A GENERALIZATION OF Rad-SUPPLEMENTED MODULES

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Abstract: We study some properties of (amply) f-Rad-supplemented modules and fwrs-modules as a proper generalization of (amply) Rad-supplemented. We show that: (1) any finite direct sum of finitely generated, projective f-Rad-supplemented modules is f-Rad-supplemented; (2) a locally Noetherian f-Rad-supplemented module is f-supplemented; (3) any generalized cover of a fwrs-module is fwrs.

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1. Introduction

Throughout this paper all rings $R$ are associative with identity element and all modules are unital left $R$–modules. Let $R$ be a ring and let $M$ be an $R$-module. The notation $N \leq M$ means that $N$ is a submodule of $M$. A submodule $N$ of a module $M$ is called small in $M$, denoted by $N \ll M$, if $N + L \neq M$ for every proper submodule $L$ of $M$, see [9]. By $\text{Rad}(M)$, i.e. the Jacobson radical of $M$, we indicate the sum of all small submodules of $M$, see [9]. Let $M$ be an $R$-module and let $U$ and $K$ be any submodules of $M$. $K$ is called a supplement of $N$ in $M$ if $M = N + K$ and $N \cap K \ll K$. In this case we say that $N$ has a supplement in $M$. Following [9], $M$ is called supplemented if every submodule of $M$ has a supplement in $M$, and $M$ is called finitely supplemented or briefly $f$-supplemented if every finitely generated submodule of $M$ has a supplement in

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$M$. $M$ is called amply supplemented if, for any submodules $U$ and $V$ of $M$ with $M = U + V$, $V$ contains a supplement of $U$ in $M$. Similarly $M$ is called amply f-supplemented if every finitely generated submodule of $M$ satisfies this condition. It is clear that (amply) f-supplemented modules are a proper generalization of (amply) supplemented modules.

Lomp [6] calls a module $M$ semilocal if $M/\text{Rad}(M)$ is semisimple and a ring $R$ is called semilocal if the left (or right) $R$-module $R$ is semilocal. He show [6, Theorem 3.5] that $R$ is semilocal if and only if every left $R$-module is semilocal.

In [4, Theorem 10.14], another generalization of supplement submodule is called as Rad-supplement (according to [10], generalized supplement). For a modules $M$ and $N$, $K$ be any submodules with $M = N + K$, $K$ is called a Rad-supplement of $N$ in $M$ if $N \cap K \subseteq \text{Rad}(K)$. $M$ is called Rad-supplemented (according to [10], amplified generalized supplemented) if every submodule has a Rad-supplement in $M$, and $M$ is called amply Rad-supplemented (according to [10], amply generalized supplemented) in case $M = K + L$ implies that $K$ has a generalized supplement $L' \leq L$. In addition, it is shown [10, Proposition 2.5 and Proposition 2.6 (1)] that the class of Rad-supplemented modules is closed under finite sums and factor modules. Every supplemented module is Rad-supplemented but it is not generally true that every Rad-supplemented module is supplemented. Let $R$ be a non-local Dedekind domain with quotient field $K$. Then the left $R$-module $K$ is Rad-supplemented, but it is not supplemented. Let $M$ be a module. $M$ is called weakly Rad-supplemented if every submodule $U$ of $M$ has a weak Rad-supplement $V$ in $M$, i.e. $M = U + V$ and $U \cap V \subseteq \text{Rad}(M)$ for some submodule $V$ of $M$. Clearly, Rad-supplemented modules and weakly supplemented modules are weakly Rad-supplemented. It is shown in [6, Proposition 2.1] that a module $M$ is semilocal if and only if the module is weakly Rad-supplemented.

This note consists of two sections. In Section 2, we introduce (amply) f-Rad-supplemented modules as a proper generalization of (amply) Rad-supplemented modules. We obtain various properties of such modules through known properties of (amply) f-supplemented. In Section 3, we study finitely weak Rad-supplemented modules which is a proper generalization of weakly Rad-supplemented modules.
2. f-Rad-Supplemented Modules and Amply f-Rad-Supplemented Modules

In this section, we define the concept of (amply) finitely Rad-supplemented modules, which is adapted from (amply) f-supplemented modules, and we give the properties of these modules.

**Definition 2.1.** Let $M$ be an $R$-module. $M$ is called **finitely Rad-supplemented** or briefly **f-Rad-supplemented** if every finitely generated submodule of $M$ has a Rad-supplement in $M$, and $M$ is called **amply f-Rad-supplemented** if every finitely generated submodule of $M$ has ample Rad-supplements in $M$.

It is clear that every Rad-supplemented module is f-Rad-supplemented and every amply Rad-supplemented module is amply f-Rad-supplemented. Also, Noetherian (amply) f-Rad-supplemented is (amply Rad-) supplemented. The following example shows that a f-Rad-supplemented module is not Rad-supplemented. Note that Von Neuman regular rings are f-Rad-supplemented.

**Example 2.2.** (see [2]) Let $F$ be any field. Consider the commutative ring $R$ which is the direct product $\prod_{i=1}^{\infty} F$, where $F_i = F$. So $R$ is a Von Neuman regular ring which is not semisimple. Thus $R$ is f-Rad-supplemented.

Since Rad-supplemented modules with zero radical is semisimple, $R$ is not Rad-supplemented.

Now we show some properties of (amply) f-Rad-supplemented modules.

We do not know whether the finite sum of f-Rad-supplemented modules is f-Rad-supplemented but we have the following theorem. Firstly we need to the following standard lemma.

**Lemma 2.3.** Let $M$ be an $R$-module and let $U$, $M_1$ be submodules of $M$ such that $U$ is finitely generated, $M_1$ is f-Rad-supplemented. If $M_1 + U$ has a Rad-supplement $X$ in $M$ such that $M_1 \cap (U + X)$ is finitely generated and $M_1 \cap (U + X)$ has a Rad-supplement $Y$ in $M_1$, then $X + Y$ is a Rad-supplement of $U$ in $M$.

**Proof.** Let $X$ be a Rad-supplement of $M_1 + U$ in $M$. Then $M = (M_1 + U) + X$ and $(M_1 + U) \cap X \subseteq \text{Rad}X$. By assumption, $M_1 \cap (U + X)$ is finitely generated submodule of $M_1$. Since $M_1$ is a f-Rad-supplemented module, $M_1 \cap (U + X)$ has a Rad-supplement $Y$ in $M_1$. Note that

$$M_1 = M_1 \cap (U + X) + Y$$

and

$$M_1 \cap (U + X) \cap Y \subseteq \text{Rad}Y.$$
Then \( M = U + X + Y \) and \( U \cap (X + Y) \subseteq \text{Rad}(X + Y) \). Thus \( X + Y \) is a Rad-supplement of \( U \) in \( M \).

**Theorem 2.4.** Let \( M \) be an \( R \)-module and \( M = M_1 \oplus M_2 \), where \( M_1, M_2 \) are finitely generated \( f \)-Rad-supplemented modules. If \( M \) is a self projective module, \( M \) is a \( f \)-Rad-supplemented module.

**Proof.** Let \( U \) be a finitely generated submodule of \( M \). Since \( M \) is a self projective module, \( M_1 \) and \( M_2 \) are \( M \)-projective. While \( M_1 \) is \( M \)-projective, \( M_1 \) is \((M_1 + U)\)-projective for a short exact sequence
\[
0 \to M_1 + U \to M \to M/(M_1 + U) \to 0 \text{ by [9, 18.2(1)]}.
\]

Note that
\[
M_1 \cong M/M_2 = (M_1 + U)/(M_2 \cap (M_1 + U))
\]
and so \((M_1 + U)/(M_2 \cap (M_1 + U))\) is \((M_1 + U)\)-projective. It follows that \( M_2 \cap (M_1 + U) \) is a direct summand of \( M_1 + U \). Then there exists a submodule \( L \) of \( M_1 + U \) such that \((M_1 + U)/L \cong M_2 \cap (M_1 + U)\). It is clear that \( M_2 \cap (M_1 + U) \) is finitely generated. Since \( M_2 \) is \( f \)-Rad-supplemented module, \( M_2 \cap (M_1 + U) \) has a Rad-supplement \( X \) in \( M_2 \). By Lemma 2.3, \( X \) is a Rad-supplement of \( M_1 + U \) in \( M \). Then \( M = M_1 + U + X \), \((M_1 + U) \cap X \subseteq \text{Rad}X \). It follows that \((M_1 + U) \cap X \subseteq \text{Rad}M_2 \). In addition \( \text{Rad}M_2 \ll M_2 \) by [9, 21.6(4)]. Then \((M_1 + U) \cap X \ll M_2 \). Since \( X \) is a direct summand of \( M_2 \), we have \((M_1 + U) \cap X \ll X \). Note that \( M/(M_1 + U) \cong X/((M_1 + U) \cap X) \). Since \( X/((M_1 + U) \cap X) \) is finitely generated and \((M_1 + U) \cap X \ll X \), \( X \) is finitely generated by [1, 16.12(1)]. While \( M_2 \) is \( M \)-projective, \( M_2 \) is \((X + U)\)-projective for a short exact sequence
\[
0 \to X + U \to M \to M/(X + U) \to 0 \text{ by [9, 18.2(1)]}.
\]
Similarly it is showed that \( M_1 \cap (U + X) \) has a Rad-supplement \( Y \) in \( M_1 \). Again by Lemma 2.3, \( X + Y \) is a Rad-supplement of \( U \) in \( M \). Therefore \( M \) is \( f \)-Rad-supplemented module.

**Corollary 2.5.** Suppose that finitely generated \( R \)-modules \( M_1, M_2, \ldots, M_n \) are projective \( f \)-Rad-supplemented and let \( M = \oplus_{i=1}^{n} M_i \). Then \( M \) is \( f \)-Rad-supplemented.

**Lemma 2.6.** (see [8], Lemma 2.3) Let \( M \) be an \( R \)-module and \( V \) be a Rad-supplement of \( U \) in \( M \). Then \((V + L)/L \) is a Rad-supplement of \( U/L \) in \( M/L \) for every submodule \( L \) of \( U \).

**Proposition 2.7.** Suppose that a submodule \( L \) of a module \( M \) is finitely generated. Then,

1. If \( M \) is a \( f \)-Rad-supplemented module, \( M/L \) is \( f \)-Rad-supplemented.
2. If \( M \) is an amply \( f \)-Rad-supplemented module, \( M/L \) is amply \( f \)-Rad-supplemented.
Proof. (1) Let $K/L$ be a finitely generated submodule of $M/L$. Then $K/L = \langle \{k_1 + L, k_2 + L, \ldots, k_n + L\} \rangle$, $k_i \in K$, $1 \leq i \leq n$ for some positive integer $n$. It follows that $K = \langle \{k_1, k_2, \ldots, k_n\} \rangle + L$. It is clear that $K$ is a finitely generated submodule of $M$. Since $M$ is a f-Rad-supplemented module, $K$ has a Rad-supplement $N$ in $M$. By Lemma 2.6, $(N + L)/L$ is a Rad-supplement of $K/L$ in $M/L$. Therefore $M/L$ is a f-Rad-supplemented module.

(2) Let $U/L$ be a finitely generated submodule of $M/K$. Suppose that $M/L = U/L + V/L$ for some submodule $V/L$ of $M/L$. Then $M = U + V$. Since $U/L$ and $L$ are finitely generated, $U$ is finitely generated submodule of $M$. Since $M$ is an amply f-Rad-supplemented module, there exists a Rad-supplement $V'$ of $U$ with $V' \subseteq V$. Again by Lemma 2.6, $(V' + K)/K$ is a Rad-supplement of $U/K$ in $M/K$. In addition, $(V' + K)/K \subseteq V/K$. Therefore $M/K$ is an amply f-Rad-supplemented module.

Proposition 2.8. Suppose that a submodule $V$ of a module $M$ is a supplement of a finitely generated submodule $U$ of $M$. If $M$ is an amply f-Rad-supplemented module, then $V$ is an amply f-Rad-supplemented module.

Proof. Let $X$ be a finitely generated submodule of $V$ and let $Y$ be a submodule of $V$ such that $V = X + Y$. By the hypothesis, we have $M = U + V$. It follows that $M = (U + X) + Y$. Since $M$ is an amply f-Rad-supplemented module, there exists a Rad-supplement $Y'$ of $U + X$ with $Y' \subseteq Y$. Then $M = U + X + Y'$ and $(U + X) \cap Y' \subseteq \text{Rad}Y'$. Note that $V = (U \cap V) + (X + Y')$.

Since $U \cap V \ll V$, we have $V = X + Y'$ and $X \cap Y' \subseteq \text{Rad}Y'$. Therefore $V$ is an amply f-Rad-supplemented module.

Corollary 2.9. Suppose that a finitely generated $R$-module $M$ is amply f-Rad-supplemented. Then, every direct summand of $M$ is amply f-Rad-supplemented.

The following lemma is well known.

Lemma 2.10. Let $M$ be a module. Suppose that a finitely generated submodule $U$ of $M$ is contained in $\text{Rad}(M)$. Then $U$ is a small submodule of $M$.

Recall from [9] that a module $M$ is called locally Noetherian if every finitely generated submodule is Noetherian. Note that over a Noetherian ring every module is locally Noetherian.

Theorem 2.11. Suppose that an $R$-module $M$ is locally Noetherian. If $M$ is (amply) f-Rad-supplemented, then $M$ is (amply) f-supplemented.
Proof. Let $M$ be a $f$-$\text{Rad}$-supplemented module and let $U$ be a finitely submodule of $M$. Then, there exists a submodule $V$ of $M$ such that $M = U + V$ and $U \cap V \subseteq \text{Rad}V$. By the hypothesis, $U$ is Noetherian. Then $U \cap V$ is finitely generated and so $U \cap V \ll V$ by Lemma 2.10. Thus $V$ is a supplement of $U$ in $M$, as required. As similar argument shows that $M$ is also an amply $f$-supplemented module.

Corollary 2.12. Let $R$ be a Noetherian ring and let $M$ be an $R$-module. If $M$ is (amply) $f$-$\text{Rad}$-supplemented, then $M$ is (amply) $f$-supplemented.

Proposition 2.13. Let $M$ be a $f$-$\text{Rad}$-supplemented module and let $N$ be a submodule of $M$ with $N \cap \text{Rad}(M) = 0$. Then $N$ is regular. In particular, if $\text{Rad}(M) = 0$, then $M$ is regular.

Proof. Let $K$ be any finitely generated submodule of $N$. Since $M$ is a $f$-$\text{Rad}$-supplemented module, then $M = K + L$, $K \cap L \subseteq \text{Rad}L$. Note that $N = K + (N \cap L)$ and $K \cap (N \cap L) = 0$. Then $N = K \oplus (N \cap L)$. Hence $N$ is regular. If $\text{Rad}M = 0$, $M$ is regular for $N = M$.

Corollary 2.14. Let $M$ be a $f$-$\text{Rad}$-supplemented module. Suppose that $\text{Rad}(M)$ is finitely generated. Then $M/\text{Rad}(M)$ is regular.

Proof. Let $\overline{M} = M/\text{Rad}(M)$. Then, by Proposition 2.7, $\overline{M}$ is $f$-$\text{Rad}$-supplemented. Note that $\text{Rad}(\overline{M}) = 0$. Hence $\overline{M}$ is regular by Proposition 2.13.

3. Finitely Weak Rad-Supplemented Modules

Recall from [1] that a module $M$ is called \textit{finitely weak supplemented} or briefly \textit{fws} if every finitely generated submodule of $M$ has a weak supplement in $M$. Motivated by this, we define the concept of finitely weak Rad-supplemented modules in this section.

Definition 3.1. Let $M$ be an $R$-module. $M$ is called \textit{finitely weak} $\text{Rad}$-supplemented or briefly \textit{fwrs} if every finitely generated submodule of $M$ has a weak $\text{Rad}$-supplement in $M$.

Clearly, both weakly $\text{Rad}$-supplemented modules and fws-modules are fwrs-modules. In addition, Example 2.2 also shows that a fwrs-module need not be weakly $\text{Rad}$-supplemented.

Lemma 3.2. (see [7], Lemma 2.1) Let $M$ be an $R$-module and let $V$ be a weak $\text{Rad}$-supplement of $U$ in $M$. Then, $(V + L)/L$ is a weak $\text{Rad}$-supplement of $U/L$ in $M/L$ for every submodule $L$ of $U$.

The following fact is a modification of Proposition 2.7.
Proposition 3.3. Suppose that a submodule $L$ of a module $M$ is finitely generated. If $M$ is fwrs, then $M/L$ is fwrs.

Proof. Let $K/L$ be a finitely generated submodule of $M/L$. Since $L$ is finitely generated, $K$ is finitely generated. Since $M$ is a fwrs-module, $K$ has a weak Rad-supplement $U$ in $M$. By Lemma 3.2, $(U + L)/L$ is a weak Rad-supplement $K/L$ in $M/L$. Hence $M/L$ is a fwrs-module.

Proposition 3.4. Let $M$ be an $R$-module. If $N \subseteq \text{Rad} M$ and $M/N$ is a fwrs-module, then $M$ is fwrs.

Proof. Let $U$ be a finitely generated submodule of $M$. Then $(U + N)/N$ is a finitely generated submodule of $M/N$. Since $M/N$ is a fwrs-module, we have $M/N = (U + N)/N + V/N$ and $(U + N)/N \cap V/N \subseteq \text{Rad}(M/N)$. It follows that $M = U + V$, $U \cap V \subseteq U \cap V + N \subseteq \text{Rad} M$. Hence $M$ is fwrs.

Recall from [11] that an epimorphism $\alpha : P \to M$ is called a generalized cover if $\ker \alpha \subseteq \text{Rad} P$.

Corollary 3.5. Let $M$ be a fwrs-module and let $f : K \rightarrow M$ be a generalized cover. Then $K$ is a fwrs-module.

Theorem 3.6. Let $M$ be a locally Noetherian module. If $M$ is fwrs, then $M$ is fws.

Proof. Let $U$ be a finitely generated submodule of $M$. Since $M$ is a fwrs-module, there exists a submodule $V$ of $M$ such that $M = U + V$ and $U \cap V \subseteq \text{Rad} M$. Since $M$ is a locally noetherian module, $U \cap V$ is a finitely generated. Then $U \cap V \ll M$ by Lemma 2.10. Hence $M$ is fws.

Corollary 3.7. Let $R$ be a Noetherian ring and let $M$ be an $R$-module. Then $M$ is fwrs if and only if it is fws.

Proposition 3.8. Let $M$ be a module with small radical. Then $M$ is a fws-module if and only if it is fwrs.

Proof. The necessity of the condition is obvious. Conversely, suppose that $M$ is fwrs. Then, for any submodule $U$ of $M$, $M = U + V$ and $U \cap V \subseteq \text{Rad} M$ for some submodule $V$ of $M$. Since $\text{Rad} M \ll M$, by [9, 21.5], $U \cap V \ll M$.

Theorem 3.9. Suppose that a submodule $V$ of a module $M$ is a Rad-supplement in $M$. If $M$ is fwrs, then $V$ is a fwrs-module.

Proof. Let $V$ be a Rad-supplement of $U$ in $M$ and $K$ is a finitely generated submodule of $V$. Since $M$ is a fwrs-module, there exists a submodule $L$ of $M$ such that $M = K + L$, $K \cap L \subseteq \text{Rad} M$. It follows that $V = K + (V \cap L)$ and $K \cap (V \cap L) \subseteq \text{Rad} M$. Since $V$ is a Rad-supplement of $U$ in $M$, we have $K \cap (V \cap L) \subseteq \text{Rad} V$. Thus $V$ is a fwrs-module.
A ring $R$ is called a left V-ring if every simple left $R$-module is injective. It is well known that $R$ is V-ring if and only if, for every left $R$-module $M$, $\text{Rad}(M) = 0$. This fact gives the following corollary which is obvious.

**Corollary 3.10.** Let $R$ be a left V-ring and let $M$ be an $R$-module. Then the following statements are equivalent.

1. $M$ is f-Rad-supplemented.
2. $M$ is $f$-supplemented.
3. $M$ is fws.
4. $M$ is fwrs.
5. $M$ is regular.

**References**


