V_i] \), i.e. \( \mu(Y, \hat{Y}) = \text{ind}(\Lambda[V_i]) - \text{ind}(-G_lQ_lG_l) \), where \( G_l = G_j - G_p \) and \( \text{rank}(G_lQ_lG_l) < \text{rank}(G_jQ_jG_j) \). Repeating this process several times we find \( \mu(Y, \hat{Y}) = \text{ind}(\Lambda[V_i]) \) where \( \text{rank}(G_lQ_lG_l) = 0 \) (see (2.14)).

For the case when \( Y := Y_i \) is a conjoined basis of (1.4) and \( j = j(i) \) is an integration path such a process leads to the evaluation of a special integration path and \( m_i(Y), m_i^*(Y) \) according to Corollary 3.8. For the particular case of the Sturm-Liouville problem (1.5) when \( \text{rank}(B_i) = 1 \) we compute \( m_i^*(Y) \) starting with the truncated matrix \( \Lambda[Y_i](\alpha_j \cup \{2n\}) \) of the size \( \text{rank}G_{j(i)\bar{}} + 1 \). Applying the relevant pivoting strategy (see Remark 3.3) we use Corollary 3.8 at the final step evaluating the index of a symmetric matrix in the form

\[
\begin{bmatrix}
0 & b_i \\
\overline{b_i} & a_i
\end{bmatrix},
\]

where the zero block of the size \( n - \text{rank}(X_i) \) is bordered by only one row and column (see the next section).
4. Applications to the Sturm-Liouville Eigenvalue Problems

In this section we consider applications of the results of Section 3 in algorithms for computing eigenvalues of the $2n$-order Sturm-Liouville problem (1.5). Discrete eigenproblem (1.5) corresponds to eigenvalue problem (1.1), where (see [1])

\[
A_i = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \quad B_i = A_i \cdot \frac{1}{r_i^{(n)}} \cdot \text{diag } (0, \ldots, 0, 1), \\
C_i = \text{diag } (r_i^{(0)}, \ldots, r_i^{(n-1)}) \cdot A_i, \\
D_i = \text{diag } (r_i^{(0)}, \ldots, r_i^{(n-1)}) \cdot B_i + (A_i)^{-T}, \\
W_i = \text{diag } (1, 0, \ldots, 0),
\]

and

\[
x_i = [y_i, \Delta y_{i-1}, \ldots, \Delta^{n-1} y_{i-n+1}]^T, \\
u_i = \left[ \sum_{\mu=\nu+1}^{n} (-\Delta)^{\mu-\nu-1} \left( r_i^{(\mu)} \Delta^{\mu} y_{i+1-\mu} \right), \ldots, r_i^{(n)} \Delta^{(n)} y_{i-n+1} \right]^T.
\]

For (1.5) we formulate the following version of Theorem 2 in [7] in terms of $m^*(Y^{(0)})$.

**Theorem 4.1.** Let $l^*(Y^{(0)}(\lambda_1), n, N)$ be the number of focal points for the principal solution $Y_i^{(0)}$ of (1.1), (4.1), (4) in $[n, N+1)$. Then, for any $\lambda_1 \in \mathbb{R}$

\[
l^*(Y^{(0)}(\lambda_1), n, N) = \# \{ \mu \in \sigma | \mu < \lambda_1 \},
\]

where $\# \{ \mu \in \sigma | \mu < \lambda_1 \}$ denotes the number of eigenvalues of (1.5) less than $\lambda_1$.

**Proof.** According to (2.5) we have

\[
l^*(Y^{(0)}(\lambda_1), 0, N) = \# \{ \mu \in \sigma | \mu < \lambda_1 \} + p^*,
\]

where $\lambda_{\text{min}}$ is the minimal eigenvalue of (1.1). By formula (2.3), $p^* - p = \text{rank}(X^{(0)}_{N+1}(\lambda_0)), \lambda_0 < \lambda_{\text{min}}$ where by (2.4) $p = l(Y^{(0)}(\lambda_0), 0, N)$. According to [15, Theorem 4] $\text{rank}(X^{(0)}_{N+1}(\lambda_0)) = n, \lambda_0 < \lambda_{\text{min}}$ and, by [15, Lemma 2], $p = l(Y^{(0)}(\lambda_0)), 0, N) = 0$. Then, for problem (1.5) we have (2.5) with $p^* = n$. Using [1, Lemma 4] we also have $m_i(Y^{(0)}(\lambda_1)) = 0, \text{rank}(X^{(0)}_{i+1}(\lambda_1)) - \text{rank}(X^{(0)}_{i}(\lambda_1)) = \ldots$. 
1, \ i = 0, \ldots, n - 1 \text{ for any } \lambda_1 \in \mathbb{R}. \text{ So we derive } m^*(Y^{(0)}(\lambda_1)) = 1, \ i = 0, \ldots, n - 1 \text{ because of (2.2), i.e. } l^*(Y^{(0)}(\lambda_1), 0, n - 1) = n. \text{ Finally, it follows from (2.5) that } l^*(Y^{(0)}(\lambda_1), 0, N) - p^* = l^*(Y^{(0)}(\lambda_1), 0, N) - l^*(Y^{(0)}(\lambda_1), 0, n - 1) = l^*(Y^{(0)}(\lambda_1), n, N) = \# \{ \mu \in \sigma | \mu < \lambda_1 \} \text{ and the proof is completed.} \]

Now we consider Theorem 3.7 and Corollary 3.8 for (1.5). Note that for the given particular case we have by (4.1) \( B_i = A_i \cdot \frac{1}{r_i} \cdot \text{diag}(0, \ldots, 0, 1) \) and \( A_i \neq 0. \) Then, applying Lemma 3.4 to (3.18) for the case \( L = I, \ C_1 = I, \ C_2 = A_i^{-T} \text{diag}(1, 1, \ldots, 1, r_i^{(n)}) \) with \( A_i \) given by (4.1) we have (3.18) in the form

\[
\Lambda[V_i \text{diag}(I, C_2)] = \begin{bmatrix} G_{j(i)}Q_{j(i)}G_{j(i)} & P_iC_2 \\ C_i^TP_i & C_2^TR_iC_2 \end{bmatrix},
\]

\[
P_iC_2 = -G_{j(i)}(I - Q_{j(i)}F_{j(i)})\text{diag}(0, 0, 0, 1),
\]

\[
C_2^TR_iC_2 = \text{diag}(0, 0, 0, 1)F_{j(i)}Q_{j(i)}F_{j(i)}\text{diag}(0, 0, 0, 1)
\]

\[
+ \text{diag}(0, 0, 0, r_i^{(n)}).
\]

Finally, according to Remark 3.9 after deleting zero rows and columns we have the following truncated \( \Lambda[V_i] \)

\[
\Lambda[V_i]|_{tr} = \begin{bmatrix} Q_{j(i)}(\alpha_{j(i)}) & b_i \\ b_i^T & a_i \end{bmatrix},
\]

\[
a_i = (F_{j(i)}Q_{j(i)}F_{j(i)})(n, n) + r_i^{(n)},
\]

\[
b_i = (-I + G_{j(i)}Q_{j(i)}F_{j(i)})(\alpha_{j(i)}, n),
\]

where \( A(\alpha) := A(\alpha, \alpha) \) and \( A(\alpha, \beta) \) denotes the submatrix of \( A \) whose rows and columns are indexed by the index sets \( \alpha, \beta \subseteq \{1, 2, \ldots, n\}, \) and the index set \( \alpha_{j(i)} = \{\alpha_1(i) < \cdots < \alpha_p(i)\} \subseteq \{1, 2, \ldots, n\} \) is defined by the binary representation of the integration path \( j(i) = 2^{n-\alpha_1(i)} + \cdots + 2^{n-\alpha_p(i)} \) (see Section 2). So, (4.3) is obtained by symmetric bordering of \( Q_{j(i)}(\alpha_{j(i)}) \) by only one row and column.

As a corollary of Theorem 3.7 for the 2n order Sturm-Liouville problem we have

**Corollary 4.2.** Let \( j = j(i) \) be an integration path for a conjoined basis \( Y_i \) of (1.1), (4.1). Then for any \( k_i \times k_i \) nonsingular principal submatrix \( H_i \) of \( Q_{j(i)}(\alpha_{j(i)}) \) in (4.3) we have \( m^*(Y_i) = \text{ind}(\Lambda[V_i]|_{tr}/H_i) - \text{ind}(Q_{j(i)}(\alpha_{j(i)})/H_i). \) If \( Q_{j(i)}(\alpha_{j(i)})/H_i = 0, \) i.e. \( k_i = \text{rank}(Q_{j(i)}(\alpha_{j(i)})), \) then \( m^*(Y_i) = \text{ind}(\Lambda[V_i]|_{tr}/H_i), \)
\[ k_i > 0 \text{ and for } k_i = 0 \text{ we have} \]
\[ m^*(Y_i) = \text{ind} \begin{bmatrix} 0 & b_i \\ b_i^T & a_i \end{bmatrix} = \begin{cases} 1, & b_i^T b_i \neq 0, \\ 1, & a_i < 0, b_i^T b_i = 0, \\ 0, & a_i \geq 0, b_i^T b_i = 0, \end{cases} \tag{4.5} \]

where the column \( b_i \) and the number \( a_i \) are given by (4.4).

For the given \( \lambda_1 \) the following algorithm based on Theorem 4.1 and Corollary 4.2 determines \( l^*(Y(0)) (\lambda_1), n, N \) and \( \# \{ \mu \in \sigma | \mu < \lambda_1 \} \).

**Step 1.** Given \( \lambda_1, W_i \) for \( 0 \leq i \leq N \) and \( Y_0(0) = [0]I \). The algorithm determines an integration path \( j = j(i) \), the matrices \( \mathfrak{N}_{j(i)} \) and the solution of (2.23) \( Q_{j(i)} \), such that condition (2.12) holds (see Lemma 2.6). For \( i = 0 \) we have \( j(0) = 2^n - 1, \mathfrak{N}_{j(0)} = J, \) and \( Q_{j(0)} = 0 \) because of the initial condition \( Y_0(0) = [0]I \). If \( j = j(i), Q_{j(i)}, \mathfrak{N}_{j(i)} \) are given, we find \( j(i + 1) \) such that

\[ |\text{det}(F_{j(i+1)}X - G_{j(i+1)}U)| = \max_{l=0,2,\ldots,2^{n-1}} |\text{det}(F_{i}X - G_{i}U)|, \]

where \( [X^T \ U^T]^T = Y := W_i \mathfrak{N}_{j(i)}[IQ_{j(i)}]^T \) and then put \( Q_{j(i+1)} = (G_{j(i+1)}X + F_{j(i+1)}U)(F_{j(i+1)}X - G_{j(i+1)}U)^{-1} \). Note that according to (2.11) this step entails a time complexity of \( O(n^32^n) \) for each \( i = 1, \ldots, N + 1 \).

**Step 2.** Given \( i, \mathfrak{N}_{j(i)}, r_i^{(n)}, Q_{j(i)} \) for \( n \leq i \leq N \), and \( tol := ||Q_{j(i)}(\alpha_{j(i)})||_{eps} \), the algorithm forms truncated matrices (4.3) and evaluates \( m^*_i(Y) \) according to the following procedure:

a) \[
\begin{align*}
&k = 0; \Lambda^{(k)} := \Lambda[V_i]_r; \\
&Q(\alpha^{(k)}) := Q_{j(i)}(\alpha_{j(i)});
\end{align*}
\]

b) if \( ||Q(\alpha^{(k)})|| > tol \)

apply the Bunch-Parlett or the Banch-Kaufman pivot strategies (see [14, p.168-169]) to choose the \( 1 \times 1 \) or \( 2 \times 2 \) pivot \( E^{(k)} \) in the block \( Q(\alpha^{(k)}) \) of \( \Lambda^{(k)} \).

Put
\[
\begin{align*}
\Lambda^{(k+1)} &:= (\Lambda^{(k)}/E^{(k)}); \\
Q(\alpha^{(k+1)}) &:= (Q(\alpha^{(k)})/E^{(k)}); \\
k &:= k + 1;
\end{align*}
\]

**goto b**
\[ M = \max_{i} \left( \frac{|\lambda_i - \tilde{\lambda}_i|}{\lambda_i} \right) \]

\[ b = 5 \quad b = 9 \quad b = 13 \]

\begin{table}
\begin{tabular}{|c|c|c|c|}
\hline
& $1$ & $2$ & $3$ & $4$ \\
\hline
$M$ & $125$ & $124$ & $125$ & $125$ \\
$\varepsilon$ & $3.0791e-014$ & $1.2958e-014$ & $3.1760e-015$ & $6.7847e-014$ \\
$b = 9$ & $5.0220e-013$ & $1.2899e-011$ & $2.7811e-012$ & $3.1289e-012$ \\
$b = 13$ & $6.6665e-012$ & $1.2899e-011$ & $2.7811e-012$ & $6.9798e-012$ \\
\hline
\end{tabular}
\end{table}

Table 1: Maximum relative error for tested matrices with the given eigenvalue distributions 1-4

c) else evaluate $m_i^*(Y) = \text{ind}(\Lambda^{(k)})$ according to (4.5).

Step 3. The algorithm determines $l^*(Y^{(0)} (\lambda_1), n, N)$ and then, by Theorem 4.1 determines the number of eigenvalues for (1.5) which are less than the given $\lambda_1$.

Combining the algorithm described above with the bisection method offered in [4] we can evaluate all eigenvalues of problem (1.5). Note that the backward error analysis (see [18, 14]) of the algorithm can be the subject of the future investigation.

4.1. Numerical Tests

First, we note the computational specifications. Our algorithm is implemented in Matlab 7.4 using 32 decimal digit Software-floats. The computations were carried out on a PC, Pentium 4 with 3 GHz.

Our tests for problems (1.5) are based on the equivalence between (1.5) and eigenvalue problems for symmetric banded matrices $A \in \mathbb{R}^{(N+1-n) \times (N+1-n)}$ with bandwidth $2n + 1$ (see [15, 17]). The algorithm have been tested on symmetric matrices of the sizes $M = N + 1 - n$ ranging from 125 to 500 and the bandwidth $b = 2n + 1$ ranging from 5 to 13 with given eigenvalue distributions.
We considered 4 types of diverse eigenvalue distributions:

1. **UNIFORM ($\varepsilon$ to 1):** $\lambda_i = \varepsilon + (i - 1) \cdot \varepsilon$, $i = 1, 2, ..., M$ where $\varepsilon = 1/M$.

2. **UNIFORM (from -1 to 1):** $\lambda_i = -1 + (i - 1) \cdot \tau$, $i = 1, 2, ..., M$ where $\tau = 2/(M - 1)$.

3. **GEOMETRIC ($\varepsilon$ to 1):** $\lambda_i = \varepsilon^{(M-i)/(M-1)}$, $i = 1, 2, ..., M$.

4. **RANDOM ($\varepsilon$ to 1):** the eigenvalues are drawn from a uniform distribution on the interval [0,1].

We use Givens rotations for generating banded matrices according to the method described in [5, p.16]. For tested matrices we compute the maximum relative error $\varepsilon = \max \left( \left| \lambda_i - \tilde{\lambda}_i \right| / \lambda_i \right)$, where $\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_M$ are computed eigenvalues obtained by the algorithm.

**References**


