**International Journal of Pure and Applied Mathematics**
**Volume 68 No. 4 2011, 477-485**

# A GENERALIZATION OF Rad-SUPPLEMENTED MODULES

Burcu Nişancı Türkmen[1 §], Ali Pancar[2]
[1,2]Department of Mathematics
Faculty of Arts and Sciences
Ondokuz Mayıs University
Samsun, 55139, TURKEY
[1]e-mail: burcun@omu.edu.tr
[2]e-mail: apancar@omu.edu.tr

**Abstract:** We study some properties of (amply) f-Rad-supplemented modules and fwrs-modules as a proper generalization of (amply) Rad-supplemented. We show that: (1) any finite direct sum of finitely generated, projective f-Rad-supplemented modules is f-Rad-supplemented; (2) a locally Noetherian f-Rad-supplemented module is f-supplemented; (3) any generalized cover of a fwrs-module is fwrs.

**AMS Subject Classification:** 16D10, 16D99
**Key Words:** Rad-supplement, f-Rad-supplemented module

## 1. Introduction

Throughout this paper all rings $R$ are associative with identity element and all modules are unital left $R-$modules. Let $R$ be a ring and let $M$ be an $R$-module. The notation $N \leq M$ means that $N$ is a submodule of $M$. A submodule $N$ of a module $M$ is called *small* in $M$, denoted by $N \ll M$, if $N + L \neq M$ for every proper submodule $L$ of $M$, see [9]. By $\mathrm{Rad}(M)$, i.e. the Jacobson radical of $M$, we indicate the sum of all small submodules of $M$, see [9]. Let $M$ be an $R$-module and let $U$ and $K$ be any submodules of $M$. $K$ is called a *supplement* of $N$ in $M$ if $M = N + K$ and $N \cap K \ll K$. In this case we say that $N$ has a supplement in $M$. Following [9], $M$ is called *supplemented* if every submodule of $M$ has a supplement in $M$, and $M$ is called *finitely supplemented* or briefly *f-supplemente*d if every finitely generated submodule of $M$ has a supplement in

Received: February 26, 2011 
[§]Correspondence author

$M$. $M$ is called *amply supplemented* if, for any submodules $U$ and $V$ of $M$ with $M = U + V$, $V$ contains a supplement of $U$ in $M$. Similarly $M$ is called *amply f-supplemente*d if every finitely generated submodule of $M$ satisfies this condition. It is clear that (amply) f-supplemented modules are a proper generalization of (amply) supplemented modules.

Lomp [6] calls a module $M$ *semilocal* if $M/\mathrm{Rad}(M)$ is semisimple and a ring $R$ is called *semilocal* if the left (or right) $R$-module $R$ is semilocal. He show [6, Thorem 3.5] that $R$ is semilocal if and only if every left $R$-module is semilocal.

In [4, Theorem 10.14], another generalization of supplement submodule is called as Rad-*supplement* (according to [10], generalized supplement). For a modules $M$ and $N$, $K$ be any submodules with $M = N + K$, $K$ is called a Rad-supplement of $N$ in $M$ if $N \cap K \subseteq \mathrm{Rad}(K)$. $M$ is called Rad-*supplemented* (according to [10], generalized supplemented) if every submodule has a Rad-supplement in $M$, and $M$ is called *amply* Rad-*supplemented* (according to [10], amply generalized supplemented) in case $M = K + L$ implies that $K$ has a generalized supplement $L' \leq L$. In addition, it is shown [10, Proposition 2.5 and Proposition 2.6 (1)] that the class of Rad-supplemented modules is closed under finite sums and factor modules. Every supplemented module is Rad-supplemented but it is not generally true that every Rad-supplemented module is supplemented. Let $R$ be a non-local Dedekind domain with quotient field $K$. Then the left $R$-module $K$ is Rad-supplemented, but it is not supplemented. Let $M$ be a module. $M$ is called weakly Rad-supplemented if every submodule $U$ of $M$ has a weak Rad-supplement $V$ in $M$, i.e. $M = U + V$ and $U \cap V \subseteq \mathrm{Rad}(M)$ for some submodule $V$ of $M$. Clearly, Rad-supplemented modules and weakly supplemented modules are weakly Rad-supplemented. It is shown in [6, Proposition 2.1] that a module $M$ is semilocal if and only if the module is weakly Rad-supplemented.

This note consists of two sections. In Section 2, we introduce (amply) f-Rad-supplemented modules as a proper generalization of (amply) Rad-supplemented modules. We obtain various properties of such modules through known properties of (amply) f-supplemented. In Section 3, we study finitely weak Rad-supplemented modules which is a proper generalization of weakly Rad-supplemented modules.

## 2. f-Rad-Supplemented Modules and Amply f-Rad-Supplemented Modules

In this section, we define the concept of (amply) finitely Rad-supplemented modules, which is adapted from (amply) f-supplemented modules, and we give the properties of these modules.

**Definition 2.1.** Let $M$ be an $R$-module. $M$ is called *finitely* Rad-*supplemented* or *briefly f*-Rad-*supplemented* if every finitely generated submodule of $M$ has a Rad-supplement in $M$, and $M$ is called*amply f*-Rad-*supplemented* if every finitely generated submodule of $M$ has ample Rad-supplements in $M$.

It is clear that every Rad-supplemented module is f-Rad-supplemented and every amply Rad-supplemented module is amply f-Rad-supplemented. Also, Noetherian (amply) f-Rad-supplemented is (amply Rad-) supplemented. The following example shows that a f-Rad-supplemented module is not Rad-supplemented. Note that Von Neuman regular rings are f-Rad-supplemented.

**Example 2.2.** (see [2]) Let $F$ be any field. Consider the commutative ring $R$ which is the direct product $\Pi_{i=1}^{\infty} F_i$, where $F_i = F$. So $R$ is a Von Neuman regular ring which is not semisimple. Thus $R$ is f-Rad-supplemented. Since Rad-supplemented modules with zero radical is semisimple, $R$ is not Rad-supplemented.

Now we show some properties of (amply) f-Rad-supplemented modules.

We do not know whether the finite sum of f-Rad-supplemented modules is f-Rad-supplemented but we have the following theorem. Firstly we need to the following standard lemma.

**Lemma 2.3.** *Let $M$ be an $R$-module and let $U$, $M_1$ be submodules of $M$ such that $U$ is finitely generated, $M_1$ is f*-Rad-*supplemented. If $M_1 + U$ has a* Rad-*supplement $X$ in $M$ such that $M_1 \cap (U + X)$ is finitely generated and $M_1 \cap (U + X)$ has a* Rad-*supplement $Y$ in $M_1$, then $X + Y$ is a* Rad-*supplement of $U$ in $M$.*

*Proof.* Let $X$ be a Rad-supplement of $M_1+U$ in $M$. Then $M = (M_1+U)+X$ and $(M_1+U)\cap X \subseteq \mathrm{Rad}X$. By assumption, $M_1 \cap (U+X)$ is finitely generated submodule of $M_1$. Since $M_1$ is a f-Rad-supplemented module, $M_1 \cap (U+X)$ has a Rad-supplement $Y$ in $M_1$. Note that

$$M_1 = M_1 \cap (U + X) + Y$$

and

$$M_1 \cap (U + X) \cap Y \subseteq \mathrm{Rad}Y.$$

Then $M = U + X + Y$ and $U \cap (X + Y) \subseteq \mathrm{Rad}(X + Y)$. Thus $X + Y$ is a Rad-supplement of $U$ in $M$. □

**Theorem 2.4.** *Let $M$ be an $R$-module and $M = M_1 \oplus M_2$, where $M_1$, $M_2$ are finitely generated f–*Rad*-supplemented modules. If $M$ is a self projective module, $M$ is a f–*Rad*-supplemented module.*

*Proof.* Let $U$ be a finitely generated submodule of $M$. Since $M$ is a self projective module, $M_1$ and $M_2$ are $M$-projective. While $M_1$ is $M$-projective, $M_1$ is $(M_1 + U)$-projective for a short exact sequence

$0 \to M_1 + U \to M \to M/(M_1 + U) \to 0$ by [9, 18.2(1)].

Note that

$$M_1 \cong M/M_2 = (M_1 + U)/(M_2 \cap (M_1 + U))$$

and so $(M_1 + U)/(M_2 \cap (M_1 + U))$ is $(M_1 + U)$-projective. It follows that $M_2 \cap (M_1 + U)$ is a direct summand of $M_1 + U$. Then there exists a submodule $L$ of $M_1+U$ such that $(M_1+U)/L \cong M_2 \cap (M_1+U)$. It is clear that $M_2 \cap (M_1+U)$ is finitely generated. Since $M_2$ is f–Rad-supplemented module, $M_2 \cap (M_1 + U)$ has a Rad-supplement $X$ in $M_2$. By Lemma 2.3, $X$ is a Rad-supplement of $M_1 + U$ in $M$. Then $M = M_1 + U + X$, $(M_1 + U) \cap X \subseteq \mathrm{Rad}X$. It follows that $(M_1 + U) \cap X \subseteq \mathrm{Rad}M_2$. In addition $\mathrm{Rad}M_2 \ll M_2$ by [9, 21.6(4)]. Then $(M_1 + U) \cap X \ll M_2$. Since $X$ is a direct summand of $M_2$, we have $(M_1 + U) \cap X \ll X$. Note that $M/(M_1 + U) \cong X/((M_1 + U) \cap X)$. Since $X/((M_1 + U) \cap X)$ is finitely generated and $(M_1 + U) \cap X \ll X$, $X$ is finitely generated by [1, 16.12(1)]. While $M_2$ is $M$-projective, $M_2$ is $(X + U)$-projective for a short exact sequence $0 \to X + U \to M \to M/(X + U) \to 0$ by [9, 18.2(1)]. Similarly it is showed that $M_1 \cap (U+X)$ has a Rad-supplement $Y$ in $M_1$. Again by Lemma 2.3, $X + Y$ is a Rad-supplement of $U$ in $M$. Therefore $M$ is f-Rad-supplemented module. □

**Corollary 2.5.** *Suppose that finitely generated $R$-modules $M_1$, $M_2$, ...$M_n$ are projective f-*Rad*-supplemented and let $M = \oplus_{i=1}^{n} M_i$. Then $M$ is f-*Rad*-supplemented.*

**Lemma 2.6.** (see [8], Lemma 2.3) *Let $M$ be an $R$-module and $V$ be a* Rad*-supplement of $U$ in $M$. Then $(V + L)/L$ is a* Rad*-supplement of $U/L$ in $M/L$ for every submodule $L$ of $U$.*

**Proposition 2.7.** *Suppose that a submodule $L$ of a module $M$ is finitely generated. Then,*

(1) *If $M$ is a f-*Rad*-supplemented module, $M/L$ is f-*Rad*-supplemented.*

(2) *If $M$ is an amply f-*Rad*-supplemented module, $M/L$ is amply f-*Rad*-supplemented.*

*Proof.* (1) Let $K/L$ be a finitely generated submodule of $M/L$. Then $K/L =< \{k_1 + L, k_2 + L, ..., k_n + L\} >$ , $k_i \in K$ , $1 \leq i \leq n$ for some positive integer $n$. It follows that $K =< \{k_1, k_2, ..., k_n\} > +L$. It is clear that $K$ is a finitely generated submodule of $M$. Since $M$ is a f-Rad-supplemented module, $K$ has a Rad-supplement $N$ in $M$. By Lemma 2.6, $(N + L)/L$ is a Rad-supplement of $K/L$ in $M/L$. Therefore $M/L$ is a f-Rad-supplemented module.

(2) Let $U/L$ be a finitely generated submodule of $M/K$. Suppose that $M/L = U/L + V/L$ for some submodule $V/L$ of $M/L$. Then $M = U + V$. Since $U/L$ and $L$ are finitely generated, $U$ is finitely generated submodule of $M$. Since $M$ is an amply f-Rad-supplemented module, there exists a Rad-supplement $V'$ of $U$ with $V' \subseteq V$. Again by Lemma 2.6, $(V' + K)/K$ is a Rad-supplement of $U/K$ in $M/K$. In addition, $(V' + K)/K \subseteq V/K$. Therefore $M/K$ is an amply f-Rad-supplemented module. □

**Proposition 2.8.** *Suppose that a submodule $V$ of a module $M$ is a supplement of a finitely generated submodule $U$ of $M$. If $M$ is an amply f-*Rad-*supplemented module, then $V$ is an amply f-*Rad-*supplemented module.*

*Proof.* Let $X$ be a finitely generated submodule of $V$ and let $Y$ be a submodule of $V$ such that $V = X + Y$. By the hypothesis, we have $M = U + V$. It follows that $M = (U + X) + Y$. Since $M$ is an amply f-Rad-supplemented module, there exists a Rad-supplement $Y'$ of $U + X$ with $Y' \subseteq Y$. Then $M = U + X + Y'$ and $(U + X) \cap Y' \subseteq \mathrm{Rad}Y'$. Note that

$$V = (U \cap V) + (X + Y').$$

Since $U \cap V \ll V$, we have $V = X + Y'$ and $X \cap Y' \subseteq \mathrm{Rad}Y'$. Therefore $V$ is an amply f-Rad-supplemented module. □

**Corollary 2.9.** *Suppose that a finitely generated $R$-module $M$ is amply f-*Rad-*supplemented. Then, every direct summand of $M$ is amply f-*Rad-*supplemented.*

The following lemma is well known.

**Lemma 2.10.** *Let $M$ be a module. Suppose that a finitely generated submodule $U$ of $M$ is contained in* $\mathrm{Rad}(M)$. *Then $U$ is a small submodule of $M$.*

Recall from [9] that a module $M$ is called *locally Noetherian* if every finitely generated submodule is Noetherian. Note that over a Noetherian ring every module is locally Noetherian.

**Theorem 2.11.** *Suppose that an $R$-module $M$ is locally Noetherian. If $M$ is (amply) f-*Rad-*supplemented, then $M$ is (amply) f-supplemented.*

*Proof.* Let $M$ be a f-Rad-supplemented module and let $U$ be a finitely submodule of $M$. Then, there exists a submodule $V$ of $M$ such that $M = U+V$ and $U \cap V \subseteq \mathrm{Rad}V$. By the hypothesis, $U$ is Noetherian. Then $U \cap V$ is finitely generated and so $U \cap V \ll V$ by Lemma 2.10. Thus $V$ is a supplement of $U$ in $M$, as required. As similar argument shows that $M$ is also an amply f-supplemented module. □

**Corollary 2.12.** *Let $R$ be a Noetherian ring and let $M$ be an $R$-module. If $M$ is (amply) f-*Rad*-supplemented, then $M$ is (amply) f-supplemented.*

**Proposition 2.13.** *Let $M$ be a f-*Rad*-supplemented module and let $N$ be a submodule of $M$ with $N \cap \mathrm{Rad}(M) = 0$. Then $N$ is regular. In particular, If* $\mathrm{Rad}(M) = 0$, *then $M$ is regular.*

*Proof.* Let $K$ be any finitely generated submodule of $N$. Since $M$ is a f-Rad-supplemented module, then $M = K + L$, $K \cap L \subseteq \mathrm{Rad}L$. Note that $N = K + (N \cap L)$ and $K \cap (N \cap L) = 0$. Then $N = K \oplus (N \cap L)$. Hence $N$ is regular. If $\mathrm{Rad}M = 0$, $M$ is regular for $N = M$. □

**Corollary 2.14.** *Let $M$ be a f-*Rad*-supplemented module. Suppose that* $\mathrm{Rad}(M)$ *is finitely generated. Then* $M/\mathrm{Rad}(M)$ *is regular.*

*Proof.* Let $\overline{M} = M/\mathrm{Rad}(M)$. Then, by Proposition 2.7, $\overline{M}$ is f-Rad-supplemented. Note that $\mathrm{Rad}(\overline{M}) = 0$. Hence $\overline{M}$ is regular by Proposition 2.13. □

## 3. Finitely Weak Rad-Supplemented Modules

Recall from [1] that a module $M$ is called *finitely weak supplemented* or *briefly fws* if every finitely generated submodule of $M$ has a weak supplement in $M$. Motivated by this, we define the concept of finitely weak Rad-supplemented modules in this section.

**Definition 3.1.** Let $M$ be an $R$-module. $M$ is called *finitely weak* Rad-*supplemented* or *briefly fwrs* if every finitely generated submodule of $M$ has a weak Rad-supplement in $M$.

Clearly, both weakly Rad-supplemented modules and fws-modules are fwrs-modules. In addition, Example 2.2 also shows that a fwrs-module need not be weakly Rad-supplemented.

**Lemma 3.2.** (see [7], Lemma 2.1) *Let $M$ be an $R$-module and let $V$ be a weak* Rad-*supplement of $U$ in $M$. Then, $(V + L)/L$ is a weak* Rad-*supplement of $U/L$ in $M/L$ for every submodule $L$ of $U$.*

The following fact is a modification of Proposition 2.7.

**Proposition 3.3.** *Suppose that a submodule $L$ of a module $M$ is finitely generated. If $M$ is fwrs, then $M/L$ is fwrs.*

*Proof.* Let $K/L$ be a finitely generated submodule of $M/L$. Since $L$ is finitely generated, $K$ is finitely generated. Since $M$ is a fwrs-module, $K$ has a weak Rad-supplement $U$ in $M$. By Lemma 3.2, $(U + L)/L$ is a weak Rad-supplement $K/L$ in $M/L$. Hence $M/L$ is a fwrs-module. □

**Proposition 3.4.** *Let $M$ be an $R$-module. If $N \subseteq \mathrm{Rad}M$ and $M/N$ is a fwrs-module, then $M$ is fwrs.*

*Proof.* Let $U$ be a finitely generated submodule of $M$. Then $(U + N)/N$ is a finitely generated submodule of $M/N$. Since $M/N$ is a fwrs-module, we have $M/N = (U + N)/N + V/N$ and $(U + N)/N \cap V/N \subseteq \mathrm{Rad}(M/N)$. It follows that $M = U + V$, $U \cap V \subseteq U \cap V + N \subseteq \mathrm{Rad}M$. Hence $M$ is fwrs.

Recall from [11] that an epimorphism $\alpha : P \to M$ is called a *generalized cover* if $\ker \alpha \subseteq \mathrm{Rad}P$.

**Corollary 3.5.** *Let $M$ be a fwrs-module and let $f : K \twoheadrightarrow M$ be a generalized cover. Then $K$ is a fwrs-module.*

**Theorem 3.6.** *Let $M$ be a locally Noetherian module. If $M$ is fwrs, then $M$ is fws.*

*Proof.* Let $U$ be a finitely generated submodule of $M$. Since $M$ is a fwrs-module, there exists a submodule $V$ of $M$ such that $M = U + V$ and $U \cap V \subseteq \mathrm{Rad}M$. Since $M$ is a locally noetherian module, $U \cap V$ is a finitely generated. Then $U \cap V \ll M$ by Lemma 2.10.. Hence $M$ is fws. □

**Corollary 3.7.** *Let $R$ be a Noetherian ring and let $M$ be an $R$-module. Then $M$ is fwrs if and only if it is fws.*

**Proposition 3.8.** *Let $M$ be a module with small radical. Then $M$ is a fws-module if and only if it is fwrs.*

*Proof.* The necessity of the condition is obvious. Coversely, suppose that $M$ is fwrs. Then, for any submodule $U$ of $M$, $M = U + V$ and $U \cap V \subseteq \mathrm{Rad}M$ for some submodule $V$ of $M$. Since $\mathrm{Rad}M \ll M$, by [9, 21.5], $U \cap V \ll M$. □

**Theorem 3.9.** *Suppose that a submodule* $V$ of a module $M$ *is a* Rad-supplement in $M$. If $M$ is *fwrs, then* $V$ is a *fwrs-module.*

*Proof.* Let $V$ be a Rad-supplement of $U$ in $M$ and $K$ is a finitely generated submodule of $V$. Since $M$ is a fwrs-module, there exists a submodule $L$ of $M$ such that $M = K + L$, $K \cap L \subseteq \mathrm{Rad}M$. It follows that $V = K + (V \cap L)$ and $K \cap (V \cap L) \subseteq \mathrm{Rad}M$. Since $V$ is a Rad-supplement of $U$ in $M$, we have $K \cap (V \cap L) \subseteq \mathrm{Rad}V$. Thus $V$ is a fwrs-module. □

A ring $R$ is called a *left V-ring* if every simple left $R$-module is injective. It is well known that $R$ is V-ring if and only if, for every left $R$-module $M$, $\mathrm{Rad}(M) = 0$. This fact gives the following corollary which is obvious.

**Corollary 3.10.** *Let $R$ be a left V-ring and let $M$ be an $R$-module. Then the following statements are equivalent.*

(1) *$M$ is f-*Rad*-supplemented.*

(2) *$M$ is f-supplemented.*

(3) *$M$ is fws.*

(4) *$M$ is fwrs.*

(5) *$M$ is regular.*

## References

[1] R. Alizade, E. Büyükaşık, Cofinitely weak supplemented modules, *Comm. Algebra*, **31** (2003), 5377-5390.

[2] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, Berlin-Heidelberg-New York (1988).

[3] E. Büyükaşık, C. Lomp, Ring whose modules are weakly supplemented are perfect. Applications to certain ring extensions, *Math. Scand.*, **105** (2009), 25-30.

[4] J. Clark, C. Lomp, N. Vajana, R. Wisbauer, *Lifting Modules, Supplements and Projectivity in Module Theory*, Birkhäuser, Basel (2006).

[5] F. Kasch, *Modules and Rings*, Academic Press, San Diego (1982).

[6] C. Lomp, Semilocal modules and rings, *Comm. Algebra*, **27**, No. 4 (1999), 1921-1935.

[7] B. Nişancı, E. Türkmen, A. Pancar, Completely weak Rad-supplemented modules, *International Journal of Computational Cognition*, **7**, No. 2 (2009).

[8] E. Türkmen, A. Pancar, On cofinitely Rad-supplemented modules, *International Journal of Pure and Applied Mathematics*, **53**, No. 2 (2009) 153-162.

[9] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach Philadelphia (1991).

[10] Y. Wang, N. Ding, Generalized supplemented modules, *Taiwanese J. Math.*, **10**, No. 6 (2006), 1589-1601.

[11] W. Xue, Characterizations of semiperfect and perfect rings, *Publ. Mat.*, **40** (1996), 115-125.