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PATH SATURATED GRAPHS

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Abstract: This paper deals with characterization for a graph to be saturated according to a path of order m. In this paper we have a full characterization for a connected graph to be P_m-saturated for $m \leq 6$ and we have some general results, closing with a conjecture for the related Turán type problem.

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1. Introduction

We study undirected graphs without loops or multiple edges. Given a graph G; $V(G), E(G), v(G)$ and $e(G)$ stand for the set of vertices, the set of edges, the *order* (number of vertices) and the *size* (number of edges) of G. $p(G)$ stands for the order of the longest path in the graph G. By $d(x)$ we denote the degree of a vertex x and $\delta(G) = min\{d(x)|x \in V(G)\}$.

K_n, E_n, P_n and S_n stand for the complete graph, the empty graph, a path and a star of order n. If H is a subgraph of the graph G we denote by $G[H]$ the subgraph of G induced by H. For two graphs G and H we denote by $G + H$ the graph obtained from the disjoint union $G \dot\cup H$ by adding all edges between G and H. By $\overline{G}$ we denote the graph on the same vertex set as G such that $e \in E(\overline{G})$ if and only if $e \notin E(G)$.

Given a set of forbidden graphs $\mathcal{R}$, we say that the graph G is $\mathcal{R}$-*saturated* if it contains no $R \in \mathcal{R}$ but the addition of any new edge gives a forbidden subgraph. We denote the set of $\mathcal{R}$-saturated graphs of order n by $SAT(n, \mathcal{R})$, and the set of connected $\mathcal{R}$-saturated graphs of order n by $SAT^*(n, \mathcal{R})$. Fur-

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thermore we denote by $ex(n, \mathcal{R})$ and $mx(n, \mathcal{R})$ the maximal and the minimal size of a $\mathcal{R}$-saturated graph of order n. By $EX(n, \mathcal{R})$ and $MX(n, \mathcal{R})$ we denote the sets of all $\mathcal{R}$-saturated graphs of order n and size $ex(n, \mathcal{R})$ or $mx(n, \mathcal{R})$, respectively. A graph $G \in EX(n, \mathcal{R})$ ($G \in MX(n, \mathcal{R})$) will be called a *maximal* (*minimal*) $\mathcal{R}$-saturated graph. If $\mathcal{R} = \{H\}$, for a certain graph H, we write H-saturated instead of $\{H\}$ - saturated. A graph G is called path-saturated if $p(G + x_i x_j) > p(G), \forall x_i x_j \in E(\overline{G})$.

The problem of determining the value of $ex(n, \mathcal{R})$ is called *a Turán-type extremal problem*. In 1941 Turán proved, see Turán [14], that the only maximal K_p-saturated graph of order n is the Turán graph $T_{p-1}(n)$ which is the complete $(p-1)$-partite graph with as equal classes as possible. The Turán graph $T_{p-1}(n)$ is the only complete $(p-1)$-partite graph with difference not greater then one between any two classes. The size of $T_{p-1}(n)$ is denoted by $t_{p-1}(n)$. Turán probably got the idea for this kind of problem from a theorem that Ramsey proved 13 years before, see Ramsey [12]. He proved that for every s and t, there exist a sufficiently large n and a graph G of order n such that $K_s \subseteq G$ or $K_t \subseteq \overline{G}$. Ramsey searched for a condition on the order of a graph G that will assure existence of a complete subgraph (of G or $\overline{G}$). On the other hand Turán searched for a condition on the size of a graph that will assure existence of a complete subgraph. Due to Ramsey's and Turán's types of problems, Erdös and Sós presented a new type of problem called *a Ramsey-Turán type extremal problem*, see Erdös et al [8]. That is finding the largest number of edges in a graph of order n not containing a complete subgraph of order t and no independent set of vertices of order s. Denoting this number by $g(n, t, s)$ they proved that $g(n, t, s) \leq \frac{ns}{2}$, and for sufficiently large n, $g(n, 5, s) = \frac{(1+o(1))n^2}{4}$ and $g(n, 4, s) \leq \frac{(1+o(1))n^2}{6}$. The last inequality was improved by Szemerédi [13]. He showed that for sufficiently large n, $g(n, 4, s) \leq \frac{(1+o(1))n^2}{8}$. In 1976 Bollobás and Erdös showed that the last inequality is actually an equality, see Bollobás et al [5].

Actually, Turán was not really the first one to deal with the type of problem that is named after him. In 1907, the theorem of Turán was already proved for the case of K_3 by Mantel [10], and in 1938 Erdös raised the question about the maximal size of a C_4-saturated graph, in connection with number theory, see Erdös [6]. The first theorem of *Turán-type* is probably: "a graph of order n and size n contains a cycle" namely $ex(n, \mathcal{C}) = n - 1$, where $\mathcal{C}$ is the set of all cycles. Furthermore it is easy to see that $SAT(n, \mathcal{C}) = \mathcal{T}_n$.

In time another type of problem was raised, finding the minimal size of a $\mathcal{R}$-saturated graph of order n for a set $\mathcal{R}$ of forbidden graphs. The first to deal with this type of problem were Erdös, Hajnal and Moon, see Erdös et al [7]. They proved that the only minimal K_p-saturated graph of order n is the graph $K_{p-2} + E_{n-p+2}$. Ollman proved, see Ollman [11], that $mx(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor$. Finally Kászonyi and Tuza proved the following results, see Kászonyi et al [9],

(i) $mx(n, S_m) = \begin{cases} \frac{(m-1)n - \lfloor \frac{m^2}{4} \rfloor}{2} & n \geq m + \lfloor \frac{m}{2} \rfloor \\ \binom{m}{2} + \binom{n-m}{2} & m+1 \leq n \leq m + \lfloor \frac{m}{2} \rfloor \end{cases}$

(ii) If $n \geq 3m - 3$, then $mx(n, mP_2) = 3m - 3$.

(iii) If $n \geq a_m$ and $m \geq 6$, then $mx(n, P_m) = n - \lfloor \frac{n}{a_m} \rfloor$ where

$$a_m = \begin{cases} 3 \cdot 2^{\frac{m-2}{2}} - 2 & m \equiv 0 \bmod 2, \\ 2^{\frac{m+1}{2}} - 2 & m \equiv 1 \bmod 2. \end{cases}$$

2. General Notations

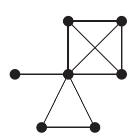
If G is P_m-saturated, then every component of G is P_m-saturated. Therefore we will focus on characterizing the connected P_m-saturated graphs, that is the set $SAT^*(n, P_m)$.

We start by denoting some general graphs.

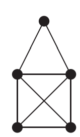
Notation 2.1. Let $t, r_1, .., r_s$ be $s + 1$ natural numbers, $t < r_1 \leq r_2 \leq .. \leq r_s$. We denote by $P_t(r_1, .., r_s)$, the graph obtained from s complete graphs, $K_{r_1}, .., K_{r_s}$ having exactly t common vertices. We call this kind of graph a t-pack.

The clique induced on the t common vertices is called the center of the pack.

Example 2.2.



$P_1(2, 3, 4)$



$P_2(3, 4)$.

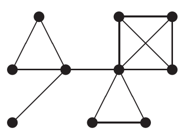
Notation 2.3. Instead of repeating s times the same number k, we write $s \times k$. For instance, instead of writing $P_2(2, 2, 2, 3, 3, 3, 3)$, we write $P_2(3 \times 2, 4 \times 3)$.

Notation 2.4. For every natural number n, a windmill of order n is

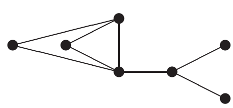
$$W_n = \begin{cases} P_1(m \times 3) & n = 2m+1 \\ P_1(2,(m-1) \times 3) & n = 2m \end{cases}$$

Notation 2.5. Let $t, r_1, .., r_s \in N$, $t < r_1 \leq r_2 \leq .. \leq r_s$ and $c, d_1, .., d_k \in N$, $c < d_1 \leq d_2 \leq .. \leq d_k$. If $P_1 = P_t(r_1, .., r_s)$ and $P_2 = P_c(d_1, .., d_k)$ are two packs, we denote by $P_1 \diamond P_2$, the graph obtained from a disjoint union of P_1 and P_2 with the addition of an edge connecting a vertex from the center of P_1 to a vertex from the center of P_2.

Example 2.6.



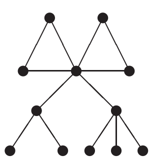
$P_1(2,3) \diamond P_1(3,4)$



$P_2(3,3) \diamond P_1(2,2)$

Notation 2.7. Let G be the graph obtained from a windmill of order t and s stars of orders $r_1, .., r_s \geq 4$ all sharing one vertex, a leaf vertex in case of the stars. We call this graph a "bird" and denote it by $B(t; r_1, .., r_s)$.

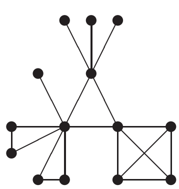
Example 2.8.



$B(5; 4, 5)$

Notation 2.9. Let $t \geq 1$ be a natural number, and let G be a graph obtained from a complete graph on t vertices, T, and t mutually disjoint 1-packs $P_1, .., P_t$. Such that the center of each pack is identified with one vertex of T. We call such a graph a t-flower, and we denote it by $F_t(P_1, .., P_s)$.

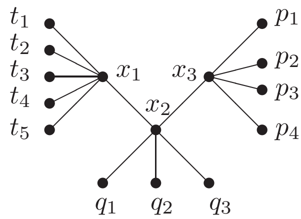
Example 2.10.



$F_3(S_4; W_6; K_4)$

Notation 2.11. Let $k, r, s \geq 0$ be three integers. we denote by $T(k, r, s)$ the tree obtained from a path of order 3, $x_1x_2x_3$, with the addition of k edges $x_1t_1, .., x_1t_k$, r edges $x_2q_1, .., x_2q_r$ and s edges $x_3p_1, .., x_3p_s$. We call this kind of graph "an eagle".

Example 2.12.



$T(5, 3, 4)$

3. Theorems

Lemma 3.1. *Let G be a connected P_m-saturated graph of order $n > m$. If $P = x_1, .., x_k$ is a maximal path in G then x_1 and x_k are not adjacent in G.*

Proof. Suppose that x_1 and x_k are adjacent in G, then G contains a cycle C of order k on $\{x_1, .., x_k\}$. Since G is connected of order $n > m > k$ there exists a vertex $t \in V(G) - V(C)$ which adjacent to a vertex of C, following that G contains a path of order $k + 1$, contradicting the maximality of P.

Corollary 3.2. *If G is a connected P_m-saturated graph of order n, such that $n > m$, then G is C_{m-1}-free.*

Theorem 3.3. *For all $n \geq 4$, $SAT^*(n, P_4) = \{S_n\}$.*

Proof. Clearly S_n is P_4-saturated for all $n \geq 4$.

Now, let G be a connected graph of order $n \geq 4$ which is P_4-saturated. Let $P = x_1 \dots x_m$ be a path in G with a maximal length. It is easy to see that m must be 3. Since G is a connected graph and from the minimality of the length of P it follows that $\forall t \in V(G), t \neq x_2, t$ is adjacent to x_2.

Notice that the graph S_n is the graph $P_1((n-1) \times 2)$.

Theorem 3.4. *For all $n \geq 5$,*

$$SAT^*(n, P_5) = \{P_1((n-3) \times 2, 3)\} \cup \{S_r \diamond S_t | 3 \leq r, t\}.$$

Proof. Clearly all the above graphs are P_5-saturated. Now let G be a connected P_5-saturated graph of order $n \geq 5$, and let $P = x_1, .., x_m$ be a maximal path in G. It is easy to see that $3 \leq m \leq 4$.

Moreover $m \neq 3$, because in that case, from the maximality of P, all the edges of G are incident to x_2, following that G is contained in S_n, which is not P_5-saturated.

Thus $m = 4$, and from the maximality of P, G does not contain an edge incident to x_1 or x_4. Furthermore, because G is P_5-free, G does not contain an edge distinct from P, and it does not contain a common neighbor of x_2 and x_3. Finally, from lemma 3.1 we derive that x_1 and x_4 are not adjacent. If x_1 is adjacent to x_3 then, clearly, $d(x_2) = 2$ and $G \subseteq P_1((n-3) \times 2, 3)$, following that $G = P_1((n-3) \times 2, 3)$. Same thing if x_4 is adjacent to x_2. Otherwise G is contained in $S_r \diamond S_t$, where r, $t \geq 3$.

Before we introduce the next theorem, note that every bird is P_6-free.

If $t = 1$ or $t = 2$, the bird $B(t; r_1, .., r_s)$ is P_6-saturated if and only if $s \geq 3$ and $r_i \geq 4$ for all $1 \leq i \leq s$.

Furthermore, if $t \geq 3$, the bird $B(t; r_1, .., r_s)$ is P_6 - saturated if and only if $s \geq 2$ and $r_i \geq 4$ for all $1 \leq i \leq s$.

Notation 3.5. Denote by $B(6)$ the set of birds which satisfy the above conditions and therefore are P_6-saturated.

Theorem 3.6. *For all* $n \geq 6$, $SAT^*(n, P_6) = \{W_n, S_{n-4} \diamond W_4, P_1((n-4) \times 2, 4), P_2((n-2) \times 3)\} \cup \{F_3(S_k, S_t, S_r) | k, t, r \geq 2\} \cup B(6)$

Proof. Clearly all the above graphs are P_6-saturated.

Let G be a connected P_6-saturated graph of order $n \geq 6$, it is sufficient to prove that G is contained in one of the above graphs. Let $P = x_1, .., x_m$ be a maximal path in G. It is easy to see that $4 \leq m \leq 5$.

In the same way as in the case of P_5, we conclude that if $m = 4$, then from the maximality of the length of P, G does not contain an edge distinct from P. Either $x_1x_3 \notin E(G)$ or $x_2x_4 \notin E(G)$. Furthermore no two vertices of P have a mutual neighbor outside of P. From all that we have that $G \subseteq S_r \diamond S_t$; $r, t \geq 2$ or $G \subseteq P_1((n-3) \times 2, 3)$ which are not P_6-saturated, therefore $m = 5$.

If G contains an edge distinct from P then at least one of its vertices is adjacent to x_3, and $G[P] \subseteq P_1(3, 3)$, hence G is a windmill or a bird.

Suppose now that G does not have an edge distinct from P.

If $e(G[p]) = 4$, it is easy to see that either G is an eagle or a proper subgraph of $P_2((n-2) \times 3)$. Therefore G is not P_6-saturated.

Suppose $e(G[p]) = 5$. if $x_1x_3 \in E(G)$ or $x_3x_5 \in E(G)$, then $G \subseteq S_k \diamond P_1((n-k-3) \times 2, 3)$ which is P_6-saturated if and only if $G = S_{n-4} \diamond W_4$. if

$x_1x_4 \in E(G)$ or $x_2x_5 \in E(G)$, then G is a proper subgraph of $P_2((n-2)\times 3)$, therefore it is not P_6-saturated. If $x_2x_4 \in E(G)$, then $G \subseteq P_2((n-2)\times 3)$ or $G \subseteq F_3(S_k; S_t; S_r), 2 \leq k, t, r.$ which are both P_6-saturated.

Now suppose $e(G[p]) \geq 6$. If $x_1x_3, x_2x_4 \in E(G)$ or $x_1x_3, x_1x_4 \in E(G)$ or $x_2x_4, x_3x_5 \in E(G)$ or $x_2x_5, x_3x_5 \in E(G)$, then $G \subseteq P_1((n-4)\times 2, 4)$. If $x_1x_3, x_3x_5 \in E(G)$ then $G \subseteq W_n$. If $x_1x_3, x_2x_5 \in E(G)$ or $x_1x_4, x_3x_5 \in E(G)$ then G contains a cycle of order 5 and that contradicts the fact that G is P_6-free.

We continue now with some general results.

Remark 3.7. Notice that $v(P_t(r_1,..,r_s)) = r_1 + .. + r_s - t(s-1)$. Furthermore $p(P_t(r_1,..,r_s)) = r_s + r_{s-1} + .. + r_{s-t} - t^2$ if and only if $s > t$.

Theorem 3.8. *Let $G = P_t(r_1,..,r_s)$ be a graph of order n. If $s \leq t$, then $p(G) = n$. Otherwise, G is path-saturated if and only if $r_1 + r_2 - t > r_{s-t}$.*

Proof. The first part of the theorem is trivial and the second derives from the fact that by adding an edge connecting two vertices from K_{r_1} and K_{r_2}, the two smallest complete subgraphs creating G, we get the smallest addition to the longest path. This addition is $r_1 + r_2 - t - r_{s-t}$, in case this expression is positive. Hence $p(G + x_{r_i}x_{r_j}) > p(G)$ if and only if $r_1 + r_2 - t > r_{s-t}$.

Theorem 3.9. *Let $n, m, s, t \in N$, $3 \leq m < n$ and let $t < r_1 \leq r_2 \leq .. \leq r_s$ be natural numbers. A graph $G = P_t(r_1,..,r_s)$ of order n is P_m-saturated if and only if:*

a. $r_s + r_{s-1} + .. + r_{s-t} - t^2 < m$.

b. $r_1 + r_2 + r_s + r_{s-1} + .. + r_{s-t+1} - t(t+1) \geq m$.

c. $s \geq t + 2$.

Proof. The theorem derived from the fact that, in case that $s \geq t+2$, by adding an edge $x_{r_i}x_{r_j}$ to G, connecting a vertex in K_{r_i} to a vertex in K_{r_j} we have $p(G + x_{r_i}x_{r_j}) \geq r_1 + r_2 + r_s + r_{s-1} + .. + r_{s-t+1} - t(t+1)$ and G contains a path of that size in case the two vertex are in K_{r_1} and K_{r_2} accordingly.

Corollary 3.10. *If $G = P_t(r_1,..,r_s)$ is a P_m-saturated graph, for some $3 \leq m < n$, then $t < \frac{m}{2}$.*

Proof. Otherwise, since $s \geq t+2$, G is not P_m-free.

Remark 3.11. Notice that for $4 \leq m = 2k$, an even natural number, if $r \geq k+1$, $P_{k-1}(r \times k)$ is P_m-saturated. For $3 \leq m = 2k+1$, an odd natural number, if $r \geq k+1$, $P_{k-1}(r \times k, k+1)$ is a P_m-saturated graph.

Lemma 3.12. *Let k be a natural number. If G is a connected graph of order ≥ 3 such that for any two vertices x and y, $d(x) + d(y) \geq k$, then G contains a path of order $k + 1$.*

Proof. See Bollobás [4] pp. 69-70.

Corollary 3.13. *Let m, n be natural numbers such that $2 \leq m < n$, and let G be a connected graph of order n. If $\delta(G) \geq \lfloor \frac{m}{2} \rfloor$, then G contains a path of order m.*

The last results leads me to the next conjecture.

Conjecture 3.14. *Let m, n be natural numbers such that $3 \leq m < n$, then*

$$EX(n, P_m) = \begin{cases} P_{k-1}((n-k+1) \times k) & m = 2k \\ P_{k-1}((n-k-1) \times k, k+1) & m = 2k+1 \end{cases}$$

Hence

$$ex(n, P_m) = \begin{cases} \frac{4mn-8n-m^2+2m}{8} & m \text{ even} \\ \frac{4mn-12n-m^2+4m+5}{8} & m \text{ odd} \end{cases}$$

Remark 3.15. By Theorems 3.3, 3.4 and 3.6 we can see that the conjecture is valid for $m \leq 6$.

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