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# NORMAL ULTRAFILTERS AND MAHLO RANK

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**Abstract:** It is straightforward that, if GCH holds then the Mahlo rank of an inaccessible cardinal $\kappa$ is less than $\kappa^{++}$. If it could be shown that the Mitchell order of a measurable cardinal is at most its Mahlo rank, it would follow that the Mitchell order is less than $\kappa^{++}$. It is shown that the Mitchell order is at most the Mahlo rank, if the Mitchell order is less than $\kappa^{+\omega}$.

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Let Inac denote the (strongly) inaccessible cardinals; $\kappa$ will invariably denote an inaccessible cardinal. Let $\mathrm{In}_\kappa$ denote $\mathrm{Inac}\cap\kappa$. For $X \subseteq \mathrm{In}_\kappa$ let $\mathrm{H}(X) = \{\lambda \in \mathrm{In}_\kappa : \lambda \in X$ and $X \cap \lambda$ is stationary$\}$. For $X, Y \subseteq \mathrm{In}_\kappa$, let $X \subseteq_{\mathrm{t}} Y$ denote that $Y - X$ is thin. For stationary $X, Y \subseteq \mathrm{In}_\kappa$, let $X \succ Y$ denote that $X \subseteq_{\mathrm{t}} \mathrm{H}(Y)$. This relation is transitive and well-founded; let $\rho$ denote the rank function (the Mahlo rank), and let $\rho(\kappa)$ denote the height.

Let Meas denote the measurable cardinals; $\kappa$ will in fact usually be measurable. For $\kappa \in \mathrm{Meas}$, and $U, W$ normal ultrafilters on $\kappa$, let $U >_M W$ denote that $W \in \mathrm{Ult}_U(V)$ (where $\mathrm{Ult}_U(V)$ is the transitive collapse of the ultrapower $V^\kappa/U$ of the universe $V$ by the ultrafilter $U$). This relation is transitive and well-founded; let $o$ denote the rank function (the Mitchell order), and let $o(\kappa)$ denote the height.

Discussions of the preceding two rank functions may be found in [3].

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Although the following lemma is undoubtedly folklore, a proof will be given; a version is stated in [1].

**Lemma 1.** *If GCH holds then for any $\kappa \in$ Inac, $\rho(\kappa) < \kappa^{++}$.*

*Proof.* If $\rho(\kappa)$ were greater than or equal to $\kappa^{++}$, there would be a stationary set $X$ with $\rho(X) = \alpha$ for each ordinal $\alpha < \kappa^{++}$, and these sets are distinct. If GCH holds, there are at most $\kappa^+$ stationary sets. □

For the following, recall that an element of a normal ultrafilter on $\kappa$ is a stationary subset of $\kappa$. Also, $\mathrm{In}_\kappa$ is an element of any such ultrafilter.

**Conjecture 2.** *Suppose $\kappa \in$ Meas, $\alpha < \kappa^{++}$, and $U$ is a normal ultrafilter on $\kappa$ with $o(U) = \alpha$. Then there is a subset $S \subseteq \mathrm{In}_\kappa$ such that $S \in U$ and $\rho(S) \geq \alpha$.*

**Theorem 3.** *If Conjecture 2 holds then $o(\kappa) < \kappa^{++}$ for any $\kappa \in$ Meas.*

*Proof.* (1) By the conjecture, if GCH then there is no cardinal $\kappa$ with $o(\kappa) = \kappa^{++}$. (2) By core model theory, if there is a cardinal with $o(\kappa) = \kappa^{++}$ then Con(ZFC+GCH+"there is a cardinal with $o(\kappa) = \kappa^{++}$"). (3) By 1 and 2, if Con(ZFC) then there is no cardinal with $o(\kappa) = \kappa^{++}$. (4) If there is an inaccessible cardinal then Con(ZFC). (5) By 3 and 4, if there is a cardinal with $o(\kappa) = \kappa^{++}$ then there is no inaccessible cardinal. (6) A cardinal with $o(\kappa) = \kappa^{++}$ is inaccessible. (7) By 5 and 6, there is no cardinal with $o(\kappa) = \kappa^{++}$. □

Conjecture 2 is stronger than necessary, in that $S \in U$ is not required. A sequence of successively stronger conclusions from the hypothesis $o(U) = \alpha$ can be given, as follows.

C1. Conjecture 2.

C2. There is a sequence $\langle S_\beta : \beta \leq \alpha \rangle$ of elements of $U \cap \mathrm{Pow}(\mathrm{In}_\kappa)$, which is ascending in the order $\succ$.

C3. There is a sequence $\langle f_\beta : \beta \leq \alpha \rangle$ of functions $f_\beta : \kappa^+ \mapsto \kappa^{++}$, such that for all $\beta \geq \alpha$ the following hold.

1. $f_\beta$ represents $\beta$ in $V^\kappa/U$.
2. If $S_\beta = \{\lambda \in \mathrm{In}_\kappa : o(\lambda) \geq f_\beta(\lambda)\}$, then the sequence $\langle S_\beta \rangle$ is ascending in the order $\succ$.

C4. There is a sequence $\langle f_\beta : \beta \leq \alpha \rangle$ of functions $f_\beta : \kappa^+ \mapsto \kappa^{++}$, such that for all $\beta \leq \alpha$ the following hold.

1. $f_\beta$ represents $\beta$ in $V^\kappa/U$.
2. If $\beta < \alpha$ then $f_{\beta+1} = f_\beta + 1$.

3. Except for a thin set of $\lambda$, there is a normal ultrafilter $U'$ on $\lambda$, of Mitchell order $f_\beta(\lambda)$, such that $f_\beta \restriction \lambda$ represents $f_\beta(\lambda)$ in $V^\lambda/U'$.

C5. There is a sequence $\langle f_\beta : \beta \leq \alpha\rangle$ of functions $f_\beta : \kappa^+ \mapsto \kappa^{++}$, such that for any normal ultrafilter $U$ on $\kappa$, for all $\beta \geq \alpha$, clauses 1 and 2 as in C4 hold, and also the following.

3. Except for a thin set of $\lambda$, there is a normal ultrafilter $U'$ on $\lambda$, of Mitchell order $f_\beta(\lambda)$, and for any such, $f_\beta \restriction \lambda$ represents $f_\beta(\lambda)$ in $V^\lambda/U'$.

**Lemma 4.** *C5 ⇒ C4 ⇒ C3 ⇒ C2 ⇒ C1.*

*Proof.* Only C4 ⇒ C3 requires any proof. Recall Lemma 19.34 of [3]: If $U$ is a normal ultrafilter on $\kappa$ then $o$ represents $o(U)$ in $V^\kappa/U$ (and $o(U) = o^{\mathrm{Ult}_U(V)}(\kappa)$). Let $S^=_\beta = \{\lambda \in \mathrm{In}_\kappa : o(\lambda) = f_\beta(\lambda)\}$; then $S^=_\alpha \in U$, whence $S_\beta \in U$ for $\beta \leq \alpha$. Since $\langle S_\beta\rangle$ is ascending in the order $\subseteq$, it suffices to show that if $\beta < \alpha$ then $S_{\beta+1} \subseteq_{\mathrm{t}} \mathrm{H}(S_\beta)$, for which it suffices to show that if $\lambda \in S_{\beta+1}$ then, except for a thin set of $\lambda$, $S_\beta \cap \lambda$ is stationary. By the hypotheses of C4, except for a thin set of $\lambda$, $S_\beta \cap \lambda \in U'$. □

Using schemes and canonical functions (q.v. see [2]), Lemma 4 can be used to show that Conjecture 2 holds for $\alpha < \kappa^+$. For a scheme $\Sigma$, let $f_\Sigma$ denote the canonical function determined by $\Sigma$ (i.e., where the increasing sequence used at the limit ordinal $\alpha$ is $\phi(\alpha)$). The notions of the exception set $T_\Sigma$ and the induced scheme $\Sigma \downarrow \lambda$ for $\lambda \notin T_\Sigma$ will be required, as well as the prefix $\Sigma_{\leq\alpha}$ of the scheme $\Sigma$.

**Lemma 5.** *Suppose $\lambda \notin T_\Sigma$.*
1. *$f_\Sigma \restriction \lambda = f_{\Sigma\downarrow\lambda}$.*
2. *$f_\Sigma(\lambda)$ equals the rank of $\Sigma \downarrow \lambda$.*

*Proof.* Part 1 is proved by induction on $\alpha \leq \sigma$, where $\sigma$ is the rank of $\Sigma$, as in [2]. Suppose $\gamma < \lambda$. In case 0 ($\alpha = 0$), $f_{\Sigma_{\leq 0}}(\gamma) = 0 = f_{\Sigma_{\leq 0}\downarrow\lambda}(\gamma)$. In case 1 ($\alpha = \beta + 1$), $f_{\Sigma_{\leq\alpha}}(\gamma) = f_{\Sigma_{\leq\beta}}(\gamma) + 1 = f_{\Sigma_{\leq\beta}\downarrow\lambda}(\gamma) + 1 = f_{\Sigma_{\leq\alpha}\downarrow\lambda}(\gamma)$. In case 2 ($\alpha = \cup_{\xi<\eta}\alpha_\xi$), $f_{\Sigma_{\leq\alpha}}(\gamma) = \sup_{\xi<\eta} f_{\Sigma_{\leq\alpha_\xi}}(\gamma) = \sup_{\xi<\eta} f_{\Sigma_{\leq\alpha_\xi}\downarrow\lambda}(\gamma) = f_{\Sigma_{\leq\alpha}\downarrow\lambda}(\gamma)$. In case 3 ($\alpha = \cup_{\xi<\kappa}\alpha_\xi$), $f_{\Sigma_{\leq\alpha}}(\gamma) = \mathrm{dsup}_{\xi<\kappa}\mathrm{f}_{\Sigma_{\leq\alpha_\xi}}(\gamma) = \mathrm{dsup}_{\xi<\lambda}\mathrm{f}_{\Sigma_{\leq\alpha_\xi}\downarrow\lambda}(\gamma) = f_{\Sigma_{\leq\alpha}\downarrow\lambda}(\gamma)$. Part 2 follows by a similar induction. □

**Lemma 6.** *Suppose $U$ is a normal ultrafilter on $\kappa$. Suppose $\Sigma$ is a scheme in $\kappa$ of rank $\sigma$. Then in $\mathrm{Ult}_U(V)$, $f_\Sigma$ represents $\sigma$.*

*Proof.* The proof is by induction on $\alpha \leq \sigma$. In case 0, $\{\gamma < \kappa : f_0(\gamma) = 0\} = \kappa \in U$. In case 1, if $[f] < [f_\alpha]$ then $[f] \leq [f_\beta]$, whence inductively $[f] = [f_\gamma]$

for some $\gamma \leq \beta$. In case 2, if $[f] < [f_\alpha]$ then by $\kappa$-completeness $[f] < [f_{\alpha_\xi}]$ for some $\xi$, whence inductively $[f] = [f_\gamma]$ for some $\gamma < \alpha_\xi$. In case 3, if $[f] < [f_\alpha]$ then by normality $[f] < [f_{\alpha_\xi}]$ for some $\xi$, whence inductively $[f] = [f_\gamma]$ for some $\gamma < \alpha_\xi$. $\square$

**Theorem 7.** *Suppose $\Sigma$ is a scheme of rank $\alpha$; then condition C5 holds for $\langle f_{\Sigma_{\leq\beta}} : \beta \leq \alpha\rangle$.*

*Proof.* Suppose $\beta < \alpha$ and $\lambda \notin T_{\Sigma_{\leq\beta}}$. Let $\theta = f_{\Sigma_{\leq\beta}}(\lambda)$, which equals the rank of $\Sigma_{\leq\beta\downarrow\lambda}$. Since $\lambda \in S_{\beta+1}$, $o(\lambda) > \theta$, and so there is a $U'$ on $\lambda$ with $o(U') = \theta$. For any such, $f_{\Sigma_{\leq\beta}} \restriction \lambda$ represents $\theta$ in $V^\lambda/U'$. $\square$

In particular, conjecture 2 holds if $\alpha < \kappa^+$. For the remainder of the paper, the methods given so far will be expanded, to show that condition C5 holds for $\alpha < \kappa^{+\omega}$.

**Lemma 8.** *Suppose, in $V^\kappa/U$, that $f$ represents $\alpha$ and $g$ represents $\beta$. Then $f + g$ represents $\alpha + \beta$.*

*Proof.* If $[h] < [f]$ then $h$ represents an ordinal less than $\alpha$ by hypothesis. Otherwise, $h(\alpha) \geq f(\alpha)$ on an element of $U$; call it $X$. Let $Y \in U$ be such that $h(\alpha) < f(\alpha) + g(\alpha)$ on $Y$. Then there is an $h'$ such that $f(\alpha) + h'(\alpha) < f(\alpha) + g(\alpha)$ on $X \cap Y$, whence $h'(\alpha) < g(\alpha)$ on $X \cap Y$. $\square$

Let $\langle \Sigma_\alpha : \alpha < \kappa^+\rangle$ be a chain of schemes; such is determined by choosing an ascending sequence for each limit ordinal $\alpha < \kappa^+$. For $\alpha < \kappa^+$ let $f_\alpha$ denote $f_{\Sigma_\alpha}$, and similarly for $T_\alpha$ and $\alpha \downarrow \lambda$.

**Lemma 9.** *Suppose, in $V^\kappa/U$, that $g$ represents $\beta$, and $\alpha < \kappa^+$. Then $g \cdot f_\alpha$ represents $\beta \cdot \alpha$.*

*Proof.* The proof is by induction on $\alpha$ as usual. The case $\alpha = 0$ is immediate. If $\alpha = \gamma + 1$, suppose $[h] < [g] \cdot [f_\gamma]$; using the induction hypothesis and preceding as in the proof of Lemma 8, it follows that $h$ represents an ordinal less than $\beta \cdot \alpha$. The remaining two cases are similar to the corresponding cases of Lemma 6. $\square$

**Lemma 10.** *Let $f_{\kappa^{+n}}$ denote $\lambda \mapsto \lambda^{+n}$; then $f_{\kappa^{+n}}$ represents $\kappa^{+n}$ in $V^\kappa/U$ for any normal ultrafilter $U$ on $\kappa$.*

*Proof.* This follows by Los' Theorem and the fact that the identity function represents $\kappa$. $\square$

**Theorem 11.** *If* $\beta = \sum_{0\le i\le n} \kappa^{+i} \cdot \alpha_i$ *let* $f_\beta = \sum_{0\le i\le n} f_{\kappa^{+i}} \cdot f_{\alpha_i}$. *Then* $\langle f_\beta : \beta < \kappa^{+\omega}\rangle$ *satisfies condition C5.*

*Proof.* That $f_\beta$ represents $\beta$ follows by Lemmas 8–10. That $f_{\beta+1} = f_\beta + 1$ is immediate. Let $T_\beta = \cup_i T_{\alpha_i}$; for $\lambda \notin T_\beta$ let $\beta \downarrow \lambda$ equals $\sum_{0\le i\le n} \lambda^{+i} \cdot (\alpha_i \downarrow \lambda)$. Clearly, for $\lambda \notin T_\beta$, $f_\beta \restriction \lambda = f_{\beta\downarrow\lambda}$ and $f_\beta(\lambda) = \beta \downarrow \lambda$. □

A few concluding remarks are as follows.

- Undoubtedly the bound for condition C5 can be increased further, but this will be left to further research.
- Even if the value is less than $\kappa^{++}$, the ordinals up to which conditions C1 to C5 hold are of interest.
- It is a question of interest, for what $\alpha$ does there exist $f$ such that $f$ represents $\alpha$ in $V^\kappa/U$ for any normal ultrafilter $U$?
- Some results on representing functions can be found in [4], given additional assumptions.

## References

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