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GENERALIZED CESÁRO INTEGRAL OPERATOR

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Abstract: In this paper, the authors introduced an integral operator which is a generalization to Cesáro integral operator and proved its properties.

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1. Introduction

The Cesáro operator $\mathcal{C}$ acts formally on the power series $f(z) = \sum_{n=0}^{\infty} a_n(f) z^n$ as

$$\mathcal{C}[f](z) = \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} \sum_{k=0}^{n} a_k(f) \Big) z^n,$$

which can be written

$$\mathcal{C}[f](z) = \frac{1}{z} \int_0^z \frac{f(\xi)}{1-\xi}. \tag{1}$$

Recently, many authors focused on the boundedness and compactness of extended Cesáro operator between several spaces of holomorphic functions. It is well known that the operator $\mathcal{C}$ is bounded on the usual Hardy spaces H^p for $0 < p < \infty$ and Bergman space, we recommend the interested readers refer to [1-4]. The Cesáro operator is unbounded on H^∞ (see [5]), so that it is reasonable to work in larger spaces of analytic functions. Moreover, the classical Cesáro means play an important role in geometric function theory (see

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[6,7,8,9]).

Let $\mathcal{H}$ be the class of functions analytic in $U := \{z \in \mathbf{C} : |z| < 1\}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$. Let $\mathcal{A}$ be the subclass of normalized functions by $f(0) = 0, f'(0) = 1$.

A function $f \in \mathcal{A}$ is said to be starlike with respect to the origin in U if it satisfies

$$f \in \mathcal{S} \Leftrightarrow \Re\{\frac{zf'(z)}{f(z)}\} > 0, \ \ (z \in U).$$

Also, a function $f \in \mathcal{A}$ is called as convex in U if it satisfies

$$f \in \mathcal{K} \Leftrightarrow \Re\{1 + \frac{zf''(z)}{f'(z)}\} > 0, \ \ (z \in U).$$

It follows that

$$f \in \mathcal{K} \Leftrightarrow zf' \in \mathcal{S}.$$

Let f be analytic and locally univalent in U. The pre-Schwarzian derivative T_f of f is defined by

$$T_f(z) = \frac{f''(z)}{f'(z)}$$

with the norm

$$\|f\| = \sup_{z \in U} |T_f|(1 - |z|^2).$$

It is known that $\|T_f\| < \infty$ if and only if f is uniformly locally univalent. It is also known that $\|T_f\| \leq 6$ for $f \in \mathcal{S}$ and that $\|T_f\| \leq 4$ for $f \in \mathcal{K}$. Moreover, it showed that when $|T_f| \leq 3.05$ then f is univalent in U. And when $|T_f| \leq 2.2, 8329..$ then f is starlike in U (see [10,11]).

Consider the general integral operator defined by the formula:

$$\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z) = \frac{1}{z}\int_0^z \Big(\frac{f_1(\xi)}{1-\xi}\Big)^{\alpha_1} ... \Big(\frac{f_m(\xi)}{1-\xi}\Big)^{\alpha_m} d\xi, \ \ (z \in U \setminus\{0\}). \quad (2)$$

where $\alpha_i \in \mathbf{R}$, $\forall i = 1, ..., m$. It is clear that when $\alpha_1 = 1$ and $\alpha_j = 0, j = 2, ..., m$ the integral operator (2) reduces to Cesáro integral operator (1).

Now a function f is called in the class Σ_ρ if and only if it has the norm

$$\|f\|_\rho = \sup_{z \in U}(1 - |z|^2)|\frac{f'(z)}{f(z)}| < \rho, \ \ \rho \in \mathbf{R}^+\setminus\{0\}.$$

2. Main Results

In this section we study some general properties for function

$$z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z)=\int_0^z\Big(\frac{f_1(\xi)}{1-\xi}\Big)^{\alpha_1}...\Big(\frac{f_m(\xi)}{1-\xi}\Big)^{\alpha_m}d\xi,\ \ (z\in U\setminus\{0\}).$$

Theorem 2.1. *Let the functions $f_j(z),\ j=1,...,m$ be starlike of order $\mu_j,\ 0\le\mu_j<1$ in U. If*

$$0\le\Re\{z\}<\frac{1}{2}\quad and\quad -\frac{1}{2}\le\alpha_j<0,$$

then $z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z)$ is convex in U and of order

$$\mu:=\sum_{j=1}^m\alpha_j(\mu_j+1)+1.$$

Proof. From the definition of the operator (2) we have

$$\frac{(z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z))''}{(z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z))'}=\alpha_1[\frac{f_1'(z)}{f_1(z)}+\frac{1}{1-z}]+...+\alpha_m[\frac{f_m'(z)}{f_m(z)}+\frac{1}{1-z}],$$

that is equivalent with,

$$\begin{aligned}&\Re\Big\{z\frac{(z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z))''}{(z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z))'}+1\Big\}\\ &=\Re\Big\{\alpha_1[\frac{zf_1'(z)}{f_1(z)}+\frac{z}{1-z}]+...+\alpha_m[\frac{zf_m'(z)}{f_m(z)}+\frac{z}{1-z}]+1\Big\}\\ &=\alpha_1[\Re\Big\{\frac{zf_1'(z)}{f_1(z)}\Big\}+\Re\Big\{\frac{z}{1-z}\Big\}]+...\\ &\qquad+\alpha_m[\Re\Big\{\frac{zf_m'(z)}{f_m(z)}\Big\}+\Re\Big\{\frac{z}{1-z}\Big\}]+1\\ &>\sum_{j=1}^m\alpha_j(\mu_j+1)+1\\ &:=\mu.\end{aligned}$$

Hence $z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z)$ is convex in U and of order $0\le\mu<1$.

Corollary 2.2. *Let the functions* $f_j(z),\ j = 1, ..., m$ *be starlike of order* μ, $0 \le \mu < 1$ *in* U. *If* $0 \le \Re\{z\} < \frac{1}{2}$ *and* $-\frac{1}{\mu+1} \le \sum_{j=1}^{m} \alpha_j < 0$, *then* $z\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z)$ *is convex in* U *order* $0 \le \sigma < 1$ *where*

$$\sigma := (\mu + 1)\sum_{j=1}^{m} \alpha_j + 1.$$

Theorem 2.3. *Let the functions* $f_j(z),\ j = 1, ..., m$ *be in the class* Σ_∞. *Then* $z\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z)$ *is uniformly locally univalent in* U.

Proof. Since $f_j \in \Sigma_\infty$ then they satisfy

$$\|f_j\|_\infty = \sup_{z\in U}(1 - |z|^2)|\frac{f_j'(z)}{f_j(z)}| < \infty, \ \ j = 1, ..., m$$

and we have

$$\frac{(z\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z))''}{(z\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z))'} = \alpha_1[\frac{f_1'(z)}{f_1(z)} + \frac{1}{1-z}] + ... + \alpha_m[\frac{f_m'(z)}{f_m(z)} + \frac{1}{1-z}].$$

Then by making a computation

$$\begin{aligned}
&\sup_{z\in U}(1 - |z|^2)|\frac{(z\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z))''}{(z\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z))'}| \\
&= \sup_{z\in U}(1 - |z|^2)|\alpha_1[\frac{f_1'(z)}{f_1(z)} + \frac{1}{1-z}] + ... \\
&+ \alpha_m[\frac{f_m'(z)}{f_m(z)} + \frac{1}{1-z}]| \\
&\le |\alpha_1|\sup_{z\in U}(1 - |z|^2)|\frac{f_1'(z)}{f_1(z)}| + |\alpha_1|\sup_{z\in U}\frac{(1 - |z|^2)}{|1-z|} + ... \\
&+ |\alpha_m|\sup_{z\in U}(1 - |z|^2)|\frac{f_m'(z)}{f_m(z)}| + |\alpha_m|\sup_{z\in U}\frac{(1 - |z|^2)}{|1-z|} \\
&\le |\alpha_1|\sup_{z\in U}(1 - |z|^2)|\frac{f_1'(z)}{f_1(z)}| + |\alpha_1|\sup_{z\in U}(1 + |z|) + ... \\
&+ |\alpha_m|\sup_{z\in U}(1 - |z|^2)|\frac{f_m'(z)}{f_m(z)}| + |\alpha_m|\sup_{z\in U}(1 + |z|),
\end{aligned}$$

we pose

$$\begin{aligned}
\|T_{z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z)}\| &< \sum_{j=1}^{m} |\alpha_j|[\|f_j\|_\infty + 2] \\
&< \infty.
\end{aligned}$$

Thus $z\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z)$ is uniformly locally univalent in U.

Theorem 2.4. *Let the functions $f_j(z)$, $j = 1, ..., m$ be in the class Σ_{ρ_j}. Then for*

$$\sum_{j=1}^{m} |\alpha_j|[\rho_j + 2] < 6,$$

the function $z\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z)$ is starlike in U.

Proof. Since $f_j \in \Sigma_{\rho_j}$ then they satisfy

$$\|f_j\|_{\rho_j} = \sup_{z\in U}(1 - |z|^2)|\frac{f_j'(z)}{f_j(z)}| < \rho_j, \ \ j = 1, ..., m.$$

From above conclusion we have

$$\begin{aligned} \|T_{z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z)}\| &< \sum_{j=1}^{m} |\alpha_j|[\|f_j\|_{\rho_j} + 2] \\ &< \sum_{j=1}^{m} |\alpha_j|[\rho_j + 2] \\ &< 6. \end{aligned}$$

Thus $z\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z)$ is starlike in U.

Corollary 2.5. *Let the functions $f_j(z)$, $j = 1, ..., m$ be in the class Σ_ρ. Then for*

$$\sum_{j=1}^{m} |\alpha_j| < \frac{3}{\frac{\rho}{2} + 1},$$

the function $z\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z)$ is starlike in U.

Theorem 2.6. *Let the functions $f_j(z)$, $j = 1, ..., m$ be in the class Σ_{ρ_j}. Then for $z \to 0$ and*

$$\sum_{j=1}^{m} |\alpha_j|[\rho_j + 1] < 3.05,$$

the function $z\mathcal{C}[f_1, ..., f_m]_{\alpha_1,...,\alpha_m}(z)$ is univalent in U.

Proof. By the assumption of the theorem we have

$$\begin{aligned}|T_{z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z)}| &= |\alpha_1[\frac{f_1'(z)}{f_1(z)}+\frac{1}{1-z}]+...+\alpha_m[\frac{f_m'(z)}{f_m(z)}+\frac{1}{1-z}]| \\ &\le |\alpha_1|[|\frac{f_1'(z)}{f_1(z)}|+1]+...+|\alpha_m|[|\frac{f_m'(z)}{f_m(z)}|+1] \\ &< |\alpha_1|[\rho_1+1]+...+|\alpha_m|[\rho_m+1] \\ &< 3.05,\end{aligned}$$

then the function $z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z)$ is univalent in U.

Corollary 2.7. *Let the functions $f_j(z),\ j=1,...,m$ be in the class Σ_ρ. Then for $z \to 0$ and*

$$\sum_{j=1}^{m}|\alpha_j| < \frac{3.05}{[\rho+1]},$$

the function $z\mathcal{C}[f_1,...,f_m]_{\alpha_1,...,\alpha_m}(z)$ is univalent in U.

Note that other work related to Cesáro can also be found in [12]-[15].

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