ON COUNTABLE SETS OF ORDER PRESERVING OPERATOR INEQUALITIES IN HILBERT SPACES

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Abstract: In this paper we show that the well-known Furuta inequality can be expressed in countable sets of operator inequalities in two forms: \((YXY)\beta\) and the \(\beta\)-power-mean. So are the ground Furuta inequality and its generalization, and the chaotic order for two operators. Generally speaking, each Furuta-type operator inequality has such expression, and they are equivalent to one another, indeed.

AMS Subject Classification: 47A63, 47A6
Key Words: Hilbert space, positive operator, \(\beta\)-power-mean, the operator expansion of the form \((YXY^*)\beta\), Furuta inequality, grand Furuta inequality and its generalization, and chaotic order for two operators

1. Introduction

Motivated by the \(\beta\)-power-mean for two operators in a Hilbert space and the operator expansion of the operator form \((YXY^*)\beta\), the purposes of this article are to show that the Furuta inequality can be expressed in countable sets of operator inequalities in two forms: one is \((YXY)^\beta\) and the other the \(\beta\)-power-mean. So are the ground Furuta inequality and its generalization, and the chaotic order for two operators. Related topics could be found in the author’s previous article in [8], [9].
Throughout this paper the capital letters mean bounded and positive linear operators in a Hilbert space, unless otherwise stated, and $I$ is the identity operator.

Recall that the $\beta$-power-mean introduced by Kubo-Ando [7] was given by

$$A^\#_\beta B = A^{1/2} (A^{-1/2} B A^{-1/2})^\beta A^{1/2},$$

(\#)

for any real number $\beta$ and $A$ and $B$ are invertible.

The $\beta$-power-mean is a useful tool in expressing alternatively the Furuta-type inequalities and chaotic order of two operators as we have seen in the literature in the past twenty some years. The next lemma is the operator expansion of operator form $(Y XY^*)^\beta$.

**Lemma A.** (see [3], Lemma 1) Let $X$ be positive and invertible, and $Y$ be invertible. For any real number $\beta$ we have

$$(Y XY^*)^\beta = Y X^{1/2} (X^{1/2} Y Y X^{1/2})^{\beta-1} X^{1/2} Y^*.$$

\section{2. Some Well-Known Furuta-Type Operator Inequalities}

Let us list some well-known Furuta-type operator inequalities, which will be used in the Sections 3 and 4. Indeed, the Furuta-type operator inequalities all started with the next theorem, which was called the order preserving operator inequality by Furuta himself.

**Theorem B.** (The Furuta Inequality, see [1]) If $A \geq B \geq 0$, then for each $r \geq 0$,

$$B^{r/2} A^p B^{r/2} \geq B^{\frac{p+r}{q}} \text{ and } A^{\frac{p+r}{q}} \geq (A^{r/2} B^p A^{r/2})^{1/q}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p + r$.

**Theorem C.** (The Grand Furuta Inequality, see [3]) If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0, 1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^{r/2} (A^{-t/2} B^p A^{-t/2})^{s} A^{r/2}\}^{\frac{1-t+r}{(p-t)\frac{s}{q}+r}}$$

for $s \geq 1$ and $r \geq t$.

It should be noted that the exponents $1/q$ in Theorem B and $\frac{1-t+r}{(p-t)\frac{s}{q}+r}$ in Theorem C are the best possible for the given conditions on $p$, $q$ and $r$ in
Theorem B (see [10]), and on \( p, r, s \) and \( t \) in Theorem C, see [11]. Incidentally, the proofs of both cases were nontrivial. Actually, in this paper we consider the validity of the operator inequalities in the Section 3 for the exponents \( \frac{1}{q} \pm i \) and \( \frac{1-\ell+r}{(p-t)\sigma+r} \pm i \) and \( \frac{1-\ell+r}{(p-t)\sigma+r} \pm j \) in Theorem 2, where \( i \) is an even number and \( j \) is an odd number.

**Theorem D.** (Characterization of the Chaotic Order, see [2]) Let \( A \) and \( B \) be invertible. Then the following are equivalent.

(i) \( A \gg B \), i.e. the chaotic order \( \log A \geq \log B \).

(ii) \( A^r \geq (A^{r/2}B^pA^{r/2})^{r}_{p+r} \) for \( p \geq 0 \) and \( r \geq 0 \).

Just recently Furuta gave a generalization of Theorem C in [5] as follows.

**Theorem E.** (see [5], Theorem 3.3) Let \( A \geq B \geq 0 \) with \( A > 0 \), \( t \in [0,1] \), and \( p_1, p_2, \cdots, p_{2n} \geq 1 \) for a natural number \( n \). Then the following inequality holds for \( r \geq t \).

\[
A^{1-t+r} \geq \{A^{r/2}[A^{-t/2}[A^{t/2}\cdots A^{t/2}[A^{-t/2}(A^{-t/2}B^{p_1}A^{-t/2})^{p_2}A^{t/2}]^{p_3}] \cdots [A^{-t/2}[A^{t/2}]^{p_5} \cdots A^{-t/2}]^{p_{2n-2}}A^{t/2}]^{p_{2n-1}}A^{-t/2}]^{p_{2n}}A^{r/2}\}^q,
\]

where \( q = \frac{1-t+r}{(\cdots((p_1-t)p_2+t)p_3-t)\cdots p_{2n-2}(p_{2n-1}-t)p_{2n}+t}\).

More precisely, in Theorem E there are \( n \) terms of \( A^{-t/2} \) and \( n-1 \) terms of \( A^{t/2} \) alternatively arranged on the left side of the term \( B^{p_1} \), and the same arrangement on the right side of the term \( B^{p_1} \). As for the denominator of \( q \), there are \( n \) terms of \( -t \) and \( n-1 \) terms of \( t \) alternatively arranged.

Notice that Theorem C is a special case of Theorem E, which is obtained by letting \( p_1 = p, p_m = 1 \) for \( m = 2, 3, \cdots, 2n-1 \), and \( p_{2n} = s \) in Theorem E. In Theorem 4 in the Section 3 we consider the validity of the operator inequalities for the powers \( q \pm i \) and \( q \pm j \).

Without loss of generality we may assume that \( A \) and \( B \) are invertible in Theorem B, Theorem C and Theorem E, cf. [1].
3. The $\beta$-Power-Mean and Countable Sets of the Operator Form $(YXY)^\beta$

In what follows we write $i = 0, 2, 4, 6, \ldots$, i.e., $i$ is an even number, and $j = 1, 3, 5, \ldots$, i.e., $j$ is an odd number. Also, we assume that $A$ and $B$ are positive and invertible.

We start with two lemmas showing that a $\beta$-power-mean can be expressed in countable sets of the operator form $(YXY)^\beta$.

**Lemma 1.** For any real number $\beta$ we have

1. $A \natural_\beta B = (AB^{-1})^{i/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{\beta+i}A^{1/2}(B^{-1}A)^{i/2}.$
2. $A \natural_\beta B = (AB^{-1})^{k+1/2}B^{1/2}(B^{1/2}A^{-1}B^{1/2})^{\beta+j}B^{1/2}(B^{-1}A)^{k+1/2}.$

**Proof.** The reversed equality in Lemma A will be used, i.e.,

$$ (X^{1/2}Y*X^{1/2})^\beta = X^{-1/2}Y^{-1}(YXY^*)^{\beta+1}(Y^*)^{-1}X^{-1/2}. \quad (**) $$

(1) Since $A \natural_\beta B = A^{1/2}(A^{-1/2}BA^{-1/2})^\beta A^{1/2}$ by (*), so it holds for $i = 0$.
Let the equality be true for $i = k = \text{an even number}$, i.e.,

$$ A \natural_\beta B = (AB^{-1})^{k/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{\beta+k}A^{1/2}(B^{-1}A)^{k/2}. $$

Then

$$ A \natural_\beta B = (AB^{-1})^{k+2}B^{1/2}(B^{1/2}A^{-1}B^{1/2})^{\beta+k+1}B^{1/2}(B^{-1}A)^{k+2} \quad \text{by } (**), $$

$$ = (AB^{-1})^{k+2}A^{1/2}(A^{-1/2}BA^{-1/2})^{\beta+k+2}A^{1/2}(B^{-1}A)^{k+2}, \quad \text{by } (**), $$

which shows that the equality holds for $i = k + 2 = \text{an even number}$..

(2) By induction again similar to (1), and the proof should be omitted. \ \Box

**Lemma 2.** For any real number $\beta$ we have:

1. $A \natural_\beta B = (BA^{-1})^{i/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{\beta-i}A^{1/2}(A^{-1}B)^{i/2}.$
2. $A \natural_\beta B = (BA^{-1})^{k-1/2}B^{1/2}(B^{1/2}A^{-1}B^{1/2})^{\beta-j}B^{1/2}(A^{-1}B)^{k-1/2}.$

**Proof.** Let us prove (2) only:

$$ A \natural_\beta B = A^{1/2}(A^{-1/2}BA^{-1/2})^\beta A^{1/2} \quad \text{by } (*) $$
\[ A_{\beta}B = (BA^{-1})^{\frac{k-1}{2}}B^{1/2}(B^{1/2}A^{-1}B^{1/2})^{\beta} B^{1/2} \] by Lemma A.

So it holds for \( j = 1 \). Let the equality hold for \( j = k \) an odd number, i.e.
\[ A_{\beta}B = (BA^{-1})^{\frac{k-1}{2}}B^{1/2}(B^{1/2}A^{-1}B^{1/2})^{\beta-k} B^{1/2}(A^{-1}B)^{\frac{k-1}{2}}. \]

Then
\[ A_{\beta}B = (BA^{-1})^{\frac{k+1}{2}}A^{1/2}(A^{-1/2}BA^{-1/2})^{\beta-k-1} A^{1/2}(A^{-1}B)^{\frac{k+1}{2}} \] by Lemma A
\[ = (BA^{-1})^{\frac{k+1}{2}}B^{1/2}(B^{1/2}A^{-1}B^{1/2})^{\beta-k-2} B^{1/2}(A^{-1}B)^{\frac{k+1}{2}} \] by Lemma A again,

and hence the equality holds for \( j = k + 2 \) an odd number.

Now, we consider the Furuta inequality \( A_{p,r}^{q} \geq (A^{r/2}B^{p}A^{r/2})^{1/q} \) in Theorem B expressed in countable sets of operator inequalities.

**Theorem 1.** Let \( A \geq B \). Then for each \( r \geq 0, p \geq 0 \) and \( q \geq 1 \) with \((1+r)q \geq p + r\) the following inequalities hold. Moreover, they are equivalent to one another.

1. \( A_{p,r}^{q} \geq (A^{r/2}B^{p}A^{r/2})^{1/q} \).
2. \( A_{p,r}^{q} \geq (A^{r/2}B^{p}A^{r/2})^{1/q} \).
3. \( A_{p,r}^{q} \geq (A^{r/2}B^{p}A^{r/2})^{1/q} \).
4. \( A_{p,r}^{q} \geq (A^{r/2}B^{p}A^{r/2})^{1/q} \).
5. \( A_{p,r}^{q} \geq (A^{r/2}B^{p}A^{r/2})^{1/q} \).

**Proof.** By (*) we rewrite the Furuta inequality in terms of the \( \beta\)-power-mean,
\[ A_{p,r}^{q} \geq (A^{r/2}B^{p}A^{r/2})^{1/q} = A^{r/2}(A^{-1}B^{1/q} B^{p})A^{r/2}. \]

(1) implies any other inequality. In fact, (2) and (3) follow by (a), (1) and (2) in Lemma 1, respectively. Similarly, use (1) and (2) in Lemma 2, respectively, we obtain inequalities (4) and (5).

(2)\( \Rightarrow \) (1). Let \( i = 0 \) in (2). Similarly (4)\( \Rightarrow \) (1).

(3)\( \Rightarrow \) (1). Let \( j = 1 \) in (3). Then
\[ A_{p,r}^{q} \geq A^{r/2}B^{p/2}(B^{p/2}A^{r/2})^{1/q+1}B^{p/2}A^{-r/2} = (A^{r/2}B^{p}A^{r/2})^{1/q}. \]
The equality is due to Lemma A.

(5)⇒(1). Let \( j = 1 \) in (5). Then

\[
A^{p/r} \geq A^{r/2}B^{p/2}(B^{p/2}A^rB^{p/2})\frac{1}{q-1}B^{p/2}A^{r/2} = (A^{r/2}B^{p}A^{r/2})^{1/q}.
\]

The equality is due to (**).

Of course, we can easily obtain four more countable sets of operator inequalities and the equivalent condition if we consider the other inequality

\[
(B^{r/2}A^pB^{r/2})^{1/q} \geq B^{\frac{p+r}{q}}
\]

in Theorem B, and we shall leave it to the reader.

Incidentally, for \( p \geq 1 \) and \( r \geq 0 \) the Furuta inequality is also given as follows in terms of the \( \beta \)-power-mean [6].

\[
A \geq A^{-r} \left( \frac{1}{\frac{p+r}{q}} \right) B^p \left( = A^{-r/2}(A^{r/2}B^pA^{r/2})^{\frac{1}{p+r}} A^{-r/2} \right).
\]

In this case we have the next result without proof.

**Corollary 1.** Let \( A \geq B \). Then for each \( r \geq 0 \) and \( p \geq 1 \) the following inequalities hold. Moreover, they are equivalent to one another.

1. \( A \geq A^{-r/2}(A^{r/2}B^pA^{r/2})^{\frac{1}{p+r}} A^{-r/2} \).
2. \( A \geq (A^{-r}B^{-p})^{i/2}A^{-r/2}(A^{r/2}B^pA^{r/2})^{\frac{1}{p+r}} A^{-r/2}(B^{-p}A^{-r})^{i/2} \).
3. \( A \geq (A^{-r}B^{-p})^{\frac{i+1}{2}}B^{p/2}(B^{p/2}A^rB^{p/2})^{\frac{1}{p+r}} A^{-r/2}(B^{-p}A^{-r})^{i+1} \).
4. \( A \geq (B^pA^r)^{i/2}A^{-r/2}(A^{r/2}B^pA^{r/2})^{\frac{1}{p+r}} A^{-r/2}(A^rB^p)^{i/2} \).
5. \( A \geq (B^pA^r)^{\frac{i-1}{2}}B^{p/2}(B^{p/2}A^rB^{p/2})^{\frac{1}{p+r}} A^{-r/2}(A^rB^p)^{\frac{i-1}{2}} \).

Countable sets of the grand Furuta inequalities are given next.

**Theorem 2.** Let \( A \geq B \). Then for \( t \in [0,1] \), \( p \geq 1 \), \( s \geq 1 \) and \( r \geq t \) the following inequalities hold. Moreover, they are equivalent to one another, where \( Z = A^{-t/2}B^pA^{-t/2} \).

1. \( A^{1-t+r} \geq \{ A^{r/2}(A^{-t/2}B^pA^{-t/2})sA^{r/2} \}^{\frac{1-t+r}{(p-t)+t+r}} \).
2. \( A^{1-t+r} \geq A^{r/2}(A^{-r}Z^{-s})^{i/2}A^{-r/2}\{ A^{r/2}(A^{-t/2}B^pA^{-t/2})sA^{r/2} \}^{\frac{1-t+r}{(p-t)+t+r}} \).
\[ A^{-r/2}(Z^{-s}A^{-r})^{i/2}A^{r/2}. \]

(3) \[ A^{1-t+r} \geq A^{r/2}(A^{-r}Z^{-s})^{(r+1)/2}Z^{s/2}A^{r/2}Z^{s/2}\{A^{r/2}B^{p}A^{-t/2}\}^{1-1/(p+1)(t+r)} \]

(4) \[ A^{1-t+r} \geq A^{r/2}(Z^{s}A^{r})^{i/2}A^{-t/2}\{A^{r/2}(A^{-t/2}B^{p}A^{-t/2})^{s}A^{r/2}\}^{1-1/(p+1)(t+r)} \]

(5) \[ A^{1-t+r} \geq A^{r/2}(Z^{s}A^{r})^{i/2}Z^{s/2}\{Z^{s/2}A^{r}Z^{s/2}\}^{1-1/(p+1)(t+r)} \]

Proof. As \( Z = A^{-t/2}B^{p}A^{-t/2} \), so (1) is by (*) the same as

(b) \[ A^{1-t+r} \geq (A^{r/2}Z^{s}A^{r/2})^{1-1/(p+1)(t+r+1)} = A^{r/2}(A^{-t/2}Z^{s}A^{r/2})^{1-1/(p+1)(t+r+1)} \]

Now, clearly inequalities (2) and (3) follow by (b), (1) and (2) in Lemma 1, respectively. While inequalities (4) and (5) follow by (b), (1) and (2) in Lemma 2, respectively.

We shall omit proofs of equivalent statement as they are similar to the proof in Theorem 1.

The next result is a characterization of the chaotic order in Theorem D in terms of countable sets of operator inequalities.

**Theorem 3.** Let \( p, r \geq 0 \). Then the following are equivalent.

(1) \( A \gg B \).

(2) \( A^{r} \geq (A^{r/2}B^{p}A^{r/2})^{1-1/(p+1)(t+r+1)} \) for \( p \geq 0 \) and \( r \geq 0 \).

(3) \( I \geq (A^{-r}B^{-p})^{i/2}A^{-r/2}(A^{r/2}B^{p}A^{r/2})^{1-1/(p+1)(t+r+1)}A^{-r/2}(B^{-p}A^{-r})^{i/2} \).

(4) \( I \geq (A^{-r}B^{-p})^{i/(r+2)}B^{p/2}(B^{r/2}A^{r}B^{p/2})^{1-1/(p+1)(t+r+1)}B^{p/2}(B^{-p}A^{-r})^{i/(r+2)} \).

(5) \( I \geq (B^{p}A^{r})^{i/2}A^{-r/2}(A^{r/2}B^{p}A^{r/2})^{1-1/(p+1)(t+r+1)}A^{-r/2}(B^{-p}A^{-r})^{i/2} \).

(6) \( I \geq (B^{p}A^{r})^{i/(r+2)}B^{p/2}(B^{r/2}A^{r}B^{p/2})^{1-1/(p+1)(t+r+1)}B^{p/2}(A^{-r}B^{p})^{i/(r+2)} \).

Proof. By Theorem D, (1) if and only if (2). That \( I \geq A^{-r} B^{p} \) is due to (2) and (*). With this inequality in mind all we have to do is applying (1) and (2) in Lemma 1 to get inequalities (3) and (4), and (1) and (2) in Lemma 2 to have inequalities (5) and (6).

We omit the proofs that each inequality (3) through (6) implies (2).
An application of Theorem E is as follows.

**Theorem 4.** Let \( A \geq B, t \in [0, 1], \) and \( p_1, p_2, \ldots, p_{2n} \geq 1 \) for a natural number \( n \). Then the following inequalities hold for \( r \geq t \) with

\[
q = \frac{1-t+r}{\cdots\cdots((p_1-t)p_2+t)p_3-t\cdots t)p_{n-1}-t)p_{2n}+r}.
\]

Moreover, they are equivalent to one another, where

\[
Z = [A^{-t/2} A^{t/2} \cdots A^{t/2}(A^{-t/2} B^{p_1} A^{-t/2})^{p_2} A^{t/2}]^{p_3} \cdots A^{t/2}]^{p_{2n-1}} A^{-t/2}]^{p_{2n}}.
\]

1. \( A^{1-t+r} \geq \{A^{r/2}[A^{-t/2} A^{t/2} \cdots A^{t/2}(A^{-t/2} B^{p_1} A^{-t/2})^{p_2} A^{t/2}]^{p_3} A^{-t/2}]^{p_4} A^{t/2}]^{p_5} \cdots A^{-t/2}]^{p_{2n-2}} A^{t/2}Q_{p_{2n-1}} A^{-t/2}]^{p_{2n}} A^{r/2}\}
\]

(i.e., \( A^{1-t+r} \geq (A^{r/2}Z^{p_{2n}} A^{r/2})q\) in short).

2. \( A^{1-t+r} \geq A^{r/2}(A^r Z^{-p_{2n}})q/i A^{-r/2}(A^{r/2} Z^{p_{2n}} A^{r/2})q+i A^{-r/2}(Z^{-p_{2n}} A^{-r})q+i/2 A^{r/2}\)

3. \( A^{1-t+r} \geq A^{r/2}(A^r Z^{-p_{2n}})^{q+i} Z^{p_{2n}} Z^{p_{2n}} A^{-1/2} Z^{p_{2n}} A^{r/2} A^{r/2}\)

4. \( A^{1-t+r} \geq A^{r/2}(Z^{p_{2n}} A^{r})^{q+i} A^{-r/2}(Z^{p_{2n}} A^{-r})^{q+i/2} A^{r/2}\)

5. \( A^{1-t+r} \geq A^{r/2}(Z^{p_{2n}} A^{r})^{q+i} Z^{p_{2n}} Z^{p_{2n}} A^{-1/2} Z^{p_{2n}} A^{r/2} A^{r/2}\)

**Proof.** (1) is by (*) the same as

(c) \( A^{1-t+r} \geq (A^{r/2} Z^{p_{2n}} A^{r/2})q = A^{r/2}(A^{-r} q) Z^{p_{2n}} A^{r/2}\). Now,

(1) \(\Rightarrow (2)\). By (c) and (1) in Lemma 1.

(1) \(\Rightarrow (3)\). By (c) and (2) in Lemma 1.

(1) \(\Rightarrow (4)\). By (c) and (1) in Lemma 2.

(1) \(\Rightarrow (5)\). By (c) and (2) in Lemma 2.

We shall omit the proof of being equivalent to one another.
4. The Operator Form $(YXY)^\beta$ and Countable Sets of the $\beta$-Power-Means

Let us list corresponding results to Lemmas 1 and 2, which show that the operator form $(YXY)^\beta$ can be expressed in countable sets of $\beta$-power-means. The proofs are quite similar to that of Lemmas 1 and 2, and we shall leave them to the reader.

Lemma 3. For any real number $\beta$ we have

1. $(BAB)^\beta = B^{-1}(A^{-1}B^{-2})^{\frac{i-2}{2}}A^{-1}(B^{-2}A^{-1})^{\frac{i-2}{2}}B^{-1}$.
2. $(BAB)^\beta = B^{-1}(A^{-1}B^{-2})^{\frac{i-1}{2}}(A^{-1}B^{-1})^{\frac{i-1}{2}}B^{-1}$.

Lemma 4. For any real number $\beta$ we have

1. $(BAB)^\beta = B(AB^2)^{\frac{i}{2}}(B^{-2}A^{-1})^{\frac{i}{2}}B$.
2. $(BAB)^\beta = B(AB^2)^{\frac{i-1}{2}}A(A^{-1}B^{-1})^{\frac{i-1}{2}}B$.

Instead of countable sets of operator inequalities in the operator form of $(YXY)^\beta$ in Section 3 we could have countable sets of operator inequalities in the form of the $\beta$-power-means. Let us first take the Furuta inequality $A^{\frac{p+r}{q}} \geq (A^{r/2}B^pA^{r/2})^{1/q}$ in Theorem B for example. By Lemmas 3 and 4 the proof of the next result is straightforward and should be omitted.

Theorem 5. Let $A \geq B$. Then for each $r \geq 0$, $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p + r$ the following inequalities hold. Moreover, they are equivalent to one another.

1. $A^{\frac{p+r}{q}} \geq (A^{r/2}B^pA^{r/2})^{1/q}$.
2. $A^{\frac{p+r}{q}} \geq A^{-r/2}(B^{-p}A^{-r})^{\frac{i-2}{2}}B^{-p}(A^{-r}B^{-p})^{\frac{i-2}{2}}A^{-r/2}$.
3. $A^{\frac{p+r}{q}} \geq A^{-r/2}(B^{-p}A^{-r})^{\frac{i-1}{2}}(B^{-p}A^{-r})^{\frac{i-1}{2}}A^{-r/2}$.
4. $A^{\frac{p+r}{q}} \geq A^{r/2}(B^pA^{r})^{\frac{i}{2}}(A^{-r}B^{-p})^{\frac{i}{2}}(A^{r}B^{p})^{\frac{i}{2}}A^{-r/2}$.
5. $A^{\frac{p+r}{q}} \geq A^{r/2}(B^pA^{r})^{\frac{i}{2}}B^{-p}(A^{-r}B^{-p})^{\frac{i}{2}}A^{r/2}$.

Finally, from Theorem E we have
Theorem 6. Let $A \geq B$, $t \in [0, 1]$, and $p_1, p_2, \ldots, p_{2n} \geq 1$ for a natural number $n$. Then the following inequalities hold for $r \geq t$ with

$$q = \frac{1 - t + r}{\cdots(((p_1 - t)p_2 + t)p_3 - t)\cdots(p_{2n} - t)p_{2n} + r}.$$

Moreover, they are equivalent to one another, where

$$Z = [A^{-t/2} A^{t/2} \cdots [A^{t/2}(A^{-t/2} B p_1 A^{-t/2}) p_2 A^{t/2} p_3 \cdots A^{t/2} p_{2n-1} A^{-t/2}] p_{2n}]$$

(1) $A^{1-t+r} \geq (A^{r/2} A^{t/2})^{2n} (t q_{q+i} Z p_{2n} Z^{-p_{2n}} A^{-r} Z^{-p_{2n}})^{t-r/2}$

(2) $A^{1-t+r} \geq (A^{r/2} Z p_{2n} A^{-r}) \frac{t-r}{2} Z^{-p_{2n}} (A^{-r} Z^{-p_{2n}})^{t-r/2}$

(3) $A^{1-t+r} \geq A^{-r/2} (Z^{-p_{2n}} A^{-r}) \frac{t-r}{2} Z^{-p_{2n}} (A^{-r} Z^{-p_{2n}})^{t-r/2}$

(4) $A^{1-t+r} \geq A^{-r/2} \frac{t-r}{2} Z^{-p_{2n}} (A^{-r} Z^{-p_{2n}})^{t-r/2}$

(5) $A^{1-t+r} \geq A^{-r/2} \frac{t-r}{2} Z^{-p_{2n}} (A^{-r} Z^{-p_{2n}})^{t-r/2}$

In conclusion we remark that there are many other Furuta-type operator inequalities appeared in the literature, and for each inequality we could express it, as we have done in above, in countable sets of operator inequalities in both forms: $(XY)^{\beta}$ and $\beta$-power-means, and equivalent relationships.

References

[1] T. Furuta, $A \geq B \geq O$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc., 101 (1987), 85-88.


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