TRADING MODEL EXPLAINS TYPICAL SHARE VALUE MOVEMENTS

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Abstract: A mathematical method is presented, using functions stepwise defined over the time scale, which describes the development of share values on the base of detailed trading activities. Two simple relations, between the share value together with its temporal variation and the velocity or acceleration of trade stimulus, result in a non-linear first-order differential equation, even providing analytical solutions in the case of constant parameters. With this equation an “ordinary” trading process can be defined, illustrated by characteristic examples. For demonstrating the usefulness of the method, a calibration procedure on some time interval of the Frankfurt-Effekten-Fonds is demonstrated. Additionally to the ordinary trading process, “disturbing” activities can be taken into account, preferably in relationship to trade stimulus acceleration. In this way, a non-linear second-order differential equation results, whose solutions contain accelerated increases and diminished decreases of the share value during buying or selling, respectively. In both cases the implementation destructive process instabilities may occur, especially share value breakdowns. Some of the mathematical statements can serve as starting point for discussions from an economical point of view.

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1. Notation

\( t \) – time – with one day \([d]\) as unit;

\( M \) – basic share value (\( \mathcal{E} \)) – usually unaffected (material) basis, preferable in the view of customers, enterprise value in a general sense;

\( W(t) \) – share value at time \( t \) (\( \mathcal{E} \)) – current price for one share at time \( t \);

\( \frac{dW(t)}{dt} \) – value velocity (\( \mathcal{E}/d \)) – change of \( W(t) \) per day;

\( S \) – saturation value (\( \mathcal{E} \)) – target price, personal expectation for \( W \);

\( K(t) \) – buying stimulus (\( \mathcal{E} \)) – stimulus for the share customer;

\( \frac{dK(t)}{dt} \) – stimulus velocity (\( \mathcal{E}/d \)) – daily buying stimulus (Assumption 1 in Section 3) proportional to the actual difference \( S - W(t) \);

\( \frac{d^2K(t)}{dt^2} \) – stimulus acceleration (\( \mathcal{E}/d^2 \)) – change of the daily buying stimulus;

\( a \) – trading factor (1/d) – part \( a > 0 \) of \( S - W(t) \) the customer is willing to spend per share for buying at the given day (Assumption 1 in Section 3);

\( g \cdot \frac{d^2W(t)}{dt^2} \) – disturbing activity (\( \mathcal{E}/d^2 \)) – additional driving force in the buying stimulus acceleration with “distortion factor” \( g \) as a positive dimensionless number;

\( \alpha \) – emission agio (dimensionless positive number) – additional charge for delivery start at time \( t = 0 \), share value beginning with \( W(0) = M(1 + \alpha) \) and \( \alpha > 0 \).

2. Introduction

Normally in share trading, one of the most interesting questions is focused on the future developments of the share values. During the last decades a remarkable progress has been achieved, to answer this problem by application of modern methods of mathematical statistics. These methods provide stochas-
tic conclusions resulting from statistical evaluations of “time sequences” of the temporal value changes for the shares under examination. Another way to answer the question is technical analysis\textsuperscript{1}, which uses special discernible patterns in the share value progress to derive qualitative (partially also quantitative) forecasts for the future development of value, see e.g. Edwards et al (2007); Murphy (1999). There are many further approaches to share value forecast: time sequence analysis, stochastic processes, neuronal networks, computer simulations, fundamental analysis. We refer e.g. to Neusser (2009), Oitzl (2008), Sattler (1999), and Schlitter (2008).

In this paper, an attempt will be undertaken to describe the driving forces, which are responsible for every change of a share value during the trading process, with functional (and therefore quantitative) expressions. Surprisingly enough, simple equations could be found which provide, to some degree, conclusions about the temporal development of a share value during trading. The basis are simple, but reasonable assumptions on the relation between velocity and acceleration of stimulus as well as the share value together with its change over the time. This procedure will be shown in the following.

Looking at a “time sequence”, the most important impression comes from its “zigzag” shape, in general. De facto, the graphic demonstrates a sequence of detailed sections, with increases of value up to sharp maxima followed by corresponding decreases, both similar to the sides of a triangle. Later on, this will be demonstrated by an example. This behavior can be observed not only in a short time range, but also for moving averages of such curves over several days or even over months. Sometimes appearing deviations from such idealized shape, like breakdowns of share values, seem to indicate deviations from a “ordinary” buying or selling behavior. In this content the definition of a ordinary trading process is of great interest.

In the next sections of the paper, mathematical relations will be formulated for a single share buying or selling process. The statements lead to a differential equation with analytical solutions in case of a ordinary trading process under at least short-time constant conditions. For a trading process with “disturbing activities”, this equation will be completed by an additional term, which is assumed as acting proportional to the buying stimulus acceleration. Finally, with a parameter calibration procedure on a time sequence interval of the Frankfurt-Effekten-Fonds, as an example, a really existing ordinary trading process will be investigated.

For many of the mathematical conclusions an economical interpretation

\textsuperscript{1}Technical analysis assists forecasts to the future value development of shares, indices or other financial objects using several data.
will be given. However, some of them are still to be discussed an economical, behavior-theoretical or practical point of view.

3. Model for Buying Stimulation and Share Value Dependence

Subject of investigation is a certain, not further specified share, a fund or an index. The following considerations base on relatively simple statements for an ordinary trading process. At first, a direct relationship between buying stimulus velocity and share value is supposed:

**Assumption 1.**

\[
\frac{d}{dt} K(t) = a \cdot [S - W(t)].
\]  

(1)

The stimulus velocity \( \frac{dK(t)}{dt} \), the “buying haste” (not amount), should therefore depend on the distance between market value \( W \) and target price \( S \). According to this, the stimulus velocity should be zero in the case \( W(t) = S \), or achieve negative quantities if \( W(t) \) exceeds the value \( S \). Such negative buying values would correspond to a domination of selling. The “trading factor” \( a \), of dimension 1/d, will also be used as a positive and constant quantity, consequently as time independent like parameters \( M \) and \( S \). With this restriction, equation (1) becomes applicable only in certain time intervals, as long as the condition of constancy is fulfilled with sufficient accuracy.

By relation (1) the quantity “buying stimulus velocity” \( \frac{dK(t)}{dt} \) is defined, the stimulus per day. Due to (1), the stimulus \( K(t) \) itself is the time integral corresponding to a cumulative value with respect to time. Furthermore, the bargain is taken here as the mean over all simultaneous movements in share trade, a share selling is understood as buying with negative sign.

Moreover, it is supposed that the value velocity \( \frac{dW(t)}{dt} \) depends on the distance between the actual value \( W(t) \) and the basic value \( M \), resp., as well as on the stimulus velocity \( \frac{dK(t)}{dt} \):

**Assumption 2.**

\[
\frac{d}{dt} (W(t) - M) = \frac{W(t) - M}{M} \cdot \frac{d}{dt} K(t).
\]  

(2)

This is simply the product of both quantities. According to equation (2), the movement of \( W \) with \( dK/dt \) increases as stronger as wider the distance between \( W \) and \( M \) is, relatively and absolutely.
From (1) a correspondence between value velocity $\frac{dW(t)}{dt}$ and buying stimulus acceleration $\frac{d^2K(t)}{dt^2}$ results. Together with (2), a differential equation for the buying stimulus $K(t)$ can be derived. However, in this paper the interest is mainly focused on the share value $W(t)$. Therefore, only for $W(t)$ a relation will be deduced consisting in a nonlinear first-order differential equation.

**Definition 1.** A buying or selling process with share values $W(t)$ described by equations (1) and (2) will be defined as an **ordinary trading process**.

The “buying instinct” will be considered here as the driving force in such a process. It is assumed to be proportional to the buying stimulus acceleration. Practically, there are further influences on the value $W(t)$, generated by disturbing activities. Among them there are effects that additionally accelerate the trading process (like “greediness”, “panic”, or similar effects). It is assumed here that these activities should not work continuously, but preferably then, when the value velocity $\frac{dW(t)}{dt}$ heavily changes over the time. For this reason, another quantity is introduced in the following supposition:

**Assumption 3.**

$$\frac{d^2K(t)}{dt^2} = a \cdot \frac{d}{dt}(S - W(t)) + g \cdot \frac{d^2W(t)}{dt^2}. \quad (3)$$

Hence, if a “disturbance” during the trading process appears, the activity responsible for that is assumed to work only in presence of non-vanishing value accelerations $\frac{d^2W(t)}{dt^2}$ and to depend on their strength. It should exclusively and supplementary act on the buying stimulus acceleration $\frac{d^2K(t)}{dt^2}$, in agreement with the above given notation for the distortion factor $g$. The possibly simplest relation was chosen here, in which the corresponding term occurs as an additional summand to the time derivative of equation (1). From this, together with equation (2), a nonlinear second-order differential equation follows for $W(t)$.

**Definition 2.** A buying or selling process described by equations (2) and (3) is said to be a **general trading process**.

### 4. Differential Equation for Ordinary Trading Processes

In case of an ordinary trading process equations (1) and (2) lead to the relation

$$\frac{dW(t)}{dt} + a \cdot \frac{1}{M} \cdot (W(t))^2 - a \cdot \left(1 + \frac{S}{M}\right) \cdot W(t) + a \cdot S = 0, \quad (4)$$

a nonlinear first-order differential equation for $W(t)$. 

With $y(t) = W(t)$ and $y' = \frac{dy}{dt}$, expression (4) can be written in the easier form

$$y' + \frac{a}{M} \cdot y^2 - a \cdot \left(1 + \frac{S}{M}\right) \cdot y + a \cdot S = 0. \quad (5)$$

With time independent constants $M$, $S$ und $a$, analytical solutions exist for the differential equation (5):

$$y_{1a}(t) = M \quad \text{and} \quad y_{1b}(t) = S, \quad \text{resp. for} \quad y'(t) = 0, \quad (6)$$

i.e. the parameters $M$ and $S$ themselves. Furthermore

$$\frac{1}{M} \cdot y_2(t) = 1 + b \cdot [1 + \tanh ab(t - t_0)] \quad \text{for} \quad y'(t) > 0, \quad (7)$$

as well as

$$\frac{1}{M} \cdot y_3(t) = 1 + b \cdot [1 + \coth ab(t - t_0)] \quad \text{for} \quad y'(t) < 0. \quad (8)$$

In (7) and (8) the quantity $b = \frac{1}{2} \left(\frac{S}{M} - 1\right)$ was used. Assuming $S > M$, the restriction $b > 0$ obviously follows. The conditions indicated for $y'(t)$ apply to the whole time axis. More details on differential equation (5) and its solutions can be found in the authors’ former preprint [10].

The constants $M$ and $S$ as solutions for $y(t) = W(t)$ are not interesting here. The other both solution functions (7) and (8) contain a time dependence via the hyperbolic tangent (tanh) and the hyperbolic cotangent (coth), respectively. Their time constants $t_0$ in the arguments are fixed by the start conditions of the actual problem. The time moves only in positive direction.

Because of the fact that the parameters $M$, $S$ and $a$ are positive, one can generally conclude that the functions $y_2(t)$ with the hyperbolic tangent are solutions of equation (4) only then if they have a positive time-derivative $y'_2(t) = \frac{dy_2(t)}{dt}$. In this case, they describe an increase of $W(t)$ on the whole time axis. On the other hand, the functions $y_3(t)$ have to possess a negative derivative and therefore a decrease of $W(t)$ with respect to time. Vanishing derivatives can only lead to a solution if $y(t)$ can reach the values $S$ or $M$ according to (6). Generally spoken, continuously differentiable transitions between $y_2(t)$ and $y_3(t)$ cannot exist practically, only sharp “triangle peaks” as mentioned above.
5. Ordinary Trading Process as Series of Initial Value Problems

5.1. Simple Processes Containing Both Buying and Selling Periods

Given a trading process with constant parameters, at least in a certain time interval, one can search for functions \( y(t) \) satisfying (7) and (8), resp., which fulfil the differential equation (4) and (5), resp., as well as the initial conditions of the considered interval. In mathematical terms, such functions solve the corresponding initial value problem. In practice, the chance to dissect a time sequence into subsequently following phases with constant (and generally different between each other) parameters exists, at least approximately, as it will be shown below for a relevant real example.

To give a clear idea, above all two simple theoretical events will be described: Firstly, a single trading process with an initially dominating buying behavior, followed by a selling period. The buying process, here designed as initial value problem \( A \), should act until a certain time \( t_1 \), where the personally expected value \( S \) is assumed to drop down suddenly. The following smaller value \( S_1 \) (the new personal expectation) requires a new initial value problem \( B \) for the time interval \( t \geq t_1 \), which contains only solutions \( W(t) \) decreasing over the time. Therefore, in this trading part the selling character dominates. Secondly, an ordinary trading process with three successive initial value problems \( A, B \) and \( C \) is considered, even with the possibility of share value “breakdowns” in the last problem (despite of the absence of disturbing activities supposed here).

5.2. Ordinary Trading Process with a Period of Buying Followed by Selling

An example shall illustrate the functional course of \( W(t) \) in two sequential initial value problems \( A \) and \( B \). Equal basic share values \( M \) are supposed for both, what allows a simplified presentation using the relative solutions \( \frac{y(t)}{M} \), without a precise choice of \( M \). Figure 1 shows two pairs of solutions for this case, \( y(t) \) and \( y^*(t) \).

For the first function \( y(t) \), the following parameters are chosen in the initial value problem \( A \), in the first interval of time \( t \geq 0 \), after the share emission at \( t = 0 \):
Figure 1: Ordinary trading process with a buying period followed by selling

\[ a = 5\% / d = 0.05 / d \]  
 corresponding to 5\% of \( S - W(t) \), daily buying stimulus;

\[ S = 2M \]  
 initial value of problem \( A \);

\[ y(0) = W(0) = M(1 + \alpha) \]  
 offer agio at emission;

\[ W(t_1) = \beta S \]  
 breaking off value at \( t = t_1 \), \( 0 < \beta < 1 \);

\[ \beta = 0.9 \]  
 breaking off rate at the end of problem \( A \).

The starting condition, with \( \alpha > 0 \) leads to a solution \( W(t) \), which increases with respect to time. This comes from \( S > W \) and, consequently, from the positive stimulus velocity \( \frac{dK(t)}{dt} \). Due to equation (1) and \( W(t) > M \), one gets a positive \( \frac{dW(t)}{dt} \) in equation (2). Therefore, function \( W(t) = y_2(t) \) from (7) is the only solution of the initial value problem \( A \). It is denoted here as as \( y_{2A}(t) \):

\[
\frac{1}{M} \cdot y_{2A}(t) = 1 + b \cdot [1 + \tanh(ab(t - t_0))].
\]  
(9)

The time constant \( t_0 \) is determined by inserting the initial value \( y_{2A}(0) \) in (9), this yields

\[
t_0 = -\frac{1}{ab} \cdot \text{artanh} \left\{ \left[ \frac{1}{M} \cdot y_{2A}(0) - 1 \right] \cdot \frac{1}{b} - 1 \right\}
\]
The parameters chosen above yield $b = \frac{1}{2}$ and $t_0 = 43.9$ [d]. Taking this into account, from expression (9) and $W(t_1) = y_{2A}(t_1) = \beta S$ the break off time $t_1 = 71.6$ [d] results.²

Another also thinkable possibility, the special case $\alpha = 0$ (delivery at basic value), would yield the “solution” $W(t) = M$ only, a time independent number and therefore out of reality.³

If in the initial value problem A the parameters would always remain constant, then the value $W(t)$ according to (9) would monotonically increase over the time, until an asymptotical approach to the saturation value $S$ for $t \to \infty$. This behavior is indicated in the figure by the dashed continuation of $y_{2A}(t)$.

In practice however, changes in ratings, opinions or strategies (and therefore of the parameters) can be expected during the trading process. First of all and as mentioned above, a possibly growing tiredness in the market participants buying stimulus is supposed, especially a diminution of the expected share value (saturation value, target price) $S$ after a certain time $t_1$. In this content, the name “market participant” stands for the mean behavior of all traders active at the same time. In the case of reduction of the parameter $S$, problem A with solution $y_{2A}(t)$ is restricted to the time interval $0 \leq t \leq t_1$. The following initial value problem B starts with the now lower target price $S_1 < W(t_1) = \beta S$⁴ and the last value $W(t_1) = \beta S$⁵ of problem A as the new initial value.

Contrary to the above argumentation for $W(0) < S$ in problem A, the new condition $W(t_1) > S_1$ requires a solution, which decreases over the time monotonically. Therefore, the only possible solution of initial value problem B is function $y_3(t)$ from (8):

$$\frac{1}{M} \cdot y_3B(t) = 1 + b_1 \cdot [1 + \coth a_1 b_1 (t - t_{01})] \quad \text{for } t > t_{01}$$  \hspace{1cm} (11)

with $b_1 = \frac{1}{2} \left( \frac{S_1}{M} - 1 \right)$. Continuity at $t = t_1$ requires

$$y_3B(t_1) = y_{2A}(t_1) = W(t_1) = \beta S.$$  \hspace{1cm} (12)

²On the other hand, if the time $t_1$ would be fixed, as in the example presented below, where there parameters of a simulation are calibrated to a real time sequence, the rate $\beta$ would follow as a purely informative value.

³Vice versa one can conclude: “There is no emission equal to the basic share value, but only emissions with a certain agio”.

⁴The indices “1” at the parameters $S$, $a$, $\beta$ and $b$ refer to the new initial problem B.

⁵In the following, the “breaking off” rate (“buying stop” factor) provides only some information on that part of the corresponding target price $S$, which is reached by $W(t)$ at the end or the beginning of an initial value problem. It is not a calculation parameter which could influence the solution.
The argument of the hyperbolic cotangent in $y_{3B}(t)$ has to be positive, because the decrease of the positive part of coth with increasing time will be used here. From condition (12) it follows:

$$t_{01} = t_1 - \frac{1}{a_1 b_1} \cdot \text{arcoth} \left[ \left( \frac{y_{2A}(t_1)}{M} - 1 \right) \cdot \frac{1}{b_1} - 1 \right]. \quad (13)$$

For problem B, a reduction of the target price down to $S_1 = 1.2M < \beta S = 1.8M$ at $t = t_1$ was assumed, but (for the sake of simplicity) with an unchanged trading factor for $y_{3B}(t)$ compared with $y_{2A}(t)$. Therefore $a_1 = a = 0.05/d$ holds. With these numbers, from (13) the time constant $t_{01} = 43.2[d]$ results. Defining a new rate $\beta_1$ by $y_{3B}(t_1) = \beta_1 S_1$, from $S_1 < \beta S$ the condition $\beta_1 = \frac{S_1}{\beta S} > 1$ follows. This means, that the initial problem B starts here with a factor $\beta_1 > 1$ for the new target price $S_1$.

In Figure 1 both solutions $y_{2A}(t)$ and $y_{3B}(t)$ are presented. To illustrate the influence of the trading factor, the graph additionally contains the solution pair $y^*_2A(t)$ and $y^*_3B(t)$ for smaller values $a_1 = a = 1.5\% / d$, but identical parameters otherwise.

In summary, the figure gives a clear idea of the continuously increase of the share value $W(t)$ after the start, where $W(t)$ always remains below the target price $S$. Due to the fact that the solution (9) starts at $t = 0$ with the negative argument $-0.025 \cdot t_0$ in the hyperbolic tangent, this function moves with increasing time $t$ through its zero point in positive direction. When the argument is equal to zero, the hyperbolic tangent has its maximal ascent. Therefore, the maximal buying activity will be achieved at $t = t_0$, in the considered example about 44 days after the delivery start.

Diminishing the target price from $S = 2M$ down to $S_1 = 1.2M$, at $t = t_1$ a share value reduction begins, possibly extended over a large time interval, similar to that of the buying period. $W(t)$ shows an abrupt decrease at $t = t_1$, but with progressive time the slope becomes smoother again. Coming from the larger quantity $S = 1.8M$, it passes into an asymptotic approach to the (now lower) saturation value $S_1$. According to equation (1), the stimulus velocity is always negative, the selling behavior dominates.

Obviously, such an asymptotic character can only exist if the process B continues a sufficiently long time, i.e. the parameters remain constant over a large time period. Otherwise, with new changes of at least one parameter (in practice mostly enlargements of selling velocity or an additional decrease of the target price), a further initial value problem starts at this moment.

In summary, a value velocity jump characterizes the transition at $t = t_1$, there is no gradual change. $W(t)$ fails to be continuously differentiable at the
transition point. Both solutions, the branch of $y_{2A}(t)$ up to $t = t_1$ and the following branch of $y_{3B}(t)$, yield a course of $W(t)$ like that known from the real share trading: the above mentioned “triangle sides”.

5.3. Are Share Value Breakdowns Possible in Ordinary Trading?

The above given example was a demonstration of the most interesting problem in a trading process – the transition from a dominating buying behavior into a selling one. However, there remains one question: Can share value breakdowns occur under the condition of ordinary trading processes corresponding to Definition 1 (without disturbances in the above discussed sense)? The notion “breakdown” means such value diminutions, which become increasingly larger over the time and do not tend towards a finally “limiting” asymptote as discussed in the last example, even in the case of several successive initial value problems, where every of which contains a reduction of $S$.

Only solutions $y_3(t)$ of the differential equation (1) could describe such a breakdown, and only in that case, where the hyperbolic cotangent argument would move towards zero with increasing time $t$. This would mean the use of the cotangent into the direction of its pole. The previous example showed the opposite tendency with an argument movement away from the pole.

Possibilities of such a kind of decrease will now be investigated by the help of a further initial value problem $C$, which follows the previous problem at $t = t_{E_1}$ starting with $W(t_{E_1}) = y_{E_1}$. The solution $y_{3C}(t)$ of problem $C$ has then to begin at the transition with the value $y_{E1}$. Different parameters are chosen now: $S_2 \neq S_1$, $M_2 \neq M_1$ and $a_2 \neq a_1$. In the possible solution

$$\frac{1}{M_2} \cdot y_{3C}(t) = 1 + b_2[1 + \coth a_2 b_2(t - t_{02})]$$

with $a_2 b_2(t - t_{02}) \neq 0$, the approach to the pole explained above requires

$$t_{02} > t.$$  \hspace{1cm} (15)

This means an opposite behavior to the condition $t > t_{01}$ in the solution $y_{3B}(t)$ of problem $B$ in the last section. Analogously to what was said above, the constant $b_2$ is defined by $b_2 = \frac{1}{2} \left( \frac{S_1}{M_2} - 1 \right)$. The time constant $t_{02}$ results from relation (13), now with the new parameters $M_2$, $S_2$ and $a_2$ as well as $y(t_{E1}) = y_{E1}$:

$$t_{02} = t_{E1} - \frac{1}{a_2 b_2} \cdot \text{arcoth} \left[ \left( \frac{y_{E1}}{M_2} - 1 \right) \cdot \frac{1}{b_2} - 1 \right].$$  \hspace{1cm} (16)

\[\text{Index “2” denotes problem C, while “1” relates to the foregoing initial value problem.}\]
Due to condition $t_{02} > t > t_{E1}$ obtained from (15), considering possible break-downs of solution (14), the arcoth argument in (16) has to be negative and lower than $-1$ or positive and greater than $+1$ for $b_2 > 0$ and $b_2 < 0$, respectively. This leads to the conditions

$$
y_{E1} < M_2 \quad \text{for } b_2 > 0 \quad \text{(giving } S_2 > M_2),$$

$$S_2 > y_{E1} \quad \text{for } b_2 < 0 \quad \text{(giving } S_2 < M_2).$$

(17)

In the last case one has to take into account that the condition $b_2 < 0$ for a positive arcoth argument requires additionally $\left(\frac{y_{E1}}{M_2} - 1\right) < 0$, with absolute values of both quantities lower than one.

Due to the condition $S_2 > M_2$ (for positive $b_2$), relation (17) requires

$$y_{E1} < M_2 < S_2 \quad \text{for } b_2 > 0.$$

Analogously, for $S_2 < M_2$ (if $b_2$ is negative), the additional condition $y_{E1} < S_2$ has to be valid. Therefore

$$y_{E1} < S_2 < M_2 \quad \text{for } b_2 < 0.$$

Summarizing the derived conditions for the appearance of a “downfall” in solution (14), one can conclude:

- Downfalls with “breakdown” character are only possible with enlargements of the basic share value $M_2$ up to values greater than the break off value $y_{E1}$, i.e. for $M_2 > y_{E1}$. Under such circumstances, $S_2 > M_2$ or $S_2 < M_2$ are allowed, only the inequality $b_2 \neq 0$ has to be true.\(^7\)

- In an ordinary trading process, simultaneous reduction of basic value and target price to $M_2 < M_1$ and $S_2 < S_1$, respectively, does not result in a “downfall”: for that the conditions are not satisfied and, hence, solutions do not exist.

Figure 2 demonstrates a time sequence simulation, with three successive initial value problems, and one example for a downfall in the last.

The first section, i.e. the first initial value problem, corresponds again to problem A (cf. Section 5.2) with break off time $t = t_1$. Condition $S = 2M$ is again used, but now with the parameters $M = 100$ [€] and $S = 200$ [€]. The values $a = 5.0\%$ and $\beta = 0.9$ remain as in problem A, hence $t_1 = 72$ [d] holds. Consequently, the solution function is $y_{2A}(t)$ known from Section 5.2.

\(^7\)Some economical interpretation can be found at the end of this section.
In the second part (analogously to the above problem B) the target price is reduced, but here deeper as in B, down to $S_1 = 80\, [\text{€}]$ (instead of $120\, [\text{€}]$). Additionally, a higher trading factor $a_1 = 7.0\%$ and a basic value reduction down to $M_1 = 70\, [\text{€}]$ are assumed. The last modification should demonstrate that even the basic value is not necessarily “untouchable”. It represents only a quantity in the trader’s perception. The solution function $y_3(t)$, here denoted by $y_{3B}(t)$, shows the behavior known from problem B: $W(t)$ drops down with increasing time towards the asymptote $S_1$. This process is interrupted at time $t_{E1} = 130\, [\text{d}]$, where $W(t_{E1})$ reaches the value $y_{3B}(t_{E1}) = y_{E1} = 90.37\, [\text{€}]$.

In the third part C, four variants are demonstrated as possible continuations of the foregoing initial value problems:

1. Maintenance of the parameters from the last section, what means $S_2 = S_1$, $M_2 = M_1$ and $a_2 = a_1$. This leads to a direct continuation of solution $y_{3B}(t)$ (the third initial value problem is simply a temporal extension of the second).

2. Maintenance of the parameter $a_2 = a_1$, but increase of the target price and basic share value up to $S_{21} = 200\, [\text{€}]$ and $M_{21} = 85\, [\text{€}]$, respectively. With the last, a higher value then $M_1$ is chosen, but still lower than $y_{E1} = 90.37\, [\text{€}]$. The corresponding solution $y_{2C1}(t)$ behaves as expected:
The value $W(t) = y_{2C1}(t)$ increases again for $t > t_{E1}$ (because of the now higher target price) after its decrease in the previous time interval.

3. Use of the same parameters $a_2$ und $M_{21}$ as in the preceding variant, but with a smaller target price up to $S_{22} = 150\ [\text{€}]$. The corresponding solution $y_{2C2}(t)$ shows indeed the expected value increase, but now weaker than before according to $S_{22} < S_{21}$.

4. Choice of the parameters in such a way that the principal possibility of share value breakdowns discussed above actually occurs. Whether such a possibility exists in practice and not only mathematically remains an open question. For instance, one can choose these parameters: $a_2$ and $S_{22}$ from the preceding variant, but with an increase of the basic value up to $M_{22} = 95\ [\text{€}]$, which is slightly higher as the final value $y_{E1} = 90.37\ [\text{€}]$ of the previous interval. The corresponding solution $y_{3C2}(t)$ shows a “share value breakdown”!

Finally an attempt will be undertaken to explain the possibility of a breakdown. Above all it seems to be strange that during an ordinary trading process after a value decrease the function $y_{3C2}(t)$ points towards a share value breakdown, despite the renewed enhancement of share intention $S$ and basic value $M$ – only because of a new $M$ amount, which exceeds the previous final value $y_{E1}$. The following reasons are conceivable:

- The traders decision to continue the selling (like here in the second time interval), even if meanwhile – whether really or only in his perception – the “material” basic value $M$ as well as his own target price $S$ could be increased, the last possibly under the influence of the first. This behavior would be similar to such in an “atmosphere of panic”.

- The use of the basic value $M$ as a really given material base, with the implementation to understand reductions (like such in the second time interval) as extractions of capital. This would correspond to the particularly strong value decrease in the second interval. For the “recreation process” in the third interval, with higher share intention $S$ (usually leading to a raise of buying interest), the new investment of capital (increase of $M$) could be too early and too much at that time, a better amount of $M$ ought to be chosen, a number below the final value $y_{E1}$.

- Other reasons are thinkable.
For all it is necessary to remind once more that until now the above downfall conditions belong to an ordinary trading process according to Definition 1.

6. Consideration of Additional Disturbing Activities


To consider disturbing activities in the driving force at buying, in Section 3 an additive term was introduced into the expression of the buying stimulus acceleration $\frac{d^2 K(t)}{dt^2}$, for a hitherto ordinary trading process, as shown by equation (3). Combining the latter with relation (2), a new differential equation for $W(t)$ can be derived:

$$y''\left[g(y - M) - M\right] = a \cdot (y - M) \cdot y' - \frac{M}{y - M} \cdot y'^2,$$

(18)

where the notations $y = W(t)$, $y' = \frac{dy}{dt}$ and $y'' = \frac{d^2 y}{dt^2}$ are used. This nonlinear second-order differential equation describes a general trading process, according to Definition 2. For $g = 0$, the equation turns into the ordinary trading process relation (4) (cf. Definition 1). This can be shown with the help of some transformations.\(^8\)

From the differential equation (18) itself one can recognize possible instabilities, which can appear in the functions $y(t)$ at $g > 0$, and originate from an unlimited increase of $y''(t)$ with respect to $t$. Transferring equation (18) into

$$y'' = \frac{Z(y, y')}{N(y)}$$

(19)

with

$$Z(y, y') = a \cdot (y - M) \cdot y' - \frac{M}{y - M} \cdot y'^2$$

(20)

and

$$N(y) = g(y - M) - M$$

(21)

makes this obvious. A sign change of the denominator $N(y)$ can occur, both for buying or selling periods in a trading process with steadily over the time increasing or decreasing values $y$, respectively. The time point of this zero transition depends on the parameters in the considered time period. Disappearance

\(^8\)For $g = 0$, more details related to (4) and (18) can be found in the paper [10].
of the denominator \( N \) would correspond to a pole of \( y'' \), with heavy temporal variations of the gradation \( y' \) in its neighbourhood, and practically uncontrolled amounts \( y(t) \) there.

6.2. General Trading Process with Disturbing Activities During Bargain

As before, the interest in the following general trading processes will be focused on such time intervals only, in which the parameters \( M, S, a \), and now additionally \( g \), remain constant, in order to allow comparisons with the above given investigations of ordinary trading processes and interpretations of time sequences as serial arranged initial value problems. Nevertheless, no analytical solutions of equation (18) can be found for \( g > 0 \); results are only achievable by numerical methods.

Figure 3 demonstrates an example of such calculations. Again the same conditions and parameters are used as in the buying process of initial value problem \( A \), presented in Section 5.2, but now supplemented by the disturbing factor \( g \). Furthermore, because of the second-order of differential equation (18), an initial condition for \( y' \) is required. For this it is assumed that at the beginning of the initial value problem the disturbing activity does not exist, it appears later. Under this condition, equation (4) valid for \( g = 0 \) determines \( y'(0) \), in the actual case \( y'(0) = 0.45 \ [€/d] \). For the sake of simplicity, in the following example a temporal delayed beginning was renounced and replaced by an immediately with \( t > 0 \) starting \( g \neq 0 \) (a later initiated action would not provide other principal statements).

The curve \( y(t) = W(t) \) for \( g = 0 \) is identical with that of an ordinary buying process shown in Figure 1. The introduction of an additional disturbing activity \( g > 0 \) accelerates the enhancement of \( W(t) \), in the considered time interval clearly measurable up to factors about \( g = 1.0 \). At higher numbers, the graph shows permanent earlier starting steep increases of \( W(t) \) with growing factors \( g \), until practically uncontrollable extensions. Arrows indicate these facts in the graph. Such behavior of \( W(t) \) is a consequence of the above mentioned instabilities, which appear if the denominator \( N(y) \) of the second derivative \( y'' \) tends towards zero with increasing \( y(t) \).

For a disturbing factor \( g = 10 \) this zero transition exists directly at the beginning of the curve at \( t = 0 \). As visible from expression (21), the first term of the dominator \( N \) becomes equal to \( M \) there, and \( N \) is equal to zero, due to the basic share value \( M = 100 \ [€] \) and the start value \( y(0) = 110 \ [€] \) in the example. For lower factors \( g \), this transition point moves to higher \( y(t) \), i.e. to
Figure 3: General trading process with disturbing activity at buying times $t > 0$ for the buying process described here.

With decreasing $g$, this movement of the critical transition towards longer times extends more and more up to practically non-interesting regions. For example, the curve associated with $g = 1.0$ represents a stable trading process even at $t = 200$ [d]. At disturbing factors above 10, only for $g \geq 10.07$ some stability will be achieved again, but with share value enhancements lower than originally derived for $g = 0$. Finally for $g \to \infty$, the second derivative of $y$ vanishes, i.e. $y'' \to 0$, which corresponds to a linear shape of $W(t)$, with the constant slope $y'(t) = y'(0)$. The figure demonstrates this approximately with curve $g = 100$. The “most inconvenient” instabilities appear in the relative small region $10 < g < 10.07$: With increasing time it comes here, after a flat maximum, to strong and abrupt “downfalls” (in the graph again marked by arrows).

Summarizing the results of Figure 3, one can conclude that in comparison to an ordinary buying process disturbing activities (greediness, panic, tactical behavior, etc.) always lead to instabilities. At (very) big factors $g$, a completely irrational trading process would arise (buying with steadily value increase) – until at least one trade participant would change its attitude to the trade. Then a new initial value problem would become necessary. However, when the activities remain sufficiently restricted, then positive effects can be created with accelerated growth of $W(t)$, but in all cases only within certain time limits. If not so, furthermore enhanced activities will result in practically uncontrollable
instabilities, with possible value downfalls, at the very least with diminished value raisings.

6.3. General Trading Process with Disturbing Activities During Sale

In this section, a trading process containing a time interval with dominating selling behavior will be considered. It is assumed to have two sections, i.e. two consecutive initial value problems: the starting buying process from the above investigated problem \( A \), until the time \( t = t_1 \), and following them a time period with reduced target price \( S_1 \), similarly to the above problem \( B \) in Section 5.2, and consequently with predominant sale.\(^9\)

The disturbances act in the selling period only. But also in that case, they can initiate instabilities, as already equation (18) shows qualitatively. The selling period begins with \( y(t_1) > M \) and a negative \( y'(t_1) \). As long as the condition \( y(t) > M \) is fulfilled, the numerator \( Z(y, y') \) of \( y''(t) \) remains negative due to (20). Opposite to it and according to (21), the denominator can change its sign with advancing time, here corresponding to a diminishing value \( y(t) \), if the initial value \( y(t_1) \) exceeds \( M \left(1 + \frac{1}{g}\right) \).

In Figure 4 some numerically calculated typical results are presented, in which again the parameters from the initial value problem \( A \) were used for the first time interval. Only in the second initial value problem \( B \), the number of one parameter is changed (\( S_1 = 160 \) €).

Analogously to the foregoing example, the picture for \( g_1 = 0 \) confirms the result already known from the ordinary trading process. Disturbing activities (\( g_1 > 0 \)) improve the temporal course of \( W(t) \), here with growing factors \( g_1 \) by a smaller and smaller reduction during the decrease. The calculations show that this improvement exists only as long as the relation \( g_1 < 1.25 \) holds.

If the factor \( g_1 \) exceeds this number, then the instabilities mentioned above appear, in the actual case with \( y'(t_1) < 0 \) as initial value always connected with value downfalls. Only for \( g_1 \to \infty \), the falling curve \( y(t) = W(t) \) would again “stabilize” its course, by an asymptotical approach to a line with slope \( y'(t) = y'(t_1) = -0.8 \) [€/d]. Nevertheless, this would still imply a continuous value decrease (without temporal stabilization). In Figure 4 this behavior is approximately represented by the curve with \( g_1 = 50.0 \).

In summary, disturbing activities can have positive effects on sale, as long as they remain low enough. Raisings can again imply instabilities, here in the

\(^9\)The indices “1” at the parameters \( S, a, b \) and \( g \) refer to the second initial problem \( B \).
7. Ordinary Trading Simulation in a Real Fitting Test

7.1. Calibration of a Time Sequence Interval from the Frankfurt-Effekten-Fonds

Now the above demonstrated results shall be proofed by a concrete example. To this purpose, the redemption price of the Frankfurt-Effekten-Fonds\textsuperscript{10} will be used here as “share value” in question. Figure 5 shows its temporal course between 07/10/1997 and 30/12/2006, and Figure 6 the selected period 07/10/1997 to 10/11/1998, with the date 06/10/1997 as zero point on the time scale.

Principally, the behavior of this time sequence is characterized by temporal zigzag movements, in certain intervals with relatively large extended courses (similar to triangle flank sides), like the in Section 1 as typically described shapes and the in Section 5 given results of an ordinary trading process.

The chosen time sequence period presented in Figure 6 demonstrates clearly that such temporal share value “triangles” can really exist, as a mean over a

\textsuperscript{10}ISIN: DE0008478058
longer time – here almost one year!

Figure 5: Redemption price of the Frankfurt-Effekten-Fonds in the time interval 07/10/1997 – 29/12/2006

Fittings of the time sequence by functions $y_2(t)$ and $y_3(t)$ according to (7) and (8) have been realized in the graph of Figure 6. Consequently, the time sequence was interpreted as an ordinary trading process, without disturbing activities.

In this procedure, the calibration started at the time point $t_A = 29$ [d] (date 04/11/1997) with the value $y_2(t_A) = 138$ [€]. The time sequence period chosen in Figure 6 was divided into four sequential initial value problems, with constant but different parameters in each case. The fittings resulted in the following values:

1. $t_A \leq t \leq t_1 = 290$ d: $M = 122$ [€] and $S = 230$ [€], ($b = 0.44$); $a = 1.2$ [%/d]
   
   with $y_2(t)/M = 1 + b[1 + \tanh ab(t - t_0)]; t_0 = 193.9$ [d]

2. $t_1 \leq t \leq t_{E1} = 352$ d: $M = 122$ [€] and $S_1 = 90$ [€] ($b_1 = 0.28$); $a = 2.0$ [%/d]
   
   with $y_{31}(t)/M_1 = 1 + b_1[1 + \coth a_1b_1(t - t_{01})]; t_{01} = 225.4$ [d]

3. $t_{E1} \leq t \leq t_{E2} = 364$ d: $M_2 = 90$ [€] and $S_2 = 60$ [€] ($b_2 = 0.17$); $a_2 = 3.5$ [%/d]
   
   with $y_{32}(t)/M_2 = 1 + b_2[1 + \coth a_2b_2(t - t_{02})]; t_{02} = 319.8$ [d]
Figure 6: Redemption price of the Frankfurt-Effekten-Fonds with simulation fittings for the period 07/10/1997 – 10/11/1998

4. $t_{E2} \leq t \leq 400 \text{ d}$: $M_3 = 110 \text{ [EUR]}$ and $S_3 = 166 \text{ [EUR]}$ ($b_3 = 0.25$); $a_3 = 14.0 \% / \text{d}$

    with $y_{23}(t)/M_3 = 1 + b_3[1 + \tanh a_3 b_3(t - t_{03})]$; $t_{03} = 367.4 \text{ [d]}$

In a first implemented variant the numerical calculations were realized in an interactive way: The time sequence to be fitted was demonstrated on the screen. In the following, solutions of the differential equation have been calculated for chosen parameters, for $g > 0$ numerically from equation (18), for $g = 0$ analytically via (4) or numerically from (18), and additionally presented on the screen. Then the parameters were varied until (visually!) best coincidence was achieved. This procedure works astonishingly robust, without any remarkable uncertainties. Therefore, there was no need for a mathematical fitting (possibly with assistance of the least-squares method).

The calibration procedure was realized similar to that explained above for the initial value problems A and B. The breaking off points were directly taken from every section border: For instance, the "triangle peak" time 290 [d] (date 23/07/1998) was chosen as $t_1$. The parameters (basic value $M$, target price $S$, and trading factor $a$) were modified in all sections until a sufficient mean approximation was achieved by the theoretical curves $y_2(t), y_{31}(t), y_{32}(t)$ and $y_{23}(t)$. The decreasing part of $W(t)$ had to be divided into two sections, a first with relatively moderate decrease caused by reduction of the target price...
and of the enhancement of the trading factor (trade velocity increase), and a second one (after \(t_{E1} = 352\) [d], date 23/09/1998) with again enhanced velocity and reduced \(M\) and \(S\), where \(S < M\). After \(t_{E2} = 364\) [d] target price and basic value raised up again, the last in such a way that it remained below the breaking off value \(y_{32}(t_{E2}) = 134.54\) [€]. Additionally, the trade factor \(a\) increased remarkably. In this way, the repeated growth of \(W(t)\) in the fourth interval could be explained.

7.2. Conclusions

The considered example demonstrates that in the investigated interval the time sequence of the Frankfurt-Effekten-Fonds can be described as a result of an ordinary trading process – with only four time sections of constant parameters, despite the relatively large duration of more than one year. This is really surprising! Using equations (1) and (2) being the base of all calculations in Section 7.1, a sufficiently precise curve fitting could be obtained. Obviously, the assertions about an ordinary trading process formulated here satisfy the conditions for buying and selling as they practically exist. Even an extension onto the whole time sequence seems to be possible, subdividing it into a series of initial value problems.

7.3. Outlook

A first result of such an extension towards more subsequent initial value problems with corresponding evaluations is given in Figure 7, again based on the time sequence from Figure 5. For the period 04/11/1997–04/04/2001 (as indicated in Figure 5), the fitting parameters of 28 initial value problems are used to calculate the stimulus velocity \(dK/dt\) at the beginning and at the end of every related time interval according to equation (1). In Figure 7 squares of these \(dK/dt\) values are plotted versus the time \(t\), illustrating the absolute share trader activity at the changes between the initial value problems (the thin lines connecting the points serve only for orientation). A surprising result becomes visible in the picture: the fitting results \((dK/dt)2\) were remarkably different for both maxima of the here selected time sequence part. Obviously, the traders acted “more relaxed” during the second maximum in comparison with the first. The reason is unknown till now, however, a look for further investigation of such effects seems to be useful.
8. Summary

The main driving forces of a share trade with their effects have been mathematically formulated, and relatively simple relations between them were derived. In this way, a description of temporal changes of share values could be obtained. An ordinary trading process was defined, not involving special actions ("disturbing" activities like "greediness" or "panic") and enabling simulations of real share value time sequences. Such a process can be described by a nonlinear first-order differential equation. This yields a model for the time dependent share value $W(t)$, in which the included parameters are well suited for charactering the behavior of traders. With the assumption of temporal invariable parameters, the equation becomes even analytically solvable in the correspondent time intervals. It was shown that typical time sequence courses can be modelled. The practical applicability of such simulations was also demonstrated calibrating the parameters of the considered functions to a real example.

Moreover, "disturbing" activities were taken into consideration, which can influence the buying stimulus acceleration during a general trading process. For this, again a simple mathematical supplement was chosen in the form of an additive term. This led to a nonlinear second-order differential equation. With it, actually, hints on an accelerated value increase or retarded value decrease could be found for a buying or selling process, respectively, always connected with the danger of process instabilities, like rate breakdowns.
Presumably, every time sequence can be divided into a series of sections with constant parameters. Treating such series as successive initial value problems, with corresponding differential equations in them and with initial values in each equal to the final value of the foregoing problem, the whole time sequence can be approximated, in principal. This enables for statistical evaluations of the corresponding parameters. The advantage of such a procedure consists in the possibility to interpret time sequences on the basis of measurable parameters characterizing the behavior of the traders. This also facilitates the forecast of future share values. Of course, the proposed approach cannot remove the most difficult problem for a prognosis: the well-timed recognition of “turn over points”, the dates at which trends begin to change.

References


