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SUBORDINATION FOR MEROMORPHIC FUNCTIONS DEFINED BY AN OPERATOR

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Abstract: By making use of a linear operator that is defined by means of the Hadamard product (or convolution), we investigate the properties of a certain family of meromorphically multivalent functions. Some subordination relations are established and some known results are extended.

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1. Introduction

Let $\Sigma_{p,\alpha}$ denote the class of functions of the form

$$f(z) = \frac{1}{z^{p+\alpha}} + \sum_{n=1}^{\infty} a_n z^{n-p-\alpha} \quad \Big(0 \le \alpha \le 1, \, p \in \mathbf{N}, \;\; n > p+\alpha\Big), \tag{1}$$

which are analytic in the punctured unit disk $U := \{z \in \mathbf{C}, 0 < |z| < 1\}$. The convolution of two power series $f(z)$, given by (1) and $g(z) = \frac{1}{z^{p+\alpha}} + \sum_{n=1}^{\infty} b_n z^{n-p-\alpha}$ is defined as the following power series:

$$f(z) * g(z) = \frac{1}{z^{p+\alpha}} + \sum_{n=1}^{\infty} a_n b_n z^{n-p-\alpha}.$$

Recalling the principle of subordination between analytic functions, let the functions f and g be analytic in $\triangle := \{z \in \mathbf{C}, |z| < 1\}$ Then we say that the

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function f is *subordinate* to g if there exists a Schwarz function $w(z)$, analytic in $\triangle$ such that $f(z) = g(w(z)),\ z \in \triangle$. We denote this subordination by $f \prec g \ or \ f(z) \prec g(z),\ z \in \triangle$. If the function g is univalent in $\triangle$ the above subordination is equivalent to $f(0) = g(0) \ and \ f(\triangle) \subset g(\triangle)$.

A function $f(z) \in \Sigma_{p,\alpha}$ belongs to the class $\mathcal{S}_{p,\alpha}(\mu)$ the class of meromorphically $p+\alpha$-valent starlike functions of order μ where $0 \le \mu < p+\alpha$, if and only if if $f \neq 0$, and $-\Re\{\frac{zf'(z)}{f(z)}\} > \mu,\ (z \in U)$. A function $f(z) \in \Sigma_{p,\alpha}$ belongs to the class $\mathcal{C}_{p,\alpha}(\mu)$ the class of meromorphically $p+\alpha$-valent convex functions of order μ where $0 \le \mu < p+\alpha$, if and only if if $f' \neq 0$, and $-\Re\{1+\frac{zf''(z)}{f'(z)}\} > \mu$. Define the function $\phi_{p,\alpha}(a, c; z)$ by

$$\phi_{p,\alpha}(a, c; z) := \frac{1}{z^{p+\alpha}} + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-p-\alpha} \qquad (z \in U),$$

where $a \in \mathbf{R}, c \in \mathbf{R}\backslash\{0, -1, -2, ...\}$ and $(a)_n$ is the Pochhammer symbol. Note that $\phi_{p,\alpha}(a, c; z) = z^{-p-\alpha} \ {}_2F_1(1, a, c; z)$ where ${}_2F_1(1, a, c; z) = \sum_{n=0}^{\infty} \frac{(1)_n(a)_n}{(c)_n} \frac{z^n}{n!}$. Corresponding to the function $\phi_{p,\alpha}(a, c; z)$, we define an operator $\mathcal{L}_{p,\alpha}$ on $\Sigma_{p,\alpha}$

$$\mathcal{L}_{p,\alpha}(a, c)f(z) = \phi_{p,\alpha}(a, c; z) * f(z) = \frac{1}{z^{p+\alpha}} + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_n z^{n-p-\alpha} \qquad (z \in U). \quad (2)$$

Operator (2) reduces to one defined by Liu and Srivastava [1] when $\alpha = 0$ for meromorphic function in U. Moreover, it was studied by many authors see ([2,3,4]). It is clear that $\mathcal{L}_{p,\alpha}(a, a)f(z) = f(z)$. The next result is an extension of results can be find in [5,6]

Lemma 1. *If* $f \in \Sigma_{p,\alpha}$ *then*

$$z\Big(\mathcal{L}_{p,\alpha}(a, c)f(z)\Big)' = a\mathcal{L}_{p,\alpha}(a+1, c)f(z) - (a+p+\alpha)\mathcal{L}_{p,\alpha}(a, c)f(z) \qquad (z \in U).$$

Now we define new classes containing the operator (2).

Definition 1. Let $f \in \Sigma_{p,\alpha}$. Then f is said to be belong to $\mathcal{MS}_{p,\alpha}(\mu)$ if and only if

$$-\Re\Big\{\frac{z(\mathcal{L}_{p,\alpha}(a, c)f(z))'}{\mathcal{L}_{p,\alpha}(a, c)f(z)}\Big\} > \mu \quad (z \in U).$$

Definition 2. Let $f \in \Sigma_{p,\alpha}$. Then f is said to be belong to $\mathcal{MC}_{p,\alpha}(\mu)$ if and only if

$$-\Re\Big\{1 + \frac{\Big(z(\mathcal{L}_{p,\alpha}(a,c)f(z))'\Big)'}{\Big(\mathcal{L}_{p,\alpha}(a,c)f(z)\Big)'}\Big\} > \mu \quad (z \in U).$$

In the present paper, we establish some sufficient conditions for $f \in \Sigma_{p,\alpha}$ to satisfy

$$-[\frac{\mu}{p+\alpha}\frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))'}{\mathcal{L}_{p,\alpha}(a,c)f(z)}] \prec q(z) \quad (z \in U), \tag{3}$$

where $q(z)$ is a given univalent function in U. Moreover, we give applications for these results in fractional calculus. Note that in [7], Ravichandran et al. studied sufficient conditions for subordination for class $(\Sigma_{1,0})$, of meromorphic functions. We need the following results in the sequel.

Lemma 2. (see [8]) *Let a and b be complex constants, and let $h(z)$ be convex (univalent) in $\triangle$ with $h(0) = 1$ and $\Re[\beta h(z) + \gamma] > 0$. If $q(z) = 1 + q_1 z + q_2 z^2 + ...$ be analytic in $\triangle$, then $q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} \prec h(z)$ $(z \in \triangle)$, implies $q(z) \prec h(z)$ $(z \in \triangle)$.*

Lemma 3. (see [9]) *Let γ, δ be any complex numbers, $\delta \neq 0$. Let $q(z) = 1 + q_1 z + q_2 z^2 + ...$ be univalent in $\triangle$, $q(z) \neq 0$. Let $Q(z) = \frac{\delta z q'(z)}{q(z)}$ be starlike and $\Re\{\frac{\gamma}{\delta}q(z) + \frac{zQ'(z)}{Q(z)}\} > 0$, $(z \in \triangle)$. If $\psi(z) = 1 + c_1 z + c_2 z^2 + ...$ is analytic in $\triangle$ and satisfies $\gamma\psi(z) + \delta\frac{z\psi'(z)}{\psi(z)} \prec \gamma q(z) + \delta\frac{zq'(z)}{q(z)}$, $(z \in \triangle)$, then $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant.*

Lemma 4. (see [10]) *Let $q(z)$ be univalent in $\triangle$, and let $\varphi(z)$ be analytic in a domain containing $q(\triangle)$. If $zq'(z)\varphi(q(z))$ is starlike and $z\psi'(z)\varphi(\psi(z)) \prec zq'(z)\varphi(q(z))$ $(z \in \triangle)$, then $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant.*

2. Differential Subordination Theorems

In this section, we pose some sufficient conditions for subordination of analytic functions belong to the class $\Sigma_{p,\alpha}$.

Theorem 1. *Let $h(z)$ be convex univalent in U with $h(0) = 1$ and $\Re\{h(z)\}$ is bounded in U. If $f(z) \in \Sigma_{p,\alpha}$ satisfies the conditions $-\frac{z\Big(\mathcal{L}_{p,\alpha}(a+1,c)f(z)\Big)'}{p\mathcal{L}_{p,\alpha}(a+1,c)f(z)} \prec$*

$h(z)$ $(z \in U)$, then $-\frac{z\left(\mathcal{L}_{p,\alpha}(a,c)f(z)\right)'}{p\mathcal{L}_{p,\alpha}(a,c)f(z)} \prec h(z)$ $(z \in U)$, *for* $\Re\{h(z)\} < \frac{a+p+\alpha}{p}$ *provided* $\mathcal{L}_{p,\alpha}(a,c)f(z) \neq 0 \, in \, U$.

Proof. Define the function $q(z)$ by $q(z) := -\frac{z\left(\mathcal{L}_{p,\alpha}(a,c)f(z)\right)'}{p\mathcal{L}_{p,\alpha}(a,c)f(z)}$. By applying Lemma 1, we receive

$$pq(z) - (a + p + \alpha) = -a\frac{\mathcal{L}_{p,\alpha}(a+1,c)f(z)}{\mathcal{L}_{p,\alpha}(a,c)f(z)}. \tag{4}$$

Taking logarithmic derivatives in both sides of (4) and multiplying by z, we have

$$q(z) + \frac{zq'(z)}{-pq(z) + (a + p + \alpha)} = -\frac{z\left(\mathcal{L}_{p,\alpha}(a+1,c)f(z)\right)'}{p\mathcal{L}_{p,\alpha}(a+1,c)f(z)} \prec h(z) \quad (z \in U)$$

in virtue of Lemma 2, it follows that $q(z) \prec h(z)$ for $0 < \Re\{-ph(z) + (a + p + \alpha)\} < \infty$.

Remark 1. Theorem 1, reduces to [11 Theorem 2.1] when $\alpha = 0$.

Theorem 2. *Let $q(z)$ satisfies the conditions of Lemma 3. If $f \in \Sigma_{p,\alpha}$ satisfies* $\frac{zf''(z)}{f'(z)} - \mu\frac{zf'(z)}{f(z)} \prec \frac{zq'(z)}{q(z)} - (p + \alpha - \mu)$. *Then* $-\frac{z^{p+\alpha-\mu}f'(z)}{p[f(z)]^{\frac{\mu}{p}}} \prec q(z)$ *and q is the best dominant.*

Proof. Define the function $\psi(z)$ by $\psi(z) := -\frac{z^{p+\alpha-\mu}f'(z)}{p[f(z)]^{\frac{\mu}{p}}}$, $(z \in U)$. Then a computation gives

$$\frac{zf''(z)}{f'(z)} - \mu\frac{zf'(z)}{f(z)} = \frac{z\psi'(z)}{\psi(z)} - (p + \alpha - \mu).$$

Thus we receive the relation $\frac{z\psi'(z)}{\psi(z)} \prec \frac{zq'(z)}{q(z)}$. By an application of Lemma 3, with $\gamma = 0$ and $\delta = 1$, it follows that, $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant.

Remark 2. Theorem 2, reduces to [12 Theorem 2.1] when $\alpha = 1$.

Corollary 1. *Let $-1 \leq B < A \leq 1$. If $f \in \Sigma_{p,\alpha}$ satisfies*

$$\frac{zf''(z)}{f'(z)} - \mu\frac{zf'(z)}{f(z)} \prec -(p + \alpha - \mu) + \frac{(A - B)z}{(1 + Az)(1 + Bz)}.$$

Then $-\frac{z^{p+\alpha-\mu}f'(z)}{p[f(z)]^{\frac{\mu}{p}}} \prec \frac{1+Az}{1+Bz}$ *and* $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. Define the function $q(z)$ by $q(z) := \frac{1+Az}{1+Bz}$, $(z \in U)$. Then a computation shows that $Q(z) := \frac{zq'(z)}{q(z)} = \frac{(A-B)z}{(1+Az)(1+Bz)}$. Since $-1 \le B < A \le 1$ yields that $\Re\{Q(z)\} > 0$. Hence $Q(z)$ is starlike in U. The result now follows from Theorem 1.

The next result can be found in [7]. Assume $p = \mu = \alpha = 1$.

Corollary 2. *Let the hypothesis of Theorem 1 be hold. Then* $-[\frac{z(f(z))'}{f(z)}] \prec q(z)$, $(z \in U)$, *and* $q(z)$ *is the best dominant.*

The next results show sufficient conditions of subordinations involving the differential operator (2).

Theorem 3. *Let* q *satisfies the conditions of Lemma 4. If* $f \in \Sigma_{p,\alpha}$ *satisfies*

$$\frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))''}{(\mathcal{L}_{p,\alpha}(a,c)f(z))'} - \frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))'}{(\mathcal{L}_{p,\alpha}(a,c)f(z)} \prec \frac{zq'(z)}{q(z)} - 1.$$

Then $-[\frac{\mu}{p+\alpha}\frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))'}{\mathcal{L}_{p,\alpha}(a,c)f(z)}] \prec q(z)$ *and* q *is the best dominant.*

Proof. Define $\psi(z)$ as follows $\psi(z) := -[\frac{\mu}{p+\alpha}\frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))'}{(\mathcal{L}_{p,\alpha}(a,c)f(z)}]$. A computation gives

$$\frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))''}{(\mathcal{L}_{p,\alpha}(a,c)f(z))'} - \frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))'}{(\mathcal{L}_{p,\alpha}(a,c)f(z)} = \frac{z\psi'(z)}{\psi(z)} - 1,$$

which implies $\frac{z\psi'(z)}{\psi(z)} \prec \frac{zq'(z)}{q(z)}$. By an application of Lemma 4, with $\varphi(w) = \frac{1}{w}$, it follows that, $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant.

Corollary 3. *Let* $-1 \le B < A \le 1$. *If* $f \in \Sigma_{p,\alpha}$ *satisfies*

$$\frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))''}{(\mathcal{L}_{p,\alpha}(a,c)f(z))'} - \frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))'}{(\mathcal{L}_{p,\alpha}(a,c)f(z)} \prec \frac{p(A-B)z}{(1+Az)(1+Bz)} - 1.$$

Then $-[\frac{\mu}{p+\alpha}\frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))'}{\mathcal{L}_{p,\alpha}(a,c)f(z)}] \prec p[\frac{1+Az}{1+Bz}]$ *and* $p[\frac{1+Az}{1+Bz}]$ *is the best dominant.*

Remark 3. Corollary 3 reduces to result in [11] when $\frac{\mu}{p+\alpha} = 1$.

Corollary 4. *If* $f \in \Sigma_{p,\alpha}$ *satisfies*

$$\frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))''}{(\mathcal{L}_{p,\alpha}(a,c)f(z))'} - \frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))'}{(\mathcal{L}_{p,\alpha}(a,c)f(z)} \prec \frac{2z}{(1+z)(1-z)} - 1.$$

Then $-[\frac{\mu}{p+\alpha}\frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))'}{\mathcal{L}_{p,\alpha}(a,c)f(z)}] \prec \frac{1+z}{1-z}$ *and* $\frac{1+z}{1-z}$ *is the best dominant.*

In general we have the following result

Corollary 5. *Let w be analytic in U, such that $0 < |w(z)| < 1$. If $f \in \Sigma_{p,\alpha}$ satisfies*

$$\frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))''}{(\mathcal{L}_{p,\alpha}(a,c)f(z))'} - \frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))'}{(\mathcal{L}_{p,\alpha}(a,c)f(z)} \prec \frac{2z}{(1+w(z))(1-w(z))} - 1.$$

Then $-[\frac{\mu}{p+\alpha}\frac{z(\mathcal{L}_{p,\alpha}(a,c)f(z))'}{\mathcal{L}_{p,\alpha}(a,c)f(z)}] \prec \frac{1+w(z)}{1-w(z)}$ *and* $\frac{1+w(z)}{1-w(z)}$ *is the best dominant.*

Corollary 6. *Let the hypothesis of Theorem 3 be hold. Then* $-[\frac{z(f(z))'}{f(z)}] \prec q(z)$ *and q is the best dominant.*

The next result can be found in [11]. Assume $\mu = 1$ and $\alpha = 0$.

Corollary 7. *Let the hypothesis of Theorem 3 be hold. Then* $-[\frac{1}{p}\frac{z(\mathcal{L}_{p,\alpha}f(z))'}{\mathcal{L}_{p,\alpha}f(z)}] \prec q(z)$ *and q is the best dominant.*

Recently, fractional power are defined and studied for meromorphic functions and normalized functions in the unit disk (see [13,14]) respectively. Further, a different studies regarding fractional and other operators can also be found in [15]-[17].

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