International Journal of Pure and Applied Mathematics
Volume 69 No. 1 2011, 1-14

A GALERKIN'S PERTURBATION TYPE METHOD TO APPROXIMATE A FIXED POINT OF A COMPACT OPERATOR

L. Grammont
Laboratoire de Mathématiques
l'Université de St-Étienne
EA3989, Membre d'Université de Lyon
23, Rue Dr. Paul Michelon, St-Étienne, 42023, FRANCE
e-mail: laurence.grammont@univ-st-etienne.fr

Abstract: We propose a fixed point approximation of a compact operator. It is adapted from the method proposed by R.P. Kulkarni for linear operator equations. It is proved to be superconvergent, while the iterated Galerkin method, proposed by K.E. Atkinson and F.A. Potra, needs an additional assumption in order to be superconvergent.

AMS Subject Classification: 41A35, 47H10, 47J25
Key Words: fixed point equation, superconvergence, Galerkin approximation, iterated Galerkin approximation

1. Introduction

Consider the nonlinear equation

$$x = \mathcal{K}(x), \tag{1}$$

where $\mathcal{K}$ is a *compact* operator defined on $\overline{\mathcal{O}}$, $\mathcal{O}$ being an open set of the Banach space $\mathcal{X}$. $\|.\|$ denotes a norm of $\mathcal{X}$ and also the induced norm over the set of bounded linear operators on $\mathcal{X}$. $\mathcal{R}$ denotes the range of a linear operator.

We assume that for all x in $\overline{\mathcal{O}}$, $\mathcal{K}(x)$ belongs to $\mathcal{Y}$, a *closed* subspace of $\mathcal{X}$.

Our assumptions are the followings:

Received: March 2, 2010

$$(H1)\left|\begin{array}{ll} \text{(i)} & \text{Equation (1) has a fixed point } x^* \text{ in } \mathcal{O}. \\ \text{(ii)} & \mathcal{K} \text{ is compact, is Fréchet differentiable on } \mathcal{O} \\ & \text{and the Fréchet derivative of } \mathcal{K},\ D\mathcal{K} \text{ is } q\text{-Lipschitz in } \mathcal{O}. \\ \text{(iii)} & 1 \text{ is not in the spectrum of } L := D\mathcal{K}(x^*). \end{array}\right.$$

The condition (iii) implies that x^* is an isolated fixed point of $\mathcal{K}$. So there is a neighborhood $\mathcal{V}$ of x^*, in which (1) has a unique solution.

Let us remark that the Fréchet derivative inherits the property of compactness from the operator $\mathcal{K}$.

The purpose of the paper is the numerical approximation of x^*, a fixed point of $\mathcal{K}$. In the classical projection methods, one needs a sequence $\mathcal{X}_n$ of finite dimensional subspaces, and a sequence P_n of bounded linear operators on $\mathcal{X}$ satisfying the following properties:

$$(H2)\left|\begin{array}{l} \text{(i) } \mathcal{R}(P_n) = \mathcal{X}_n. \\ \text{(ii) } P_n x = x \text{ for each } x \in \mathcal{X}_n. \\ \text{(iii) } \exists p > 0 \text{ such that } \|P_n\| \leq p \text{ for each } n \in \mathbb{N}. \\ \text{(iv) } P_n y \to y \text{ for each } y \in \mathcal{Y}. \end{array}\right.$$

The assumption (iii) is needed for a numerical reason.

The assumption (iv) is weaker than the assumption that P_n converges pointwise towards the identity operator on $\mathcal{X}$. Sometimes the Banach space $\mathcal{X}$ is not separable so that we cannot have such an assumption. But because of the regularity properties of the operator $\mathcal{K}$, one can prove that $\mathcal{K}(x)$ is in a separable subspace $\mathcal{Y}$ of $\mathcal{X}$ so that the assumption (iv) is plausible.

The classical Galerkin method consists in solving in $\mathcal{X}_n$:

$$x_n^G = P_n \mathcal{K}(x_n^G). \tag{2}$$

This method was analyzed by Krasnosel'skii in [10]. It has been proved that, if x^* is an isolated fixed point, then, for sufficiently large n, equation (2) has at least one solution $x_n^G \in \mathcal{X}_n \cap \mathcal{O}$ such that $\|x_n^G - x^*\|$ tends to 0 as $n \to \infty$. With additional asumptions on $\mathcal{K}$, the solution of (2) is unique and the rate of convergence of x_n^G to x^* is the same as that of $P_n x^*$ to x^*.

In [3], Atkinson and Potra investigate the iterated Galerkin method given by

$$x_n^S = \mathcal{K}(x_n^G). \tag{3}$$

This solution was introduced by Sloan for linear equations [14]. See also Chandler [4].

If $\mathcal{X}$ is an Hilbert space and P_n the orthogonal projection, x_n^S is shown to be superconvergent to x^* (see [3]), in the sense that,

$$\frac{\|x_n^S - x^*\|}{\|P_n x^* - x^*\|} \to 0 \text{ as } n \to \infty.$$

For general P_n, the iterated Galerkin approximation is superconvergent if and only if

$$\frac{\|D\mathcal{K}(x^*)(I - P_n)x^*\|}{\|(I - P_n)x^*\|} \to 0 \quad \text{as } n \to \infty. \tag{4}$$

This condition may fail as the following example shows.

Example 1. Let $\mathcal{X} = C^0[0,1]$ be the set of continuous functions on $[0,1]$, and let $\mathcal{X}_n$ be the space of continuous piecewise linear functions with respect to the uniform partition

$$0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n}{n} = 1. \tag{5}$$

Let P_n be the interpolatory operator defined on $C^0[0,1]$ with the range as $\mathcal{X}_n$. Let $(l_i)_{i=0,\ldots,n}$ be the set of hat functions. Then P_n has the following representation:

For all $x \in \mathcal{X}$, $P_n x = \sum_{i=0}^{n} x(\frac{i}{n}) l_i$.

It is not difficult to prove that P_n satisfies $(H2)$.

Let $\mathcal{K}$ be the following nonlinear operator defined from $C^0[0,1]$ to $C^0[0,1]$:

$$\mathcal{K}(x)(s) = (\int_0^1 x(t)dt + 1)s^2, \quad s \in [0,1]. \tag{6}$$

Note that $\mathcal{K}$ is compact, Fréchet differentiable and for all $x \in C^0[0,1]$ and $h \in C^0[0,1]$, $(D\mathcal{K}(x).h)(s) = (\int_0^1 h(t)dt)s^2$, $s \in [0,1]$.

It is easy to show that (1) has a unique solution x^* in $C^0[0,1]$: $x^*(s) = \frac{3}{2}s^2$, $s \in [0,1]$ and that 1 is not an eigenvalue of $D\mathcal{K}(x^*)$.

We have $\|D\mathcal{K}(x^*)(I - P_n)(x^*)\|_\infty = \frac{1}{4n^2}$ and as $\|(I - P_n)(x^*)\|_\infty \leq \frac{3}{4n^2}$, $\frac{\|D\mathcal{K}(x^*)(I-P_n)(x^*)\|_\infty}{\|(I-P_n)(x^*)\|_\infty} \geq \frac{1}{3}$, so that the ratio does not tend to 0. Hence the necessary and sufficient condition of superconvergence of the iterated Galerkin method is not fulfilled.

In the linear case, a new method based on projection has been proposed recently by Kulkarni in [12] and [13]. In [13], the author considers the operator equation

$$x - Tx = f, \tag{7}$$

where T is a compact linear integral operator defined on a Banach space $\mathcal{X}$.

If P_n is a projection from $\mathcal{X}$ onto $\mathcal{X}_n$, converging pointwise towards the identity, the author proposes to approximate the operator T by

$$T_n^K = P_n T P_n + (I - P_n) T P_n + P_n T (I - P_n). \tag{8}$$

Note that in the classical Galerkin method, T is approximated by $T_n^G = P_n T P_n$ and in the iterated Galerkin by $T_n^S = P_n T P_n + (I - P_n) T P_n$.

The resulting solution x_n^K of

$$x_n^K - T_n^K x_n^K = f, \tag{9}$$

may converge faster than the Galerkin solution x_n^G and than the iterated Galerkin x_n^S.

The result can be explained by the following error bounds: If x^* is the exact solution of (7)

$$\|x^* - x_n^G\| \leq c\|x^* - P_n x^*\|, \tag{10}$$

$$\|x^* - x_n^S\| \leq c\|T(I - P_n)\| \, \|x^* - P_n x^*\|, \tag{11}$$

$$\|x^* - x_n^K\| \leq c\|(I - P_n)T\| \, \|x^* - P_n x^*\|. \tag{12}$$

As T is compact, $\|(I - P_n)T\|$ tends to 0. Thus x_n^K is superconvergent to x^* without any supplementary conditions. On the other hand $\|T(I - P_n)\|$ may not tend to 0 when P_n is not the orthogonal projection.

The purpose of the present paper is to adapt the method proposed by Kulkarni in [13] to the nonlinear case. It turns out to be a theoretical improvement on the classical Galerkin and iterated Galerkin method in the sense that the resulting solution is superconvergent to the exact solution without any additional conditions.

The paper is organized as follows. In Section 2, the new method to approximate an isolated fixed point of a compact operator is proposed. We prove the local existence and uniqueness of the approximation as well as its superconvergence. While we were writing this paper, we came across a paper [7] which proposes a class of accelerated projections. Among them, some can be compared to our method. In Section 3, we discuss the similarities and the differences between the results of Dellwo and Friedman and ours.

2. The New Method

We approximate $\mathcal{K}$ by $\mathcal{K}_n^K$ defined by

$$\mathcal{K}_n^K(x) = P_n \mathcal{K}(x) + \mathcal{K}(P_n x) - P_n \mathcal{K}(P_n x), \tag{13}$$

for all $x \in \mathcal{X}$. We remark that, if $\mathcal{K}$ is linear, then this approximation is equivalent to (8).

We propose to approximate (1) by

$$x = \mathcal{K}_n^K(x). \tag{14}$$

The first task is to prove that for an integer n large enough, (14) has a unique solution x_n^K and that $\|x_n^K - x^*\|$ tends to 0.

The following lemmas state some properties needed to prove the existence and the convergence of x_n^K.

Lemma 1. *Under the assumptions $(H1)$, $D\mathcal{K}(x^*)$ is compact.*

Proof. See Theorem 17.1, p. 77 in [9]. □

Lemma 2. *Under the assumptions $(H1)$ and $(H2)$, $\mathcal{R}(D\mathcal{K}(x^*))$ is included in $\mathcal{Y}$, and $a_n := \|(I - P_n)D\mathcal{K}(x^*)\|$ tends to 0 as $n \to \infty$.*

Proof. We have for all $h \in \mathcal{X}$,

$$D\mathcal{K}(x^*)h = \lim_{t \to 0} \frac{\mathcal{K}(x^* + th) - \mathcal{K}(x^*)}{t} = \lim_{n \to \infty} \frac{\mathcal{K}(x^* + \frac{h}{n}) - \mathcal{K}(x^*)}{\frac{1}{n}}.$$

The sequence $f_n = \frac{\mathcal{K}(x^* + \frac{h}{n}) - \mathcal{K}(x^*)}{\frac{1}{n}}$ is in $\mathcal{Y}$, which is a closed set, so that its limit is in $\mathcal{Y}$.

Because of property (iv) in $(H2)$, as $D\mathcal{K}(x^*)h \in \mathcal{Y}$ for each $h \in \mathcal{X}$, then, $(I - P_n)D\mathcal{K}(x^*)h \to 0$ for all $h \in \mathcal{X}$. Moreover $\overline{D\mathcal{K}(x^*)(B(0,1))}$ is compact, where $B(0,1)$ is the unit ball in $\mathcal{X}$. Hence the convergence of $(I - P_n)D\mathcal{K}(x^*)$ to 0 is uniform. □

Lemma 3. *Let $\eta_1 > 0$ and $\varepsilon > 0$ such that $B(x^*, \eta_1 p + \varepsilon)$ is included in $\mathcal{O}$.*

Under the assumptions $(H2)$, there exists a positive integer n_1 such that for $n \geq n_1$, $P_n x \in \mathcal{O}$ for all $x \in B(x^*, \eta_1)$.

Proof. As P_n converges pointwise to identity in $\mathcal{Y}$ and $x^* \in \mathcal{Y}$, there exists an integer n_1 such that for $n \geq n_1$, $\|P_n x^* - x^*\| \leq \epsilon$.

Let x be in $B(x^*, \eta_1)$. We have $\|P_n x - x^*\| \leq \|P_n x - P_n x^*\| + \|P_n x^* - x^*\|$. Then for $n \geq n_1$, $\|P_n x - x^*\| \leq p\|x - x^*\| + \varepsilon \leq p\eta_1 + \varepsilon$. Thus $P_n x$ is in $\mathcal{O}$. □

Lemma 4. *Under the assumptions* $(H1)$ *and* $(H2)$*, for* $n \geq n_1$*,* $\mathcal{K}_n^K$ *is Fréchet differentiable on* $B(x^*, \eta_1)$*. Moreover there exists a positive integer* $n_2 \geq n_1$ *such that for* $n \geq n_2$*, 1 is not in the spectrum of* $D\mathcal{K}_n^K(x^*)$*, and*

$$\|(I - D\mathcal{K}_n^K(x^*))^{-1}\| \leq 2c,$$

where $c := \|(I-L)^{-1}\|$*.*

Proof. According to the chain rule, $\mathcal{K}_n^K$ is Fréchet differentiable on $B(x^*, \eta_1)$ and for all $x \in B(x^*, \eta_1)$, $D\mathcal{K}_n^K(x) = P_n D\mathcal{K}(x) + (I - P_n)D\mathcal{K}(P_n x)P_n$, so that $I - D\mathcal{K}_n^K(x^*) = (I-L) + D_n$, where $D_n = (I-P_n)L(I-P_n) + (I-P_n)(D\mathcal{K}(x^*) - D\mathcal{K}(P_n x^*))P_n$.

Let $a_n = \|(I-P_n)L\|$. Since $D\mathcal{K}$ is q-Lipschitz, we can derive $\|D_n\| \leq (1+p)a_n + (1+p)pq\|P_n x^* - x^*\|$.

We apply the following classical perturbation result: If M an N are bounded linear operators on a normed space $\mathcal{X}$ such that N has a bounded inverse N^{-1} and $\|M - N\| \leq \frac{1}{\|N^{-1}\|}$, then M is invertible, M^{-1} is bounded and $\|M^{-1}\| \leq \frac{\|N^{-1}\|}{1 - \|N^{-1}\|\|M-N\|}$.

As $\|D_n\| \to 0$ as $n \to \infty$, there exists an integer $n' \geq n_1$ such that for $n \geq n'$, $\|D_n\| \leq \frac{1}{\|(I-L)^{-1}\|}$, so that $I - D\mathcal{K}_n^K(x^*)$ is invertible and $\|(I - D\mathcal{K}_n^K(x^*))^{-1}\| \leq \frac{\|(I-L)^{-1}\|}{1 - \|(I-L)^{-1}\|\|D_n\|}$.

Then there exists $n_2 \geq \max(n_1, n')$ such that for $n \geq n_2$, $\|(I - D\mathcal{K}_n^K(x^*))^{-1}\| \leq 2c$. □

Lemma 5. *For* $n \geq n_1$*,* $D\mathcal{K}_n^K$ *is Lipschitz continuous on* $B(x^*, \eta_1)$*, with Lipschitz constant* $m \leq qp + qp^2(l + p)$

Proof. Obvious. □

We recall the generalized mean value theorem needed later:

Lemma 6. *If* T *is Fréchet differentiable on* Ω*, and* $[x_1, x_1 + h] \subset \Omega$*, then* $\|T(x_1 + h) - T(x_1)\| \leq \|h\| \sup\{\|DT(x_1 + \theta h)\|,\ 0 < \theta < 1\}$*.*

We now prove our main result about the local existence and uniqueness of the fixed point of $\mathcal{K}_n^K$ defined by (13) and give an estimation of its rate of convergence.

Theorem 1. *Under the assumptions* $(H1)$ *and* $(H2)$*, there exist a real number* $\eta_0 > 0$ *and a positive integer* n_0 *such that* $\mathcal{K}_n^K$ *has a unique fixed point* x_n^K *in* $\mathcal{B}(x^*, \eta_0)$ *for all* $n \geq n_0$ *and* $\frac{2}{3}\alpha_n \leq \|x_n^K - x^*\| \leq 2\alpha_n$ *where* $\alpha_n = \|(I - D\mathcal{K}_n^K(x^*))^{-1}[x^* - \mathcal{K}_n^K(x^*)]\|$*.*

Also

$$\alpha_n \leq 2c((1+p)r_n + a_n)\|P_n x^* - x^*\|, \tag{15}$$

where

$$a_n = \|(I - P_n)L\|, \tag{16}$$

$$r_n = \frac{\|\mathcal{K}(P_n x^*) - \mathcal{K}(x^*) - L(P_n x^* - x^*)\|}{\|P_n x^* - x^*\|}, \tag{17}$$

so that $\frac{\|x_n^K - x^*\|}{\|P_n x^* - x^*\|} \to 0.$

Remark 1. In fact, we only need that P_n be a bounded linear operator from $\mathcal{X}$ into $\mathcal{X}_n$, converging pointwise to the identity on $\mathcal{Y}$ and uniformly bounded in n. We do not need that P_n be a projection nor do we need assumption (ii) of $(H2)$.

Proof. The proof is inspired from [16], [10] and [11].

We prove that $\mathcal{K}_n^K$ is a contraction in an invariant neighborhood of x^*.

Let $A_n = I - \mathcal{K}_n^K$. Since by Lemma 4, $DA_n(x^*) = I - D\mathcal{K}_n^K(x^*)$ is invertible for $n \geq n_2$ and $A_n x = 0$ is equivalent to $x = B_n x$, where $B_n x = x^* - DA_n(x^*)^{-1}\left\{A_n x^* + A_n x - A_n x^* - DA_n(x^*)(x - x^*)\right\}$.

Hence for $n \geq n_2$

$$\|B_n x - x^*\| \leq \|DA_n(x^*)^{-1}\left\{A_n x^*\right\}\| + \|DA_n(x^*)^{-1}(A_n x - A_n x^* - DA_n(x^*)(x - x^*))\|. \tag{18}$$

Since $A_n x^* = (P_n - I)[\mathcal{K}(P_n x^*) - \mathcal{K}(x^*) - L(P_n x^* - x^*)] + (P_n - I)L(P_n x^* - x^*)$, then, for $n \geq n_2$,

$$\|DA_n(x^*)^{-1}A_n x^*\| = \alpha_n \leq 2c[(1+p)r_n + a_n]\|P_n x^* - x^*\|.$$

To bound the second term in (18), we apply Lemma 6, with $T = A_n - DA_n(x^*)$ and $[x_1, x_1 + h] = [x^*, x]$ with x in $B(x^*, \eta_1)$. We obtain

$$\|A_n x - A_n x^* - DA_n(x^*)(x - x^*)\| \leq \|x - x^*\| \sup_{0<\theta<1} \|D\mathcal{K}_n^K(x^*) - D\mathcal{K}_n^K(x^* + \theta(x - x^*))\|.$$

According to Lemma 5, for $n \geq n_2$,

$$\|D\mathcal{K}_n^K(x^*) - D\mathcal{K}_n^K(x^* + \theta(x - x^*))\| \leq m \;\; \theta\|x - x^*\|.$$

Then

$$\|A_n x - A_n x^* - DA_n(x^*)(x - x^*)\| \leq m\|x - x^*\|^2.$$

Hence for $n \geq n_2$ and $x \in B(x^*, \eta_1)$,

$$\|B_n x - x^*\| \leq 2c\ \partial_n\ \|P_n x^* - x^*\| + 2cm\ \|x - x^*\|^2,$$

where $\partial_n = (1+p)r_n + a_n$.

There exists $\eta_0 < \eta_1$ such that $2cm\eta_0 < \frac{1}{2}$.

Let $n_0 \geq n_2$ be an integer such that for all $n \geq n_0$, $2c\partial_n \|P_n x^* - x^*\| \leq \frac{\eta_0}{2}$.

Then for $n \geq n_0$ and for all x in $B(x^*, \eta_0)$,

$$\|B_n x - x^*\| \leq \eta_0 \frac{1}{2} + \frac{1}{2}\eta_0 = \eta_0,$$

which proves that B_n maps $B(x^*, \eta_0)$ into itself.

B_n is a contraction on $B(x^*, \eta_0)$: in fact for $x_1, x_2 \in B(x^*, \eta_0)$,

$$\begin{aligned} B_n x_1 - B_n x_2 = DA_n(x^*)^{-1}[A_n x_2 - A_n x_1 - DA_n(x^*)(x_2 - x_1)] \\ = DA_n(x^*)^{-1}[Tx_2 - Tx_1], \end{aligned}$$

so that, applying Lemma 6 and Lemma 5, for $n \geq n_0$,

$$\begin{aligned} \|B_n x_1 - B_n x_2\| &\leq 2c \sup_{0<\theta<1} \|DA_n(x_1 + \theta(x_2 - x_1)) - DA_n(x^*)\| \|x_1 - x_2\| \\ &\leq 2c0 < \theta < 1\sup \|D\mathcal{K}_n^K(x^*) - D\mathcal{K}_n^K(x_1 + \theta(x_2 - x_1))\| \|x_1 - x_2\| \\ &\leq m2c0 < \theta < 1\sup \|x_1 + \theta(x_2 - x_1) - x^*\| \|x_1 - x_2\| \\ &\leq m2c((1-\theta)\eta_0 + \theta\eta_0)\|x_1 - x_2\| \leq m2c\eta_0\|x_1 - x_2\| \leq \frac{1}{2}\|x_1 - x_2\|. \end{aligned}$$

By the contraction mapping theorem, B_n has a unique fixed point x_n^K in $B(x^*, \eta_0)$.

We have

$$\begin{aligned} x_n^K - x^* &= B_n x_n^K - x^* \\ &= -DA_n(x^*)^{-1}A_n x^* - DA_n(x^*)^{-1}(A_n x_n^K - A_n x^* - DA_n(x^*)(x_n^K - x^*)), \end{aligned}$$

so that

$$\begin{aligned} \|DA_n(x^*)^{-1}A_n x^*\| - 2cm\|x_n^K - x^*\|^2 \\ \leq \|x_n^K - x^*\| \leq \|DA_n(x^*)^{-1}A_n x^*\| + 2cm\|x_n^K - x^*\|^2. \end{aligned}$$

Thus, as $2cm\|x_n^K - x^*\| \leq 2cm\eta_0 < \frac{1}{2}$, we obtain

$$\frac{2}{3}\alpha_n \leq \|x_n^K - x^*\| \leq 2\alpha_n. \qquad \square$$

Remark 2. Note that the rate of convergence of $\|x_n^K - x^*\|$ is bounded from below and from above in terms of α_n, and that

$$\frac{\alpha_n}{\|P_n x^* - x^*\|} \leq 2c\{(1+p)r_n + a_n\} \to 0.$$

Hence

$$\frac{\|x_n^K - x^*\|}{\|P_n x^* - x^*\|} \to 0$$

under the assumptions (H_1) and (H_2). As a consequence, x_n^K is superconvergent to x^* without any additional conditions.

Remark 3. Let $k \in]0,1[$ and η_0 be such that $2cm\eta_0 < k$. Then :

$$\frac{\alpha_n}{1+k} \leq \|x_n^K - x^*\| \leq \frac{\alpha_n}{1-k}.$$

Remark 4. Our method can be seen as a *quasi* perturbed Galerkin method in the sense explained in [11]. However our approximate solution does not belong to the finite dimensional approximating space.

The approximation x_n^K provided by Theorem 1 is an improvement over the iterated Galerkin approximation x_n^S from a theoretical point of view:

Theorem 2.4 of [3] establishes that the iterated Galerkin solution x_n^S is superconvergent *if and only if* the condition (4), recalled below, is satisfied:

$$\frac{\|D\mathcal{K}(x^*)(I-P_n)x^*\|}{\|(I-P_n)x^*\|} \to 0 \quad \text{as } n \to \infty.$$

Our Theorem 1 above claims that the new approximation x_n^K is superconvergent.

The condition (4) is always satisfied when P_n is the orthogonal projection in a Hilbert space. But in Example 1, we show that in a very simple case, (4) is not satisfied. Thus for that example, x_n^K will be superconvergent but x_n^S won't be.

The reduction of equation (14) to a finite dimensional system is now considered. Applying P_n and $(I-P_n)$ to equation (14), we obtain respectively,

$$(a) \qquad P_n x_n^K = P_n P_n \mathcal{K}(x_n^K),$$

$$(b) \qquad (I - P_n)x_n^K = (P_n - P_n^2)(\mathcal{K}(x_n^K) - \mathcal{K}(P_n x_n^K)) + (I - P_n)\mathcal{K}(P_n x_n^K).$$

As the linear operators P_n satisfy assumption $(H2)$, then these equations are equivalent to :

$$(a) \qquad w_n^K = P_n \mathcal{K}(w_n^K + (I - P_n)\mathcal{K}(w_n^K)),$$

$$(b) \qquad x_n^K = w_n^K + (I - P_n)\mathcal{K}(w_n^K),$$

where $w_n^K = P_n x_n^K$.

The equation (a) is a fixed point problem in $\mathcal{R}(P_n)$ which is a finite dimensional space. One can use Newton type methods to solve it. Once we have computed $w_n^K \in \mathcal{X}_n$, we recover x_n^K with formula (b).

For the iterated Galerkin method, setting $w_n^S = P_n x_n^S$, the reduction is as follows :

$$(c) \qquad w_n^S = P_n \mathcal{K}(w_n^S),$$

$$(d) \qquad x_n^S = \mathcal{K}(w_n^S).$$

The equation (c) is a fixed point problem in $\mathcal{R}(P_n)$, the *same finite dimensional space* as previously. Notice that, in practice, equation (c) is easier and cheaper to treat than (a).

3. Results of D.R. Dellwo and M.B. Friedman

While we were writing this paper, we discovered the paper [7] of Dellwo and Friedman. We have been very surprised by the similarities between some of their results and ours though the philosophy of our method was far from theirs:

Our approach is classical: our inspiration comes from the iterated Galerkin solution of Atkinson and Potra [3] who adapted to the nonlinear case the method proposed by Sloan [14]. All these methods relies on the same philosophy: they propose to approach the operator $\mathcal{K}$ by an approximate operator $\mathcal{K}_n$. The iterated Galerkin operator $\mathcal{K}_n^S$ has been built up to be better than the Galerkin operator $\mathcal{K}_n^G$ in the sense $\|\mathcal{K} - \mathcal{K}_n^S\| < \|\mathcal{K} - \mathcal{K}_n^G\|$ and our approximate operator has been built up to be better than the iterated Galerkin operator.

In [7], Dellwo and Friedman propose accelerated projection and iterated projection methods, based upon an idea of the Lyapunov-Cesari method. The fixed point equation (1) is decomposed into a finite dimensional component belonging to $\mathcal{X}_N = \mathcal{R}(P_N)$, P_N being a finite rank projection, and an infinite dimensional one:

$$\begin{array}{ll} \text{(a)} & P_N x = P_N \mathcal{K}(P_N x + Q_N x), \\ \text{(b)} & Q_N x = Q_N \mathcal{K}(P_N x + Q_N x), \end{array}$$

where P_N converges pointwise to the identity and $Q_N = I - P_N$. This system can be rewritten as

$$\begin{array}{ll} \text{(a)} & z = P_N \mathcal{K}(z + w), \\ \text{(b)} & w = Q_N \mathcal{K}(z + w), \end{array}$$

where $z \in \mathcal{X}_N$. As $\mathcal{K}$ is compact, the operator $Q_N\mathcal{K}$, in (b), satisfies the conditions of the contraction mapping principle, so that (b) has a unique solution $w = W_N(z)$. Dellwo and Friedman define a sequence of approximate solutions $w_n = W_N^{(n)}(z)$ of (b) leading to a sequence of algebraic systems of rank N for all n,

$$z_N^{(n)} = P_N \mathcal{K}(z_N^{(n)} + W_N^{(n)}(z_N^{(n)})).$$

The approximate solution $x_N^{(n)} = z_N^{(n)} + W_N^{(n)}(z_N^{(n)})$ depends on the technique used to compute $W_N^{(n)}(z)$. Dellwo and Friedman propose two different approximate solutions: the Picard iterates and a power series.

It took us some time to realize that *the second order scheme* ($n = 2$) obtained by using the Picard approximation or the power series approximation corresponds to our new method and the *iterated frst order scheme* obtained by using the Picard approximation or the power series approximation is the iterated Galerkin method.

The results of Dellwo and Friedman are very important: they give general schemes providing approximations to any desired accuracy for a fixed computational error. It is very imaginative and original. Their theorem are very general and can be applied with various ways of computing the infinite dimensional component of the solution. As the results have a high degree of generality, the assumptions, used in the proof or in the formulations of the theorems, are quite restrictive. It is impossible for Dellwo and Friedman to formulate the weakest assumptions for all the particular cases. And that is not their goal in [7].

Our formulation allows us to *prove* completely the estimation given in Theorem 1 under very weak assumptions. Dellwo and Friedman just note that the strong assumptions under which they formulate and prove their results "can be replaced by less restrictive assumptions for a particular order by exploiting the functional character of the partial sum (4.3) corresponding to that order"([7],

p. 243). But they did not give the assumptions nor proved their estimation (4.11) (p. 244).

Our proof is given in details and is straightforward: our method can be seen as a *quasi* perturbed Galerkin method and a simple application of the contraction mapping principle is enough.

4. Conclusions

We propose a new fixed point approximation of $\mathcal{K}$ by building an approximate operator $\mathcal{K}_n^K$ closer to the operator $\mathcal{K}$ than in the classical Galerkin and iterated Galerkin approximations. Our new approximation is superconvergent without any additional assumptions while the iterated Galerkin method needs a condition to be superconvergent. In fact this approximation is not new. It is a particular case of a class of approximations proposed in [7] by Dellwo and Friedman. We show how our approach completes the approach of Dellwo and Friedman and allows a better understanding of the method.

Acknowledgments

This work was partially supported by the French program ARCUS-INDE (Actions en Régions de Coopération Universitaire et Scientifique) 2005-2008 of the Rhône-Alpes region.

I am grateful to Rekha Kulkarni whose ideas motivated this paper.

I wish to thank Mario Ahues whose advice and suggestions have been precious for the elaboration of the paper.

References

[1] M. Ahues, A. Largillier, B.V. Limaye, *Spectral Computations with Bounded Operators*, CRC, Boca Raton (2001).

[2] K.E. Atkinson, *The Numerical Solutions of Integral Equations of the Second Kind*, Cambridge University Press (1997).

[3] K.E. Atkinson, F.A. Potra, Projection and iterated projection methods for nonlinear integral equations, *SIAM J. Numer. Anal.*, **24**, No. 6 (1987), 1352-1373.

[4] G.A. Chandler, *Superconvergence of Numerical Solutions of Second Kind Integral Equations*, Ph.D. Thesis, Australian National University (1979).

[5] F. Chatelin, *Spectral Approximation of Linear Operators*, Academic Press, New York (1983).

[6] F. Chatelin, R. Lebbar, Superconvergence results for the iterated projection method applied to a Fredholm integral equation of the second kind and the corresponding eigenvalue problem, *Journal of Integral Equations*, **6** (1984), 71-91.

[7] D.R. Dellwo, M.B. Friedman, Accelerated projection and iterated projection methods with applications to nonlinear integral equations, *SIAM J. Numer. Anal.*, **28**, No. 1 (1991), 236-250.

[8] L. Grammont, R. Kulkarni, A superconvergent projection method for nonlinear compact operator equations, *C.R. Acad. Sci. Paris*, Ser. I, No. 342 (2006), 215-218.

[9] M.A. Krasnoselskii, P.P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (1984).

[10] M.A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, New-York (1964).

[11] M.A. Krasnoselskii, G. Vainikko, P. Zabreiko, Y. Rutitskii, V. Stetsenko, *Approximate Solution of Operator Equations*, Noordhoff, Groningen, The Nederlands (1972).

[12] R.P. Kulkarni, A superconvergence result for solutions of compact operator equations, *Bulletin of the Australian Mathematical Society*, **68** (2003), 517-528.

[13] R.P. Kulkarni, On improvement of the iterated Galerkin solution of the second kind integral equations, *Journal of Numerical Mathematics*, **13**, No. 3 (2005), 205-218.

[14] I.H. Sloan, Improvement by iteration for compact operator equations, *Math. Comp.*, **30** (1976), 758-76.

[15] I.H. Sloan, Superconvergence, In: *Numerical Solution of Integral Equations* (Ed. M. Golberg), Plenum Press (1990), 35-70.

[16] G.M. Vainikko, Galerkin's perturbation method and general theory of approximate methods for non-linear equations, *USSR Comput. Math. and Math. Phys.*, **7**, No. 4 (1967).