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ON A METHOD FOR SOLVING SOME SPECIAL CLASSES OF NONLINEAR SYSTEMS OF EQUATIONS

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Abstract: In this paper we consider some iterative methods for solving nonlinear systems of equations. Some successive overrelaxation modifications are proposed. Interesting numerical examples are presented.

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1. Introduction

In [3], Cvetkovic and Herceg developed the following *modified accelerated overrelaxation–Newton* (MAORN) method for solving nonlinear system of equations

$$f_i(x_1, x_2, \dots, x_n) = 0, \quad i \in N := \{1, 2, \dots, n\},$$

$$\begin{aligned} x_i^{k+1} &= x_i^k - \omega \Delta_i^k(\sigma), \\ \overline{x}_i^{k+1} &= x_i^k - \sigma \Delta_i^k(\sigma), \end{aligned}$$

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where

$$\Delta_i^k(\sigma) = \frac{f_i(\overline{x}_1^{k+1}, \ldots, \overline{x}_{i-1}^{k+1}, x_i^k, \ldots, x_n^k)}{d_i(\overline{x}_1^{k+1}, \ldots, \overline{x}_{i-1}^{k+1}, x_i^k, \ldots, x_n^k)},$$

$\sigma, \omega \in R$, $\omega \neq 0$ and $d_i(x)$, $i \in N$, $x \in R^n$ are arbitrary real nonzero functions.

In the case when $d_i(x) = \frac{\partial f_i}{\partial x_i}$, $i \in N$, $x = [x_1, \ldots, x_n]^T$ the MAORN method reduces to an *Accelerated Overrelaxation–Newton* (AORN) method, investigated by Chernyak [1]–[2].

The MAORN method has one very important advantage: we do not have to calculate the partial derivatives of the given nonlinear function.

The following modification of the Newton's method is known (see Chernyak [1]–[2]):

$$\begin{aligned} x_i^{k+1} &= x_i^k - \Delta_i^k, \\ \Delta_i^k &= \frac{f_i(x_1^k, \ldots, x_n^k)}{d_i(x_1^k, \ldots, x_n^k)}, \end{aligned} \tag{1}$$

$$i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots,$$

where $d_i(x), x \in R^n$ are arbitrary real nonzero functions.

Let $d_i(x) = \frac{\partial f_i}{\partial x_i}$, $i = 1, 2, \ldots, n$.

We shall analyze the same class of nonlinear systems as in [1], i.e. nonlinear systems of the following type

$$f_i(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} a_{ij} x_j + g_i(x) - b_i = 0, \tag{2}$$

$$i = 1, 2, \ldots, n,$$

where $d_i(t)$, are real nonlinear differentiable functions.

In this case the method (1) is reduced to

$$x_i^{k+1} = x_i^k - \frac{f_i(x_1^k, \ldots, x_n^k)}{a_{ii} + g_i'(x_i^k)}, \tag{3}$$

$$i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots.$$

Under some conditions, Chernyak proved that this iterative method converges.

For other results see Cvetkovic and Herceg [3]–[5], Herceg and Cvetkovic [6], Chernyak [1]–[2].

2. Main Results

Now, let us explore the following modification of method (3) (assume that $x_i \neq x_j$ and $x_i^0 \neq x_j^0$ for $i \neq j$):

$$x_i^{k+1} = x_i^k - \frac{f_i(x_1^k, \dots, x_n^k)}{\prod_{j \neq i}^{n}(x_i^k - x_j^k) + g_i'(x_i^k)}, \tag{4}$$

$$i = 1, 2, \dots, n; \quad k = 0, 1, 2, \dots .$$

One geometric interpretation of method (4) is the following:

Let x_i^k, $i = 1, 2, \dots, n$ and let us denote by $G_k(x)$ the polynomial

$$G_k(x) = (x - x_1^k)(x - x_2^k) \dots (x - x_n^k).$$

Then for $x = x_i^k$, we have

$$G_k'(x_i^k) = \prod_{j \neq i}^{n}(x_i^k - x_j^k)$$

and previous expression can be used for approximation of a_{ii} in the method (3).

We construct the following method

$$x_i^{k+1} = x_i^k - \frac{f_i(x_1^k, \dots, x_n^k)}{\max\left(a_{ii} + g_i'(x_i^k), \prod_{j \neq i}^{n}(x_i^k - x_j^k) + g_i'(x_i^k)\right)}, \tag{5}$$

$$i = 1, 2, \dots, n; \quad k = 0, 1, 2, \dots .$$

From (5) we have

$$\begin{aligned} |x_i^{k+1} - x_i^k| &\leq \frac{|f_i(x_1^k, \dots, x_n^k)|}{\max\{|a_{ii} + g_i'(x_i^k)|, \, |\prod_{j \neq i}^{n}(x_i^k - x_j^k) + g_i'(x_i^k)|\}} \\ &\leq \frac{|f_i(x_1^k, \dots, x_n^k)|}{|a_{ii} + g_i'(x_i^k)|}. \end{aligned}$$

Evidently, the method (5) yields considerable improvement in the rate of convergence for the iterative method (3).

We thus conclude the following

Theorem 1. *If the iteration method (3) is convergent, then the combined procedure (5) is also convergent.*

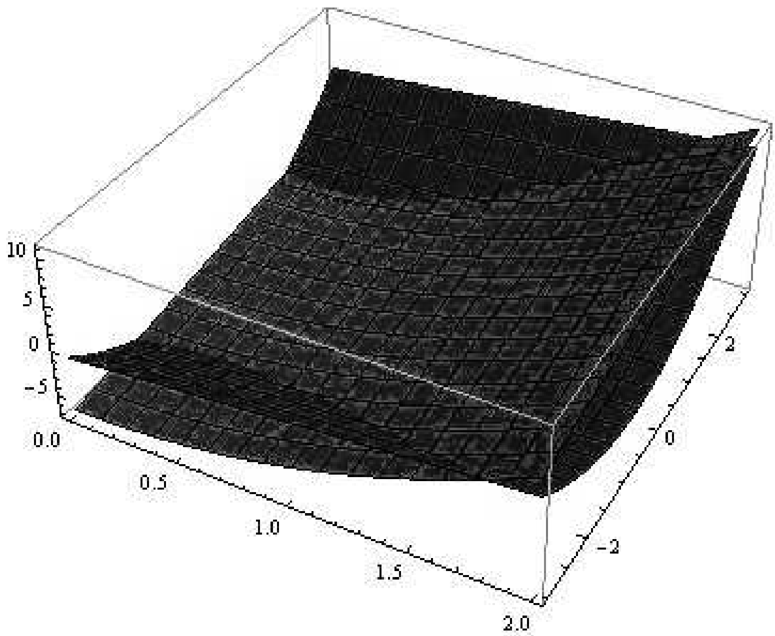

Figure 1

3. Numerical Experiments

Let us consider the system (*classical example by N. Obreschkoff* [8], displayed in Figure 1):

$$\begin{aligned} f_1(x_1,x_2) &= 3x_1^2 + x_2 - 4 = 0,\\ f_2(x_1,x_2) &= 2x_1 + x_2 - 7 + x_2^2 = 0. \end{aligned} \tag{6}$$

For initial approximation we choose $x_1^0 = 1.5$, $x_2^0 = -2.5$.

We give results of numerical experiments for each of methods (3) and (5).

I. In Table 1 the following notations are used:

— in the first column a serial number of iteration step has been used;

— in the second column results are given (array $x[\,]$) using the method (3).

The convergence test is $||x^{k+1} - x^k||_2 < 10^{-5} = \epsilon$;

II. In Table 2 the following notations are used:

— in the first column a serial number of iteration step is used;

— in second columns results are given, using modified scheme (5).

0	$X[1] = 1.5$ $X[2] = -2.5$
1	$X[1] = 1.4722222222222223$ $X[2] = -2.5625$
2	$X[1] = 1.4790356394129978$ $X[2] = -2.5750210437710437$
3	$X[1] = 1.4804308926757377$ $X[2] = -2.571699725425853$
4	$X[1] = 1.480056321713926$ $X[2] = -2.57102358070115$
5	$X[1] = 1.4799801347894834$ $X[2] = -2.5712043330496295$
6	$X[1] = 1.4800004880959106$ $X[2] = -2.5712411090390965$
7	$X[1] = 1.4800046293948483$ $X[2] = -2.5712312820891703$

Table 1

0	$XJ[1] = 1.5$ $X[2] = -2.5$
1	$XJ[1] = 1.4807692307692308$ $X[2] = -2.5625$
2	$XJ[1] = 1.4795677546700232$ $X[2] = -2.5708770396270397$
3	$XJ[1] = 1.4798396341619455$ $X[2] = -2.571440273793256$
4	$XJ[1] = 1.4799683429315615$ $X[2] = -2.57130894579352$
5	$XJ[1] = 1.4799985072591257$ $X[2] = -2.5712468027733033$
6	$XJ[1] = 1.480003151695585$ $X[2] = -2.571232238473964$

Table 2

Two arrays are necessary $x[\,]$ and $xJ[\,]$ as follows:
in $x[\,]$– data are stored, when is fulfilled

$$\max\left(a_{ii}+g_i'(x_i^k),\prod_{j\neq i}^{n}(x_i^k-x_j^k)+g_i'(x_i^k)\right)=a_{ii}+g_i'(x_i^k),$$

and in $xJ[\,]$ – data are stored, when

$$\max\left(a_{ii}+g_i'(x_i^k),\prod_{j\neq i}^{n}(x_i^k-x_j^k)+g_i'(x_i^k)\right)=\prod_{j\neq i}^{n}(x_i^k-x_j^k)+g_i'(x_i^k).$$

In Table 2 we give the results of numerical experiments.

From given results it can be seen that component x_2 is calculated only by using scheme (3), and $XJ[1]=x_1$ is calculated only by using modified method with Weierstrass correction – (5).

We note that using method (3) 7 iteration steps for receiving the solution with fixed accuracy ϵ are necessary (see Table 1).

For the same precision the modified scheme (5) consummates 6 iterations (see Table 2).

Numerical experiments demonstrate that in some aspects an improved convergence can be reached through mentioned above combined "(3)–(5)" iteration procedure.

Remark. The convergence order can be increased by calculating the new approximation x_i^{k+1} in (5) using the already calculated approximations x_1^{k+1}, x_2^{k+1}, ..., x_{i-1}^{k+1} (the so called Gauss-Seidel approach).

It is possible to use the following procedure:

$$x_i^{k+1}=x_i^k-\frac{f_i(x_1^{k+1},\dots,x_{i-1}^{k+1},x_i^k,\dots,x_n^k)}{\max\left(a_{ii}+g_i'(x_i^k),\prod_{j=1}^{i-1}(x_i^k-x_j^{k+1})\prod_{j=i+1}^{n}(x_i^k-x_j^k)+g_i'(x_i^k)\right)},\qquad(7)$$

$$i=1,2,\dots,n;\quad k=0,1,2,\dots.$$

The following theorem is valid.

Theorem 2. *If the (MAORN)-method investigated by Cvetkovic and Herceg [3] is convergent, then the combined procedure (7) is also convergent.*

0	$XJ[1] = 1.5$ $X[2] = -2.5$
1	$XJ[1] = 1.4807692307692308$ $X[2] = -2.5721153846153846$
2	$XJ[1] = 1.480311865763371$ $X[2] = -2.571081200245555$
3	$XJ[1] = 1.4800885300721067$ $X[2] = -2.571188777351939$
4	$XJ[1] = 1.4800268686887172$ $X[2] = -2.5712185455689394$
5	$XJ[1] = 1.480009851935706$ $X[2] = -2.571226761172249$

Table 3

In Table 3 we give results of numerical experiments for method (7).

The components x_1^5 and x_2^5 tend to solution of system (6) with fixed accuracy ϵ for 5 iterations.

For other nontrivial methods for solving an almost linear system of equations see monograph [7].

References

[1] V. Chernyak, O skhodimosti metoda PVR–N'yutona. *Prikladnaya Matematika*, SEI SO AN SSSR, Irkutsk (1978), 184–200.

[2] V. Chernyak, O vliyanii nelineinosti reshaemoi sistemu na skhodimost' metodov Yakobi, PVR-i UVR-N'yutona. *Chislenye Metody Analiza i ih Prilozheniya*, Akad. Nauk SSSR Sibirsk. Otdel., Energet. Inst., Irkutsk (1983), 19–150.

[3] L. Cvetkovic, D. Herceg, On a generalized vSOR–Newton's method, In: *IV Conference on Applied Mathematics (Vrdoljak, B., ed.)*, Faculty of Civil Engineering, Split (1985), 99–102.

[4] L. Cvetkovic, D. Herceg, An improvement for the area of convergence of the AOR method, *Anal. Numer. Theor. Approx.*, **16** (1987), 109–115.

[5] L. Cvetkovic, D. Herceg, Nonlinear AOR method, *ZAMM*, **68** (1988), 486–487.

[6] D. Herceg, L. Cvetkovic, On a method for solving an almost linear system of equations, Univ. u Novom Sadu, *Zb. Mat.*, **2** (1991), 15–22.

[7] A. Iliev, N. Kyurkchiev, *Nontrivial Methods in Numerical Analysis: Selected Topics in Numerical Analysis*, LAP LAMBERT Academic Publishing GmbH and Co. KG, Saarbrücken (2010).

[8] N. Obreschkoff, *Verteilung und Berechnung der Nullstellen reeller Polynome*, VEB Deutscher Verlag der Wissenschaften, Berlin (1963).