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PERTURBATION THEORY FOR DISCRETE VOLTERRA EQUATION

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Abstract: In this paper oscillatory and stability problems for some Volterra nonlinear difference equations are investigated.

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1. Introduction

There are two qualitative properties which have wide applications: the oscillation and the stability of solutions. It is natural to expect the solution in an explicit form but unfortunately it is not always possible.

During the past several years the oscillation and the stability of solutions of difference equations have been extensively studied (see for example [1]-[7] and the references cited therein).

In this paper we establish sufficient conditions for the oscillation of solutions of Volterra equation

$$(E_1) \qquad y(n) = p(n) + \sum_{s=0}^{n-1} L(n,s) f(s, y(s)) y(s), \quad n \in N_0,$$

and the stability of Volterra equation

$$(E_2) \qquad x(n) = q(n) + \sum_{s=0}^{n-1} K(n,s,x(s)) x(s), \quad n \in N_0,$$

where $N_0 = \{0, 1, 2, \ldots\}$.

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Further we obtain growth estimates on solutions of equations (E_1) and (E_2).

2. Assumptions

The following positive assumptions are made about functions in equations (E_1) and (E_2):

(i) $\{p(n)\}$ is a sequence of real numbers for $n \in N_0$.

(ii) $L : N_0 \times N_0 \to R^+$ and $L(n, s) = 0$ if $n < s$.

(iii) $f : N_0 \times R \to R$ is continuous and $\varphi f(n, \varphi) > 0$ for $\varphi \neq 0$.

(iv) $x(n)$, $q(n)$ are given real vectors in E^m - the m-dimensional real Euclidean space.

Let $|.|$ denotes a norm of vectors in E^m and the corresponding subordinate norm of a matrix acting on the corresponding Banach space E^m of all sequences $\{\varphi(n)\}$, $n \in N_0$, $\varphi(n) \in E^m$ with the norm $||\varphi|| = \sup_{n \in N_0} |\varphi(n)| < \infty$.

(v) $K(n, s, x(s))$ is for each $n \geq s \geq 0$ as $m \times m$ matrix defined for all $x \in S(b) = \{x \in E^m : ||x|| = \sup_{n \in N_0} |x(n)| \leq b\}$ for some $b > 0$ which is sufficiently smooth so that solutions of (E_2) exist. $K(n, s, x)$ is continuous in the third argument and $K(n, s, x) = 0$ for $n < s$.

(vi) In Banach space E^m equation (E_2) has a unique solution $x = \{x(n)\}$, $n \in N_0$.

By a solution of equation (E_1) ((E_2)) we mean a real sequence $\{y(n)\}$ ($\{x(n)\}$) satisfying equation (E_1) ((E_2)) for all $n \in N_0$.

3. Oscillation of Equation (E_1)

We begin with the following lemma given in paper [4].

Lemma 1. (see [4]) *Suppose that $\{y(n)\}$, $\{p(n)\}$ are nonnegative sequences defined on $N_{n_0} = \{n_0, n_0 + 1, \ldots\}$, $n_0 \in N_0$, $L(n, s) = L_1(n)L_2(s)$ is nonnegative on $N_{n_0} \times N_{n_0}$. If*

$$y(n) \leq p(n) + \sum_{s=n_0}^{n-1} L(n, s)y(s),$$

then

$$y(n) \le p(n) + L_1(n) \sum_{s=n_0}^{n-1} p(s)L_2(s) \prod_{l=s+1}^{n-1} (1 + L_1(l)L_2(l))$$

or

$$y(n) \le p(n) + L_1(n) \sum_{s=n_0}^{n-1} p(s)L_2(s)e^{\sum\limits_{l=s+1}^{n-1} L_1(l)L_2(l)}.$$

The following theorem will be used in the sequel.

Theorem 1. *Assume that:*

1° $\{y(n)\}$, $\{p(n)\}$ *are nonnegative sequences defined on* N_{n_0},

2° $L(n,s)$ *is nonnegative on* $N_{n_0} \times N_{n_0}$, $L(n,s) \le L_1(n)L_2(s)$ *for* $n \ge s \ge n_0$,

3° $1 - L_1(n)L_2(n)p(n) > 0$ *for all* $n \in N_{n_0}$,

4° $1 - \sum\limits_{s=n_0}^{n-1} L_1(s)L_2(s)H(s) \prod\limits_{l=s+1}^{n-1} (1 - L_1(l)L_2(l)p(l)) > 0$ *for all* $n \in N_{n_0}$.

If

$$y(n) \le p(n) + L_1(n) \sum_{s=n_0}^{n-1} L_2(s)y^2(s), \tag{1}$$

then

$$y(n) \le \frac{H(n)}{1 - \sum\limits_{s=n_0}^{n-1} L_1(s)L_2(s)H(s) \prod\limits_{l=s+1}^{n-1} (1 - L_1(l)L_2(l)p(l))}, \tag{2}$$

where

$$H(n) = p(n) + L_1(n) \sum_{s=n_0}^{n-1} p^2(s)L_2(s) \prod_{l=s+1}^{n-1} (1 + L_1(l)L_2(l)p(l)).$$

Proof. From (1) and Lemma 1 we obtain

$$y(n) \le p(n) + L_1(n) \sum_{s=n_0}^{n-1} p(s)L_2(s)y(s) \prod_{l=s+1}^{n-1} (1 + L_1(l)L_2(l)y(l)) \tag{3}$$

for $n \in N_0$. By setting

$$u(n) = y(n)\Big(\prod_{s=n_0}^{n-1} (1 + L_1(s)L_2(s)y(s))\Big)^{-1}, \tag{4}$$

we obtain from (3) successively

$$
\begin{aligned}
& y(n)\Big(\prod_{s=n_0}^{n-1}(1+L_1(s)L_2(s)y(s))\Big)^{-1} \le p(n)\Big(\prod_{s=n_0}^{n-1}(1+L_1(s)L_2(s)y(s))\Big)^{-1} \\
+\quad & L_1(n)\Big(\prod_{s=n_0}^{n-1}(1+L_1(s)L_2(s)y(s))\Big)^{-1} \\
\times\quad & \sum_{s=n_0}^{n-1} p(s)L_2(s)y(s)\prod_{l=s+1}^{n-1}(1+L_1(l)L_2(l)y(l)).
\end{aligned}
$$

Hence

$$
u(n) \le p(n)\Big(\prod_{s=n_0}^{n-1}(1+L_1(s)L_2(s)y(s))\Big)^{-1} + L_1(n)\sum_{s=n_0}^{n-1} p(s)L_2(s)u(s).
$$

In view of Lemma 1 we get

$$
\begin{aligned}
& u(n) \le p(n)\Big(\prod_{s=n_0}^{n-1}(1+L_1(s)L_2(s)y(s))\Big)^{-1} \\
+\quad & L_1(n)\sum_{s=n_0}^{n-1} p(s)\Big(\prod_{l=n_0}^{s-1}(1+L_1(l)L_2(l)y(l))\Big)^{-1} p(s)L_2(s) \\
\times\quad & \prod_{l=s+1}^{n-1}(1+L_1(l)L_2(l)p(l)).
\end{aligned} \tag{5}
$$

Let $v(n) = 1 - \Big(\prod_{s=n_0}^{n-1}(1+L_1(s)L_2(s)y(s))\Big)^{-1}$, then

$$
\Delta v(n) = \frac{L_1(n)L_2(n)y(n)}{1+L_1(n)L_2(n)y(n)}\Big(\prod_{s=n_0}^{n-1}(1+L_1(s)L_2(s)y(s))\Big)^{-1}.
$$

From (4)

$$
L_1(n)L_2(n)u(n) = L_1(n)L_2(n)y(n)\Big(\prod_{s=n_0}^{n-1}(1+L_1(s)L_2(s)y(s))\Big)^{-1},
$$

so

$$
(1+L_1(n)L_2(n)y(n))\Delta v(n) = L_1(n)L_2(n)u(n).
$$

Hence

$$\Delta v(n) \leq L_1(n)L_2(n)u(n) \leq L_1(n)L_2(n)p(n)(1-v(n))$$
$$+ \quad L_1^2(n)L_2(n)\sum_{s=n_0}^{n-1} p^2(s)L_2(s)\prod_{l=s+1}^{n-1}(1+L_1(l)L_2(l)p(l)).$$

Therefore

$$\Delta v(n) + L_1(n)L_2(n)p(n)v(n) \leq L_1(n)L_2(n)H(n).$$

This is equivalent to

$$\Delta v(n) + L_1(n)L_2(n)p(n)v(n) = L_1(n)L_2(n)H(n) + w(n),$$

where $w(n) \leq 0$ for $n \in N_{n_0}$, the solution of which with the initial conditions $v(n_0) = 0$ ($\prod_{k=n}^{n-1} a_k := 1$) is

$$v(n) = \sum_{s=n_0}^{n-1}(L_1(s)L_2(s)H(s) + w(s))\prod_{l=s+1}^{n-1}(1 - L_1(l)L_2(l)p(l)),$$

i.e.

$$v(n) \leq \sum_{s=n_0}^{n-1}(L_1(s)L_2(s)H(s))\prod_{l=s+1}^{n-1}(1 - L_1(l)L_2(l)p(l)).$$

From (5) we have

$$u(n) \leq H(n)$$

and

$$y(n) \leq \frac{u(n)}{1-v(n)}.$$

Hence we obtain (2). □

Remark 1. If

1° $\overline{\lim_{n\to\infty}} p(n) < \infty$,

2° $\overline{\lim_{n\to\infty}} L_1(n) < \infty$,

3° $\overline{\lim_{n\to\infty}} \sum_{s=n_0}^{n-1} p^2(s)L_2(s)\prod_{l=s+1}^{n-1}(1+L_1(l)L_2(l)p(l)) < \infty$,

4° $\overline{\lim_{n\to\infty}} \Big\{1 - \sum_{s=n_0}^{n-1} L_1(s)L_2(s)H(s)\prod_{l=s+1}^{n-1}(1 - L_1(l)L_2(l)p(l))\Big\}^{-1} < \infty$, then all $\{y(n)\}$ are bounded as $n \to \infty$.

Let us first study the system (E_1).

Definition 1. A solution $\{y(n)\}$ of equation (E_1) is said to be oscillatory (around zero) if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

In other words, a solution $\{y(n)\}$ is oscillatory if for every positive integer n_1 there exists $n \geq n_1$ such that $y(n)y(n+1) \leq 0$. Otherwise, the solution is said to be nonoscillatory.

Theorem 2. *Suppose the following conditions hold:*

1° $\frac{f(n,u)}{u} \leq M$ *for some* $M > 0$, *all* $n \in N_{n_0}$ *and* $u \neq 0$,

2° *there exist positive real sequences* $\{L_1(n)\}$, $\{L_2(n)\}$, $n \in N_{n_0}$ *such that* $L(n,s) \leq L_1(n)L_2(s)$ *for* $n \geq s \geq n_0$,

3° *sequences* $\{p(n)\}$, $\{L_1(n)\}$ *are bounded for* $n \in N_{n_0}$, *i.e. there exist constants* $K > 0$ *and* $K_1 > 0$ *such that* $|p(n)| \leq K$, $L_1(n) \leq K_1$ *for all* $n \in N_{n_0}$,

4° $\lim_{n\to\infty} H(n)\Big\{1 - \sum_{s=n_1}^{n-1} K_1 M L_2(s) H(s) \prod_{l=s+1}^{n-1} (1 - K_1 C M L_2(l))\Big\}^{-1} < \infty$, *where*

$$1 - K_1 C M L_2(n) > 0, \quad n \geq n_1 \geq 0,$$

$$1 - \sum_{s=n_1}^{n-1} K_1 M L_2(s) H(s) \prod_{l=s+1}^{n-1} (1 - K_1 C M L_2(l)) > 0, \quad n \geq n_1 \geq 0,$$

C *- some positive const.,*

$$H(n) = C + K_1 M^2 C^2 \sum_{s=n_1}^{n-1} L_2(s) \prod_{l=s+1}^{n-1} (1 + K_1 M C L_2(l)).$$

Then all unbounded solution of (E_1) *are oscillatory.*

Proof. Suppose there is an unbounded nonoscillatory solution $\{y(n)\}$ of (E_1). Let $y(n) > 0$ for $n \geq n_1$, $n_1 \in N_{n_0}$, n_1 – large enough.

From equation (E_1) we have

$$|y(n)| \quad \leq \quad |p(n)| + L_1(n) \sum_{s=n_0}^{n_1-1} L_2(s) |f(s, y(s))| |y(s)|$$

$$+ \quad L_1(n) \sum_{s=n_1}^{n-1} L_2(s)|f(s, y(s))||y(s)|.$$

Now, from the assumptions 1°, 3° we have

$$y(n) \leq C + K_1 M \sum_{s=n_1}^{n-1} L_2(s)|y(s)|^2,$$

where $C = K + K_1 M_1$, $M_1 = \sum_{s=n_0}^{n_1-1} L_2(s)|f(s, y(s))||y(s)|$.

Applying Theorem 1 to the last inequality we obtain

$$y(n) \leq \frac{H(n)}{1 - \sum_{s=n_1}^{n-1} K_1 M L_2(s) H(s) \prod_{l=s+1}^{n-1} (1 - K_1 C M L_2(l))}.$$

As $n \to \infty$ condition 4° implies that $\{y(n)\}$ is bounded. This contradiction completes the proof of the theorem. □

Remark 2. Suppose that the conditions of Theorem 2 are satisfied. Then all nonoscillatory solutions of (E_1) are bounded.

Example. The Volterra summation equation

$$y(n) = \frac{n+1}{2^n} + \frac{1}{2^{2n}} - 1 + \frac{3}{4^l} \sum_{s=0}^{n-1} \frac{1}{(s+1)^2} y'(s) y(s),$$

where $l \in N$, l is large enough, satisfies all assumptions of Theorem 2. Hence all nonoscillatory solutions of the equation are bounded. In particular,

$$y(n) = \frac{n+1}{2^n}$$

is a bounded nonoscillatory solution of the equation.

Theorem 3. *Suppose the following conditions hold:*

1° $\frac{f(n,u)}{u} \leq M$ *for some* $M > 0$, *all* $n \in N_{n_0}$ *and* $u \neq 0$,

2° *there exist positive real sequences* $\{L_1(n)\}$, $\{L_2(n)\}$, $n \in N_{n_0}$ *such that* $L(n, s) \leq L_1(n) L_2(s)$ *for* $n \geq s \geq n_0$,

3° $\frac{p(n)}{n^\alpha}$ *and* $L_1(n)$ *are bounded for all* $n \in N_{n_0}$, $\alpha \geq 1$, 4° $\sum\limits_{n=n_0}^{\infty} L_1(n)L_2(n)|p(n)| < \infty$,

5° $\sum\limits_{n=n_0}^{\infty} L_2(n)\frac{|p(n)|^2}{n^\alpha} < \infty$,

6° $\sum\limits_{n=n_0}^{\infty} L_1(n)L_2(n)n^\alpha < \infty$,

7° $1 - \sum\limits_{s=n_0}^{n-1} ML_1(s)L_2(s)s^\alpha H(s) \prod\limits_{l=s+1}^{n-1} (1 - ML_1(l)L_2(l)|p(l)|) > 0$,
$1 - ML_1(s)L_2(s)|p(s)| > 0$ *for all* $n_0 \le s \le n < \infty$.

If $\{y(n)\}$ *is any solution of equation* (E_1) *then* $y(n) = O(n^\alpha)$, *that is* $\limsup\limits_{n\to\infty}(\frac{y(n)}{n^\alpha}) < \infty$.

Proof. From equation (E_1) and assumptions we have

$$\frac{|y(n)|}{n^\alpha} \le \frac{|p(n)|}{n^\alpha} + L_1(n)\sum_{s=n_0}^{n-1} L_2(s)s^\alpha \frac{|f(s,y(s))|}{|y(s)|}\Big(\frac{|y(s)|}{s^\alpha}\Big)^2.$$

Applying Theorem 1 to this inequality, we obtain

$$\frac{|y(n)|}{n^\alpha} \le \frac{H_1(n)}{1 - \sum\limits_{s=n_0}^{n-1} ML_1(s)L_2(s)s^\alpha H_1(s) \prod\limits_{l=s+1}^{n-1}(1 - ML_1(l)L_2(l)|p(l)|)},$$

where

$$H_1(n) = \frac{|p(n)|}{n^\alpha} + ML_1(n)\sum_{s=n_0}^{n-1} \frac{|p(s)|^2}{s^\alpha} L_2(s) \prod_{l=s+1}^{n-1} (1 + ML_1(l)L_2(l)|p(l)|).$$

As $n \to \infty$, assumptions of Theorem imply that

$$\limsup_{n\to\infty}\Big(\frac{y(n)}{n^\alpha}\Big) < \infty.$$

This completes the proof of Theorem 3.

Theorem 4. *Suppose the conditions* $1° - 5°$ *and* $7°$ *of Theorem 3 hold. In addition assume*

8° $\sum\limits_{n=n_0}^{\infty} L_2(n)n^{2\alpha} < \infty$, 9° $\limsup_{n\to\infty}(p(n)) = +\infty$, $\liminf_{n\to\infty}(p(n)) = -\infty$.

Then all solutions of equation (E_1) are oscillatory.

Proof. Condition 8° implies condition 6° of Theorem 3. Thus the conclusion of Theorem 3 holds. Without loss of generality, suppose that $n_1 \in N_{n_0}$ is large enough so that $y(n) > 0$ for $n \geq n_1$.

From equation (E_1) and assumptions of Theorem we have

$$\begin{aligned} y(n) &\leq p(n) + \sum_{s=n_0}^{n_1-1} L_1(n)L_2(s)|f(s,y(s))||y(s)| \\ &+ \sum_{s=n_1}^{n-1} L_1(n)L_2(s)s^{2\alpha}\frac{f(s,y(s))}{y(s)}\Big(\frac{y(s)}{s^\alpha}\Big)^2. \end{aligned} \tag{6}$$

Now $\{L_1(n)\}$ is bounded and, by Theorem 3, $\{\frac{y(n)}{n^\alpha}\}$ is bounded. Consequently the last two summations on the right-hand side of (6) are finite. Since $y(n) > 0$ for all $n \geq n_1$ and 9° holds, we obtain the contradiction. This completes the proof. □

4. Stability Criteria of System (E_2)

In this section we deal with stability solutions of the system (E_2).

We observe that equation (E_2) can be expressed in the form

$$(E_3) \qquad x(n) = q(n) + \sum_{s=0}^{n-1} K(n,s)x(s)$$

as $K(n,s,x(s))$ given in equation (E_2) independent of x.

As far as we know such equations have been relatively widely discussed in the literature.

Let

$$\begin{aligned} K^{(1)}(n,s,\varphi(s)) &= K(n,s,\varphi(s)), \\ K^{(r)}(n,s,\varphi(s)) &= \sum_{l=s+1}^{n-1} K^{(1)}(n,l,\varphi(l))K^{(r-1)}(l,s,\varphi(s)) \\ &= \sum_{l=s+1}^{n-1} K^{(r-1)}(n,l,\varphi(l))K^{(1)}(l,s,\varphi(s)), \end{aligned}$$

($\varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_m(s))$, $\varphi_1(s), \ldots, \varphi_m(s)$ – generally unknown functions).

We define the resolvent matrices $\{R(n,s,\varphi)\}$ (where $R(n,s,\varphi(s)) = \sum_{r=1}^{\infty} K^{(r)}(n,s,\varphi(s)))$ of the kernel K defined by $\{K(n,s,\varphi)\}$ in equation (E_2) as the solution of the following matrix equation

$$\begin{aligned} R(n,s,\varphi(s)) &= K(n,s,\varphi(s)) + \sum_{l=s+1}^{n-1} K(n,l,\varphi(l))R(l,s,\varphi(s)) \\ &= K(n,s,\varphi(s)) + \sum_{l=s+1}^{n-1} R(n,l,\varphi(l))K(l,s,\varphi(s)) \end{aligned}$$

($R(n,s,\varphi) = 0$ for $n < s$).

The resolvent matrices allow us to write down an expression for the solution of the equation (E_2) in the form

$$x(n) = q(n) + \sum_{s=0}^{n-1} R(n,s,x(s))q(s), \quad n \in N_0. \tag{7}$$

We need the following definitions.

Definition 2. The null solution $x = \{x(n)\} = 0$ of equation (E_2) (corresponding to $q = \{q(n)\} = 0$) is called stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $||q|| < \delta$ implies equation (E_2) has a unique solution x with $||x|| < \varepsilon$.

Definition 3. The null solution $x = \{x(n)\} = 0$ of equation (E_2) (corresponding to $q = 0$) is asymptotically stable if for any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exist $\delta_1(\varepsilon_1) > 0$, $\delta_2(\varepsilon_1) > 0$, $\delta_3(\varepsilon_2) > 0$ and $T = T(\varepsilon_2) \geq n_0 \geq 0$ such that $||\varphi|| = ||\varphi(0)|| \leq \delta_1$, $||\Delta\varphi|| \leq \delta_2$ for $0 \leq n_0 \leq n < \infty$, $||\Delta\varphi|| \leq \delta_3$ for $T \leq n < \infty$ implies $||x|| \leq \varepsilon_1$ for $0 \leq n_0 \leq n < \infty$, $||x|| \leq \varepsilon_2$ for $T \leq n < \infty$.

Definition 4. The null solution $x = \{x(n)\} = 0$ of equation (E_2) (corresponding to $q = \{q(n)\} = 0$) is called uniformly stable if there exist $B > 0$, $\eta > 0$ such that for all $\{x(n)\}$, $||x|| < \eta$, $\sup_{n \geq 0} \sum_{s=0}^{n-1} |R(n,s,x(s))| \leq B$.

Lemma 2. *Suppose that there exists a constant $C > 0$ such that*

$$\sup_{n \in N_0} \sum_{s=0}^{n-1} |R(n,s,x(s))| = C < \infty$$

for all $x \in R^m$.

Then for any $\varepsilon > 0$ *and* $q = \{q(n)\}$, $||q|| \leq \varepsilon(1+C)^{-1}$ *the solution* $x = \{x(n)\}$ *of equation* (E_2) *satisfies* $||x|| < \varepsilon$.

Proof. If $||q|| \leq \varepsilon(1+C)^{-1}$ then by (7)

$$|x(n)| \leq |q(n)| + \sum_{s=0}^{n-1} |R(n,s,x(s))||q(s)|, \quad n \in N_0,$$

hence $||x|| \leq \varepsilon$ and this completes the proof. □

Theorem 5. *If* $P(n,s,x)$ *is an* $m \times m$ *matrix which satisfies the conditions*

1° $P(n,s,x(s)) = E + \sum\limits_{l=s}^{n-1} R(n,l,x(l))$, $x \in S(b)$, E *- identity matrix,*

2° $\lim\limits_{n\to\infty} P(n,0,x(0)) = 0$, $x \in S(b)$,

3° $\lim\limits_{n\to\infty} \sum\limits_{s=0}^{n_1-1} |P(n,s,x(s))| = 0$, $n_1 \in N$, $x \in S(b)$,

4° $\sum\limits_{s=0}^{n-1} |P(n,s,x(s))| \leq M$ *for* $n \in N$, $x \in S(b)$,

and

$$\lim_{n\to\infty} \Delta q(n) = 0,$$

then the solution of the equation (E_2) *is asymptotically stable.*

Proof. Let $q(n) = q(0) + \sum\limits_{s=0}^{n-1} \Delta q(s)$. In the case 1° it can easily be seen from (7) that

$$x(n) = P(n,0,x(0))q(0) + \sum_{s=0}^{n-1} P(n,s,x(s))\Delta q(s) \tag{8}$$

for $x \in S(b)$. Using the assumptions of the theorem and (8) we complete the proof of the theorem. □

Theorem 6. *Let* $P(n,s,x)$ *be as in Theorem 5 and* $q(n) \equiv const$ *for* $n \geq n_0 \geq 0$.

1° *The null solution of system* (E_2) *is stable in* E^m *if there exists a constant* $M(n_0) > 0$ *such that* $||P(n,n_0,x(n_0))|| \leq M$ *for* $n \geq n_0 \geq 0$, $x \in S(b)$.

2° *The null solution of system* (E_2) *is uniformly stable in* E^m *if there exists a constant* $M > 0$, *independent of* n_0 *such that* $||P(n, n_0, x(n_0))|| \leq M$, $n \geq n_0 \geq 0$, $x \in S(b)$.

3° *The null solution of system* (E_2) *is asymptotically stable in* E^m *if*

$$\lim_{n\to\infty} P(n, n_0, x(n_0)) = 0 \text{ for } x \in S(b).$$

Proof. According (8) the solution of (E_2) for $q \in E^m$, $q(n) = const.$, $n \in N_0$ is given by $x(n) = P(n, 0, x(0))q(0)$. Then the stability, the asymptotic stability and the uniformly stability follow immediately. □

Theorem 7. *Suppose that:*

1° $\lim q(n) = 0$ *as* $n \to \infty$,

2° *there is a constant* $C_1 > 0$ *such that* $|P(n, 0, x(0))| \leq C_1$ *for all* $n \in N_0$ *and* $x \in S(b)$,

3° *there is a constant* C_2 *such that* $\sup_{n\in N_0} \sum_{s=0}^{n-1} |R(n, s, x(s))| = C_2 < \infty$,

4° $\lim_{n\to\infty} R(n, s, x(s)) = 0$ *for each* $s \geq 0$ *and* $x \in S(b)$.

Then the solution $\{x(n)\}_{n\in N_0}$ *of* (E_2) *satisfies* $\lim x(n) = 0$ *as* $n \to \infty$.

Proof. For any $\{q(n)\}_{n\in N_0}$ from (7) we have

$$x(n) = P(n, 0, x(0))q(n) - \sum_{s=0}^{n-1} R(n, s, x(s))(q(n) - q(s)).$$

For any $\varepsilon > 0$ there exists $n_1 > 0$ such that the following inequalities $|q(n)| < \frac{\varepsilon}{3C_1}$, $|q(n) - q(s)| < \frac{\varepsilon}{3C_2}$ hold for $n > s > n_1$.

Now for this fixed $n_1 > 0$ it follows from the assumptions 3° and 4° that there exists $n_2 > 0$ such that

$$\sum_{s=0}^{n_1-1} |R(n, s, x(s))||q(n) - q(s)| < \frac{\varepsilon}{3}$$

for $n > n_2$.

Let $n_3 = \max(n_1, n_2)$. If $n > n_3$ then

$$|x(n)| \leq |P(n, 0, x(0))||q(n)| + \sum_{s=0}^{n_1-1} |R(n, s, x(s))||q(n) - q(s)|$$

$$+ \sum_{s=n_1}^{n-1} |R(n,s,x(s))||q(n)-q(s)| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The proof of the theorem is complete. □

Corollary 1. *Assume the following conditions are satisfied:*

1° *there exists a function y which satisfies*

$$\lim_{n\to\infty} \{q(n) - (y(n) - \sum_{s=0}^{n-1} K(n,s,x(s))y(s))\} = 0$$

for $x \in S(b)$,

2° $\sup_{n\ge 0} \sum_{s=0}^{n-1} |R(n,s,x(s))| = C < \infty$ *for $x \in S(b)$,*

3° $\lim_{n\to\infty} |R(n,s,x(s))| = 0$ *for each fixed $s \ge 0$, $x \in S(b)$.*

Then the function $z(n) = x(n) - y(n)$ where $\{x(n)\}$ is the solution of (E_2) satisfies

$$\lim_{n\to\infty} z(n) = 0.$$

Proof. We observe that equation (E_2) can be expressed in the form

$$z(n) = w(n) + \sum_{s=0}^{n-1} K(n,s,x(s))z(s), \tag{9}$$

where

$$w(n) = q(n) - (y(n) - \sum_{s=0}^{n-1} K(n,s,x(s))y(s)).$$

Using the relation (7) to (9) we obtain

$$z(n) = w(n) + \sum_{s=0}^{n-1} R(n,s,x(s))w(s).$$

Now see the proof of Theorem 7. □

Theorem 8. *Assume that the following conditions are valid:*

1° $|K(n,s,x)| \le L(n,s)$ *for $0 \le s \le n < \infty$, $x \in S(\eta)$, η - small enough,* 2° $\sum_{s=0}^{n-1} |\overline{R}(n,s)| \leq M$, *where* $\{\overline{R}(n,s)\}$ *are the resolvent matrices of the kernel* $L(n,s)$ *defined by* $\{L(n,s)\}$ *in equation*

$$x(n) = q(n) + \sum_{s=0}^{n-1} L(n,s)x(s), \quad n \geq 0,$$

as the solution of the following matrix equation

$$\overline{R}(n,s) = L(n,s) + \sum_{l=s+1}^{n-1} L(n,l)\overline{R}(l,s)$$

or

$$\overline{R}(n,s) = L(n,s) + \sum_{l=s+1}^{n-1} \overline{R}(n,l)L(l,s).$$

Then for any $\varepsilon > 0$ *and* $\{q(n)\}$, $||q|| < \delta$ *the unique solution* $x = \{x(n)\}$ *of* (E_2) *satisfies* $||x|| < \varepsilon$.

Proof. For $\varepsilon > 0$ let $\delta = \min(\frac{\eta}{1+M}, \frac{\varepsilon}{1+M})$. Then for $|q(n)| < \delta$ $|x(0)| = |q(0)| < \delta(1+M) = \min(\eta, \varepsilon)$. Assume that there exists $n_1 > 0$ such that $|x(n_1)| = \delta(1+M)$ and $|x(n)| < \delta(1+M)$ for $n = 0, 1, \ldots, n_1 - 1$. From (E_2) and the assumptions we have $|x(n)| \leq \delta + \sum_{s=0}^{n-1} L(n,s)|x(s)|$ for $0 \leq n \leq n_1$. Hence $|x(n)| = \delta - \varphi(n) + \sum_{s=0}^{n-1} L(n,s)|x(s)|$, where $\varphi(n) \geq 0$. The solution of this equation can be represented in the form

$$|x(n)| = \delta - \varphi(n) + \sum_{s=0}^{n-1} \overline{R}(n,s)(\delta - \varphi(s)).$$

Therefore

$$|x(n_1)| = \delta(1+M) = \delta - \varphi(n_1) + \sum_{s=0}^{n_1-1} \overline{R}(n_1,s)(\delta - \varphi(s)) < \delta(1+M)$$

which is a contradiction. Thus the solution x of (E_2) satisfies the inequality $|x(n)| < \delta(1+M) \leq \varepsilon$ for all $n \in N_0$. □

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