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RULED SURFACES AS A TANGENT DEVELOPABLE IN POSITIVE CHARACTERISTIC AND OSCULATING PLANES TO A SUBCURVE OF THEM

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Abstract: Let $Y \subset \mathbb{P}^n$ be an integral curve contained in a surface S. When is the osculating plane of Y at a general $P \in Y$ equal to the tangent plane of S at P? In positive characteristic we find for each ruled surface S a curve Y as above and classify all such Y if S is either a smooth quadric or a cone.

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1. Introduction

We work over an algebraically closed field $\mathbb{K}$ such that $p := \mathrm{char}(\mathbb{K}) > 0$. For any integral curve $Y \subset \mathbb{P}^n$, let $T(Y) \subset \mathbb{P}^n$ denote the tangent developable of Y, i.e. the closure in $\mathbb{P}^n$ of the union of all tangent lines T_PY, $P \in Y_{reg}$. If Y is not a line, then $T(Y)$ is an integral surface. For any $P \in Y_{reg}$ and any integer t such that $1 \le t \le n-1$ let $O(Y,t,P)$ the osculating t-dimensional linear subspace of Y at P. We started from the observation that any ruled surface $S \subset \mathbb{P}^n$ is the tangent developable $T(Y)$ of some curve $Y \subset S$ (Proposition 1). Obviously, this is a positive characteristic phenomenon, since the same statement is obviously false in characteristic zero. We look at curves $Y \subset S$ such that $S = T(Y)$ and $Y \cap S_{reg} \neq \emptyset$. Assume $S = T(Y)$ and $Y \cap S_{reg} \neq \emptyset$. It is natural to ask if $O(Y,2,P) = T_PS$ for a general $P \in Y$. We first prove the following result.

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Theorem 1. *Fix a ruled and non-degenerate surface $S \subset \mathbb{P}^n$ and an integral and curve $Y \subset S$ such that $S = T(Y)$ and $Y \cap S_{reg} \neq \emptyset$. Then $O(Y, 2, P) = T_P S$ for a general $P \in Y$.*

In the case in which S is a smooth quadric, then we may drop the assumption $T(Y) = S$ and prove a very detailed result (see Propositions 2 and 3).

2. Other Results

Let $\epsilon_i(Y)$, $0 \leq i \leq n$, be the order sequence of Y (see [4], [2]). Thus $\epsilon_i(Y)$ is strictly increasing, $\epsilon_i(Y) = i$ for $i = 0, 1$ and $\epsilon_n(Y) \leq \deg(Y)$.

Proposition 1. *Let D be a smooth and projective curve and E a rank 2 vector bundle on D. Set $S := \mathbb{P}(E)$. Let $\pi : S \to D$ be the ruling associated to E. Fix a morphism $u : D \to \mathbb{P}^n$, $n \geq 2$, birational onto its image and such that $u(T)$ is a line for every general fiber T of π. Fix a general $Q \in u(S)$. Then there exists an integral curve $Y \subset u(S)$ such that $Y \cap u(S)_{reg} \neq \emptyset$, $Q \in Y$ and $S = T(Y)$.*

Proof. It is sufficient to prove the existence of an integral curve $X \subset S$ such that X is not a fiber of π and $\pi|X : X \to D$ is inseparable (hint: take $Y := u(X)$). The morphism u is induced by an $(n+1)$-dimensional linear subspace V of the space of sections L of a line bundle L on S whose restriction to each fiber of π has degree 1. Fix a basis $z_0, \dots, z_n$ of L. The sections $z_0^p, \dots, z_n^p \in H^0(S, L^{\otimes p})$ have no base points and spans an $(n-1)$-dimensional linear subspace $V^{(p)}$ of $H^0(S, L^{\otimes p})$ inducing a generically injective morphism $u_p : S \to \mathbb{P}^n$ with inseparable degree p. Fix a very ample $M \in \text{Pic}(D)$ such that $h^0(D, M) \geq n + 1$. The projection formula gives $H^0(S, L^{\otimes p} \otimes \pi^*(M)) \cong H^0(D, u_*(L^{\otimes p}) \otimes M)$. Hence we may see any mz_i^p, $m \in H^0(D, M)$, as a section of $H^0(S, L^{\otimes p} \otimes \pi^*(M))$. Fix general $m_i \in H^0(D, M)$, $0 \leq i \leq n$, and let $X \subset S$ be the zero-locus of the section $m_0 z_0^p + \cdots + m_n z^p$. Since $m_i \neq 0$, X is an effective divisor such that $\pi|X$ has degree p and differential identically zero. Thus it is sufficient to prove that for general $m_0, \dots, m_n$ the divisor is integral. Assume that this is not the case. Since $\pi|X$ has degree p and differential identically zero, the divisor X would be of the form pA with A irreducible and $\deg(\pi|A) = 1$. Since $\mathcal{O}_S(X) \cong L^{\otimes p} \otimes \pi^*(M)$, we get $L^{\otimes p} \otimes \pi^*(M) \cong \mathcal{O}_S(A)^{\otimes p}$. This is absurd if $\deg(M)$ is not divisible by p.

Since we may find a non-constant family of such curves Y, the general element of this family will contain a general point, Q, of $u(S)$. □

Proposition 2. *Fix a hyperplane $H \subset \mathbb{P}^n$, $n \geq 3$, $O \in \mathbb{P}^n \setminus H$ and an integral curve $D \subset H$ spanning H. Let G be the cone with vertex O and D as its basis.*

(i) Let $Y \subset G$ be an integral curve spanning $\mathbb{P}^n$ and such that $T(Y) = G$ (or, equivalently, a strange curve with O as its strange point). Then $O(Y, 2, P) = T_P G$ for a general $P \in Y$.

(ii) Assume $\epsilon_2(D) = 2$. Let $E \subset G$ be an integral and non-degenerate curve such that $T(E) \neq G$. Then $T_P E \neq T_P G$ for a general $P \in E$.

Proof. The parenthetical sentence is obvious. Let $\ell : \mathbb{P}^n \setminus \{O\} \to H$ be the linear projection from O. Fix a general $P \in Y$. Let $f : Y \setminus \{O\} \to D$ be the morphism induced by ℓ. Even when $O \in Y$ the set $f(Y \setminus \{O\})$ contains all except finitely many points of D. For general P we may assume $f(P) \in D_{reg}$. For general P we may also assume the existence of a connected subscheme Z of Y with Z as its reduction and with $f(Z)$ the degree 2 Cartier of D with $f(P)$ as its support. Let $M := \langle \{O\} \cup T_{f(P)}D \rangle$ be the plane spanned by O and $T_{f(P)}D$. We have $M = T_P G$. Since $f(Z)$ spans $T_{f(P)}D$, Z is not contained in the line $\langle \{P, O\} \rangle = T_P Y$. Thus $M = O(Y, 2, P)$.

Now assume $\epsilon_2(D) = 2$ and fix an integral and non-degenerate curve $E \subset G$. Call $m : E \setminus \{O\} \to D$ the morphism induced by ℓ. Fix a general $P \in E$. Hence $m(P)$ is general in D. Hence the order of contact $\epsilon_2(D, m(P))$ of the tangent line of D at $m(P)$ with D is equal to the generic one, i.e. $\epsilon_2(D, m(P)) = \epsilon_2(D) = 2$. Since O is not a strange point of E and P is general, $T_P E \neq \langle \{P, O\} \rangle$, i.e. m induces an étale map at P. The plane $T_P G$ is the linear span A of $T_P D$ and O. Assume $T_P G = A$. Since m is étale at P, we get $\epsilon_2(D, m(P)) \geq \epsilon_3(E, P) \geq 3$, contradiction. □

Proposition 3. *Let $S \subset \mathbb{P}^3$ be a smooth quadric surface. For all positive integers a, b let $|\mathcal{O}_X(a, b)$ the set of all divisors of type (a, b) on S and $S(a, b)$ the subset of it parametrizing the integral curves. Let $\pi_i : S \to \mathbb{P}^1$, $i = 1, 2$, be the projection onto the i-th factor, i.e. the projection with as fibers the lines of type $(1, 0)$ (case $i = 1$) or of type $(0, 1)$ (case $i = 2$). Let $S_i(a, b)$ the subset of $S(a, b)$ formed by the curves A such that the lines of type $(1, 0)$ (case $i = 1$) or $(0, 1)$ (case $i = 2$) on S are the tangent lines to A at its smooth points (or at its general points), i.e. the curves $A \in S(a, b)$ such that $\pi_i | A$ is not separable. For each $A \in S_i(a, b)$ let $\sigma_{A,i}$ denote the inseparable degree of the morphism $\pi_i | A$*

(a) $S_i(a_1, a_2) \neq \emptyset$ if and only $a_i \equiv 0 \pmod p$. There is $A \in S_i(a_1, a_2)$ such that $\sigma_{A,i} = p^e$ if and only if $a_i \equiv 0 \pmod{p^e}$. For any $A \in S_i(a, b)$ we have $\sigma_{A,i} = \epsilon_2(A)$.

(b) Fix $Y \in S'(a_1, a_2)$. The following conditions are equivalent:

(b1) $Y \in (S_1(a_1, a_2) \cup S_2(a_1, a_2))$*;*

(b2) $S = T(Y)$*;*

(b3) $O(Y, 2, P) = T_P S$ *for a general* $P \in Y$*;*

(b4) $Y \in (S_1(a_1, a_2) \cup S_2(a_1, a_2))$*, say* $Y \in S_i(a_1, a_2)$*, and* $\sigma_{Y,i}$ *divides* $\epsilon_3(Y)$*.*

Proof. The two existence statement in part (a) are known for all Hirzebruch surfaces. We will prove it in a different way in Example 1. The last sentence in part (a) is obvious. Fix $Y \in S(a_1, a_2)$ and a general $P \in Y$. Since the lines of S are the fibers of the projections, $Y \in (S_1(a_1, a_2) \cup S_2(a_1, a_2))$ if and only if $S = T(Y)$, i.e. (b1) and (b2) are equivalent. Since (b1) and (b2) are equivalent, [2], Proposition 3.3, shows that (b3) and (b4) are equivalent. Assume $T_P S = O(Y, 2, P)$. Let Z be the connected component of the scheme $T_P S \cap Y$ with $\{P\}$ as its reduction. We have $T_P S = O(Y, 2, P)$ if and only if Z spans the plane $T_P S$. Since Y is smooth at P, Z is curvilinear. First assume $Y \in (S_1(a_1, a_2) \cup S_2(a_1, a_2))$, say $Y \in S_1(a_1, a_2)$. Hence $T_P Y$ is the line $\pi_1^{-1}(\pi_1(P))$. In order to obtain a contradiction we assume $T_P S \neq O(Y, 2, P)$. Since $T_P Y = \pi_1^{-1}(\pi_1(P)) \subset O(Y, 2, P)$, the scheme-theoretic intersection $S \cap O(Y, 2, P)$ is the union of $\pi_1^{-1}(\pi_1(P))$ and a line R of type $(0, 1)$ not containing P. Since $Y \cap O(Y, 2, P) \subseteq \S \cap O(Y, 2, P)$ we get $Z \subset \pi_1^{-1}(\pi_1(P))$. Hence Z does not span $O(Y, 2, P)$, contradiction.

Now assume $T_P S \neq O(Y, 2, P)$. Hence $T_P S \cap O(Y, 2, P)$ is a line and $O(Y, 2, P) \cap S$ is a smooth conic containing P and tangent to $T_P S \cap O(Y, 2, P)$ at P. Since the smooth conic $T_P S \cap O(Y, 2, P)$ is a divisor of type $(1, 1)$, it intersects the two lines $\pi_i^{-1}(\pi_i(P))$ in a length one scheme. Thus $T_P Y \neq \pi_i^{-1}(\pi_i(P))$. Thus both $\pi_1|Y$ and $\pi_2|Y$ are separable, i.e. $Y \notin S_1(a_1, a_2) \cup S_2(a_1, a_2)$. □

By [3], Example 4.1, there are $Y \in S_1(a, b)$ which are smooth.

Example 1. Fix two distinct points P_1, P_2 of $\mathbb{P}^2$. Let $\pi : \Pi \to \mathbb{P}^2$ be the blowing up of these two points and $u : \Pi \to S$ the contraction of the strict transform of the line $\langle \{P_1, P_2\} \rangle$. The surface S is a smooth quadric surface and the two rulings of S are induced by the pencil of lines in $\mathbb{P}^2$ passing through P_1 or P_2. For all positive integers a_1, a_2 there is a bijection between $Y \in S'(a_1, a_2)$ and the integral plane curves C_Y of degree $a_1 + a_2$ such that P_i has multiplicity a_{3-i} at P_i. We have $Y \in S_i(a_1, a_2)$ if and only if C_Y is a strange curve with P_i as its strange point. Moreover, $\sigma_{A,i} = \epsilon_2(C_A)$ for all $A \in S_i(a_1, a_2)$. For the construction of all plane strange curves with fixed degree, fixed strange point and fixed ϵ_2, see [1].

Proof of Theorem 1. Fix a general $P \in Y$. Since $Y \cap S_{reg} \neq \emptyset$, both Y and S are smooth at P. Notice that $T_P S$ is the only plane M containing the line $T_P Y$ and such that the schemes $M \cap S$ and $T_P Y$ are different in a neighborhood of P. Since $Y \subset S$ and the connected component of $O(Y, 2, P) \cap Y$ with $\{P\}$ as its reduction has length $\epsilon_3(Y) > \epsilon_2(Y)$, we get $O(Y, 2, P) = T_P S$. □

Remark 1. It is very easy to construct smooth and non-degenerate surfaces $S \subset \mathbb{P}^n$, $n \geq 5$, such that for any non-degenerate curve $Y \subset S$ we have $T_P S \neq O(Y, 2, P)$ for a general $P \in Y$. It is sufficient that S contains finitely many plane curves and the existence of an open subset U of S such that $S \setminus U$ is finite and for all $P \in U$ the scheme $S \cap T_P S$ has as connected components support by P the first infinitesimal neighborhood $\{2P, S\}$ of P in S (use that if $P \in Y_{reg}$ the scheme $Y \cap O(Y, 2, P)$ is a curvilinear scheme and hence $\text{length}((Y \cap O(Y, 2, P)) \cap \{2P, S\}) \leq 2)$. To get such a surface S start from an arbitrary smooth surface $X \to \mathbb{P}^r$, take a Veronese embedding $j : \mathbb{P}^r \hookrightarrow \mathbb{P}^N$, $N := \binom{r+d}{r} - 1$, of order $d \geq 3$ and then take a general linear projection of $j(X)$.

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References

[1] V. Bayer, A. Hefez, Strange curves, *Comm. Algebra*, **19**, No. 11 (1991), 3041-3059.

[2] A. Hefez, N. Kakuta, Duality of osculating developables of projective curves, *Comm. Algebra*, **29**, No. 1 (2001), 285-301.

[3] H. Kaji, On the tangentially degenerate curves, *J. London Math. Soc.*, **33**, No. 2 (1986), 430-440

[4] D. Laksov, Wronskians and Plücker formulas for linear systems on curves, *Ann. Scient. École Norm. Sup.*, **17** (1984), 565-579.